MPC Advanced Topics

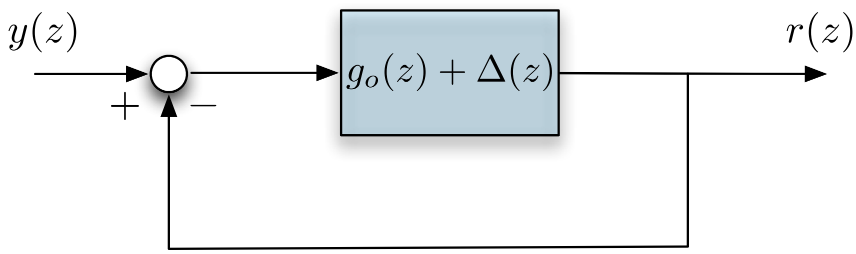
Robust MPC

- The principal reason for feedback control is to deal with uncertainty
  - Models of real-world processes cannot be known exactly
  - In addition, environmental factors introduce disturbances and sources of noise which cannot be known with precision a priori
- Typical approaches to dealing with this reality involve characterizing a nominal plant with a (simplified) mathematical model and then generating a set of other allowable plant models around this nominal plant
- Robust control system design attempts to deliver some prescribed level of stability and performance so long as the plant remains within the uncertainty set
- Most of the theory developed for robust control assumes a linear plant model
  - But model predictive control behaves in a non-linear fashion when constraints are active
- Nonetheless, predictive controllers operate linearly when formulated with quadratic costs, linear models and linear constraints, so much of the robust theory still applies

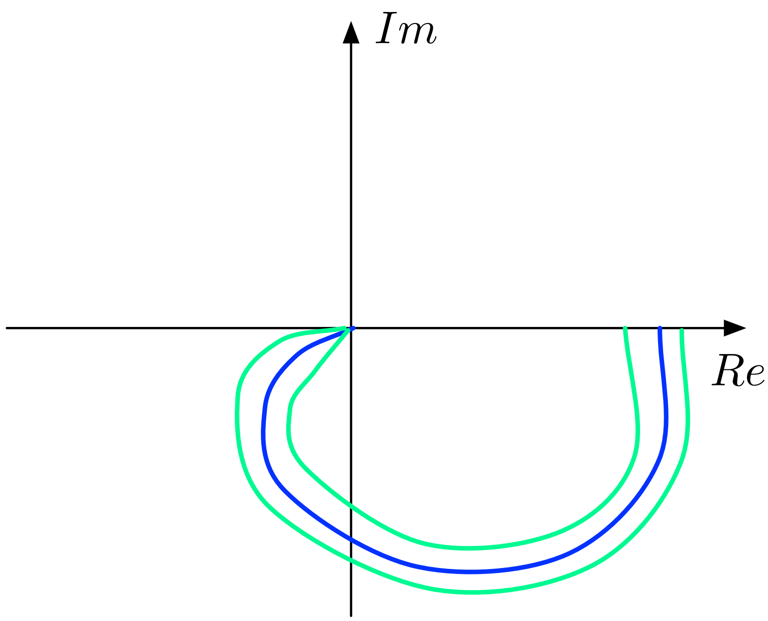
- So, we examine the question: *Can MPC be tuned systematically to give robust performance?*

Norm-bounded Uncertainty

- *Norm-bounded* uncertainty assumes the real plant is characterized within a 'set'
  - Set includes a nominal model of the plant and is bounded by a suitable uncertainty description
  - This is depicted notionally below for the single-input-single-output case



- A frequency domain depiction might look like the following



- In the general sense, the uncertainty parameter $\triangle$ is a stable bounded operator and $g_o(z)$ is the nominal plant model

- Obviously, one does not know exactly what $\triangle$ is
- A number of assumptions can be made about the nature of $\triangle$, but the most useful is a bound of some type on the 'size' of the uncertainty
- Typically, this comes in the form of a matrix norm:

$$\|\triangle\|_p \leq \delta$$

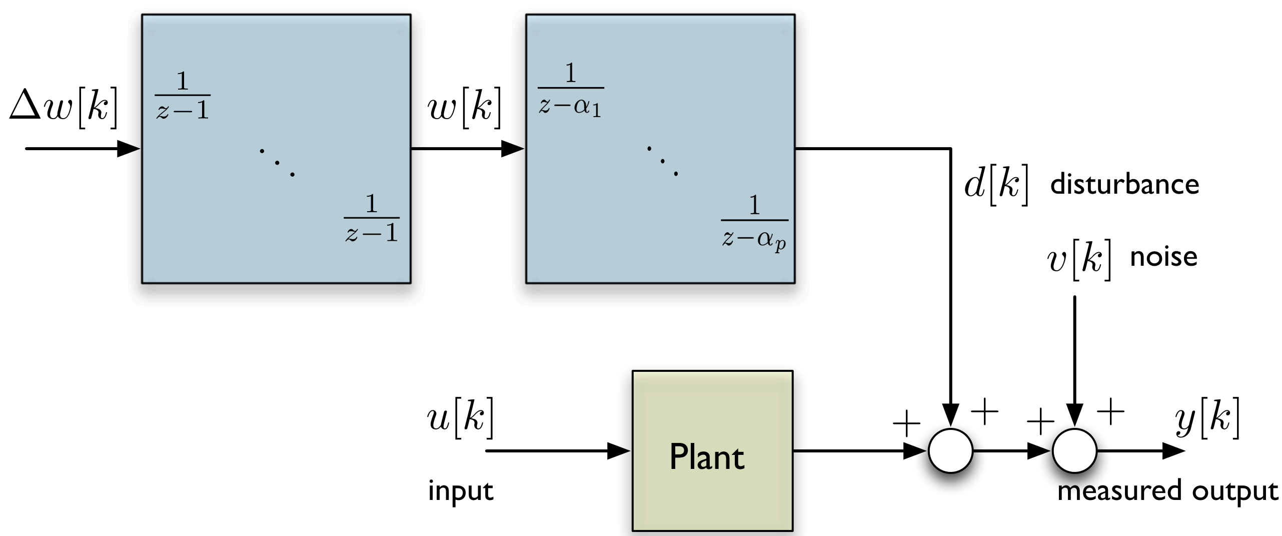
  where δ is a scalar value giving an upper bound on the norm size.
- Various norms can be used for this measure, with the most common being the 1, 2, and ∞ norms
- Note also that in general, this measure will be frequency-dependent; as such the norm bound expression will hold for a discrete frequency values
- A controller $k(z)$ designed for the nominal plant $g_o(z)$ will, if designed properly, give good stable performance for the nominal plant. But the performance will differ for other plants allowed by the uncertainty description
- A stability test can be derived from the resulting closed-loop relationships

Tuning Procedure of Lee and Yu

- Lee and Yu developed an approach in which they adopt a disturbance and noise model which allow systematic tuning of the controller to improve overall system robustness properties
- In their formulation,
  - Only output disturbances are assumed to act on the plant, but are not assumed to be step disturbances

- Disturbances assumed to be generated by white noise processes ($\triangle w[k]$ in figure)
- $\triangle w[k]$ integrated to form stochastic process $w[k]$
- $w[k]$ processed by first-order low-pass filters to form stochastic disturbance $d[k]$
- Plant output also corrupted by white measurement noise $v[k]$

• The disturbance model is shown below



• Model is similar to that used in Dynamic Matrix Control (DMC)

• Using integrated white noise ensures offset-free tracking performance for constant disturbances and plant-model steady-state gain mismatch

• The filter poles, α_i, and process covariances can be used as tuning parameters to enhance the robustness of the controlled system

• An appropriate estimator and predictor can be obtained using Kalman filter theory

- Using a standard state-space formulation,

$$\begin{aligned} \boldsymbol{x}[k+1] &= A\boldsymbol{x}[k] + Bu[k] \\ \boldsymbol{y}(k) &= C\boldsymbol{x}[k] + \boldsymbol{d}[k] + \boldsymbol{v}[k] \end{aligned}$$

where $\boldsymbol{x}[k]$ is the state of the plant, $d[k]$ is an output disturbance and $v[k]$ represents measurement noise

- The disturbance evolves as

$$\begin{aligned} \boldsymbol{x}_w[k+1] &= A_w\boldsymbol{x}_w[k] + \boldsymbol{w}[k] \\ \boldsymbol{d}[k] &= \boldsymbol{x}_w[k] \end{aligned}$$

where,

$$A_w = \begin{bmatrix} \alpha_1 & & & \\ & \alpha_2 & & \\ & & \ddots & \\ & & & \alpha_p \end{bmatrix}$$

- Augmenting the original state vector, we can write

$$\begin{bmatrix} \boldsymbol{x}[k+1] \\ \boldsymbol{x}_w[k+1] \end{bmatrix} = \begin{bmatrix} A & 0 \\ 0 & A_w \end{bmatrix} \begin{bmatrix} \boldsymbol{x}[k] \\ \boldsymbol{x}_w[k] \end{bmatrix} + \begin{bmatrix} B \\ 0 \end{bmatrix} u[k] + \begin{bmatrix} 0 \\ I \end{bmatrix} w[k]$$

$$\boldsymbol{y}[k] = \begin{bmatrix} C & I \end{bmatrix} \begin{bmatrix} \boldsymbol{x}[k] \\ \boldsymbol{x}_w[k] \end{bmatrix} + \boldsymbol{v}[k]$$

- Since $\boldsymbol{\Delta w}[k] = \boldsymbol{w}[k] - \boldsymbol{w}[k-1]$ we can use the first differences to write

$$\begin{bmatrix} \boldsymbol{\Delta x}[k+1] \\ \boldsymbol{\Delta x_w}[k+1] \\ \boldsymbol{\eta}[k+1] \end{bmatrix} = \begin{bmatrix} A & 0 & 0 \\ 0 & A_w & 0 \\ CA & A_w & I \end{bmatrix} \begin{bmatrix} \boldsymbol{\Delta x}[k] \\ \boldsymbol{\Delta x_w}[k] \\ \boldsymbol{\eta}[k] \end{bmatrix}$$

$$+\begin{bmatrix} B \\ 0 \\ CB \end{bmatrix} \triangle u[k] + \begin{bmatrix} 0 \\ I \\ I \end{bmatrix} \boldsymbol{\Delta w}[k]$$

$$\boldsymbol{y}(k) = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \boldsymbol{\Delta x}[k] \\ \boldsymbol{\Delta x_w}[k] \\ \boldsymbol{\eta}[k] \end{bmatrix} + \boldsymbol{v}[k]$$

- This is now a standard model similar to that developed earlier
- Note that the stochastic disturbance $\boldsymbol{\Delta w}[k]$ does not excite the plant state $\boldsymbol{\Delta x}[k]$;
  - This means unstable modes in the plant model will not be stabilized by the Kalman filter gain
  - Hence this disturbance model must be used with stable plants
- We must also specify the covariance models for $\boldsymbol{\Delta w}[k]$ and $\boldsymbol{v}[k]$
- We will assume these are diagonal:

$$W = \begin{bmatrix} \rho_1 & & & \\ & \rho_2 & & \\ & & \ddots & \\ & & & \rho_p \end{bmatrix} \qquad V = \begin{bmatrix} \sigma_1 & & & \\ & \sigma_2 & & \\ & & \ddots & \\ & & & \sigma_q \end{bmatrix}$$

- Now the Kalman filter gain can be computed by solving the discrete-time algebraic Ricatti equation

$$P_\infty = AP_\infty A^T - AP_\infty C^T \left[CPC^T + V\right]^{-1} CP_\infty A^T + \Gamma W \Gamma^T$$

- It can be shown that the Ricatti equation solution for this problem has the following structure:

$$P = \begin{bmatrix} 0 & 0 & 0 \\ 0 & P_{22} & P_{23} \\ 0 & P_{23}^T & P_{33} \end{bmatrix}$$

and the Kalman filter gain matrix is given by:

$$L = \begin{bmatrix} 0 \\ P_{23} \\ P_{33} \end{bmatrix} [P_{33} + V]^{-1}$$

- Since A_w, W and V are all diagonal, the blocks comprising P are all diagonal (square).
- This gives

$$L = \begin{bmatrix} 0 \\ L_{\triangle w} \\ L_{\eta} \end{bmatrix} = \begin{bmatrix} 0 \\ \mathrm{diag}\{\phi_i\} \\ \mathrm{diag}\{\varphi_i\} \end{bmatrix}$$

where

$$\phi_i = \frac{\varphi_i^2}{1 + \alpha_i - \alpha_i \varphi_i}$$

- It can be shown that

$$\varphi_i \rightarrow 0 \text{ as } {}^{\rho_i}/_{\sigma_i} \rightarrow 0$$
$$\varphi_i \rightarrow 0 \; as \; \sigma_i \rightarrow 0$$

and that $0 < \varphi_i \leq 1$ for all values of ${}^{\rho_i}/_{\sigma_i}$.

- The tuning procedure is accomplished in two steps:

1. Choose cost function weights and horizons to obtain nominal stability

2. De-tune the controller so as to obtain robustness with respect to plant-model mismatch

- For the simple case where only one output is being controlled, then there is only a single pole α_1 and a single gain φ_1 to adjust.
- Generally speaking, increasing φ_1 (or ρ_i/σ_i) increases the bandwidth of the closed loop system with the predictive controller in place
- That is, it increases the frequency at which the gain of the sensitivity function $|S(e^{j\omega T_s})|$ becomes close to 1, and at which the gain of the complementary sensitivity function $|T(e^{j\omega T_s})|$ begins to deviate significantly from 1.
- Increasing α_1 effectively improves (reduces) sensitivity at low frequencies, but increases both the sensitivity and complementary sensitivity at high frequencies and thereby reduces robustness to modeling errors.

Linear Quadratic Gaussian/Loop Transfer Recovery (LQG/LTR)

- Loop Transfer Recovery is a tuning technique used together with the Linear Quadratic Gaussian design approach to improve robustness properties
- LQG can be summarized as follows:
  - Linear plant model (A, B, C, D)
  - Quadratic cost function
  - Gaussian disturbance and noise
  - Kalman filter for state estimation
  - Deterministic optimal control solution

- Before proceeding further, let us first establish the equivalence between the standard MPC form and the LQ problem so that we may properly apply the results

Equivalence Between MPC and LQ Problem Formulation

- Recall that we originally developed our standard MPC problem using the following cost function

$$J = (\boldsymbol{R}_s - \boldsymbol{Y})^T(\boldsymbol{R}_s - \boldsymbol{Y}) + \triangle \boldsymbol{U}^T \bar{R} \triangle \boldsymbol{U}$$

- This is equvalent to

$$J = (\boldsymbol{Y} - \boldsymbol{R}_s)^T Q(\boldsymbol{Y} - \boldsymbol{R}_s) + \triangle \boldsymbol{U}^T \bar{R} \triangle \boldsymbol{U}$$

where we can off-weight between Q and R (although we generally normalize one of the two).

- We further recall that $\boldsymbol{Y}$ and $\triangle \boldsymbol{U}$ have the following defintions:

$$\boldsymbol{Y} = \begin{bmatrix} y([k+1|k_i] \\ y[k_i+2|k_i] \\ y([k+3|k_i] \\ \vdots \\ y([k+N_p|k_i] \end{bmatrix}; \qquad \triangle \boldsymbol{U} = \begin{bmatrix} \triangle u[k_i] \\ \triangle u[k+1] \\ \triangle u([k+3] \\ \vdots \\ \triangle u[k_i+N_c-1] \end{bmatrix}$$

- Alternatively, we can express the cost J in terms of a summation:

$$J\left(N_p, N_c\right) = \sum_{j=1}^{N_p} \boldsymbol{y}^T[k_i+j|k_i] Q \boldsymbol{y}[k_i+j|k_i] + \sum_{j=1}^{N_c} \boldsymbol{\Delta u^T}[k_i+j-1] R \boldsymbol{\Delta u}[k_i+j-1]$$

- So long as $N_c < N_p$ the cost function can be expressed as a function of N_p
- Now, since we are only interested in the measured output, $\boldsymbol{y}_z[k] = C_z\boldsymbol{x}[k]$, we have,

$$J\left(N_p\right) = \sum_{j=1}^{N_p} \boldsymbol{x}^T([k+j|k_i]C_z^T Q C_z \boldsymbol{x}[k_i+j|k_i]$$
$$+\sum_{j=1}^{N_c} \Delta\boldsymbol{u}^{\boldsymbol{T}}([k+j-1]R\Delta\boldsymbol{u}[k_i+j-1)$$

or for the infinite horizon problem,

$$J_{MPC} = \sum_{j=1}^{\infty}\{\boldsymbol{x}^T([k+j|k_i]C_z^T Q C_z \boldsymbol{x}[k_i+j|k_i]$$
$$+\Delta\boldsymbol{u}^{\boldsymbol{T}}[k_i+j-1]R\Delta\boldsymbol{u}[k_i+j-1]\}$$

- For the LQ problem, we utilize a cost function in the form

$$J_{LQ} = \sum_{j=0}^{\infty}\left\{\boldsymbol{x}^T[j]Q\boldsymbol{x}[j] + \boldsymbol{u}^T(j)R\boldsymbol{u}[j]\right\}$$

- The LQ problem weights the state $\boldsymbol{x}[k]$ and the input $\boldsymbol{u}[k]$ while the MPC problem weights the measured output $C_z\boldsymbol{x}[k]$ and the input increment $\Delta\boldsymbol{u}[k]$
- Therefore, we need to place the MPC standard problem into an equivalent LQ framework so that we can utilize the proper analogies.
- First we will introduce the augmented state $\boldsymbol{\xi}$ and corresponding state matrices $\tilde{A}$ and $\tilde{B}$

$$\boldsymbol{\xi}(k) = \begin{bmatrix} \boldsymbol{x}[k] \\ \boldsymbol{u}[k-1] \end{bmatrix}$$

$$\tilde{A} = \begin{bmatrix} A & B \\ 0 & I \end{bmatrix}$$

$$\tilde{B} = \begin{bmatrix} B \\ I \end{bmatrix}$$

- Then the two models

$$\boldsymbol{x}[k+1] = A\boldsymbol{x}[k] + B\boldsymbol{u}[k] \text{ and } \boldsymbol{\xi}[k+1] = A\boldsymbol{\xi}[k] + B\boldsymbol{\Delta u}[k]$$

  are equivalent.

- Note we have defined (as before): $\triangle u[k] = u[k] - u[k-1]$
- We must also define $\tilde{Q}$ appropriately to maintain the equivalence

$$\|y_z(k)\|_Q = \|\xi(k)\|_{\tilde{Q}}$$

- This is accomplished by defining

$$\tilde{Q} = \begin{bmatrix} C_z^T Q C_z & 0 \\ 0 & 0 \end{bmatrix}$$

- With these equivalent state-space models, our standard predictive control problem is the same as the linear quadratic problem if we replace A and B in the plant model with $\tilde{A}$ and $\tilde{B}$, respectively, and the weight Q with $\tilde{Q}$ in the cost function
- When we apply the receding horizon control strategy, we always apply the first part of the equivalent finite-horizon control strategy, i.e.,

$$\boldsymbol{\Delta u}[k] = -\tilde{K}_{N-1}\boldsymbol{\xi}[k]$$

where $\tilde{K}$ is the state-feedback gain matrix obtained from solving the finite horizon LQ optimal control problem (with no contraints). See below.

Linear Quadratic Gaussian Problem Formulation

- Given the plant model

$$\begin{aligned} \boldsymbol{x}(k+1) &= A\boldsymbol{x}(k) + Bu(k) + \Gamma\boldsymbol{w}(k) \\ y(k) &= C\boldsymbol{x}(k) + \boldsymbol{v}(k) \end{aligned}$$

- We define $\boldsymbol{w}$ and $\boldsymbol{v}$ as plant and measurement noise processes respectively, with covariance matrices given by

$$E\{\boldsymbol{w}\boldsymbol{w}^T\} = W \geq 0$$
$$E\{\boldsymbol{v}\boldsymbol{v}^T\} = V > 0$$

where

$$E\{\boldsymbol{w}\boldsymbol{v}^T\} = 0$$

meaning the two noise processes are uncorrelated (mutually independent).

- The performance index for optimal control is defined as:

$$J = E\left\{\sum_{j=0}^{\infty}\left[\boldsymbol{y_z}^T(j)Q\boldsymbol{y_z}(j) + \boldsymbol{u}^T(j)R\boldsymbol{u}(j)\right]\right\}$$

where $\boldsymbol{z} = C_z\boldsymbol{x}$ represents the measured output and

$$Q = Q^T \geq 0$$
$$R = R^T > 0$$

- The LQG problem can then be divided into two subproblems:

– LQ optimal state feedback control

– State estimation with disturbances

- The states can be estimated optimally if a Kalman filter, rather than an observer, is used

Kalman Filter

- An optimal estimate of the state x is obtained by iterating the following Kalman filter equations:

$$\hat{\mathrm{x}}(k|k) = \hat{\boldsymbol{x}}(k|k-1) + L'(k)[y(k) - C\hat{\boldsymbol{x}}(k|k-1) \quad \text{(Correction)}$$
$$\hat{\boldsymbol{x}}(k+1|k) = A\hat{\boldsymbol{x}}(k|k) + Bu(k) \quad \text{(Prediction)}$$

- The Kalman filter gain L' is given by

$$L'(k) = P(k)C^T\left[CP(k)C^T + V\right]^{-1}$$

and

$$P(k+1) = AP(k)A^T - AP(k)C^T\left[CP(k)C^T + V\right]^{-1}CP(k)A^T + \Gamma W\Gamma^T$$

- In the case where the model and covariances are constant, then the Ricatti difference equation above is solved for a constant $P_\infty \geq 0$

  – This is found as a solution to the *Algebraic Ricatti Equation* (ARE):

$$P_\infty = AP_\infty A^T - AP_\infty C^T\left[CPC^T + V\right]^{-1}CP_\infty A^T + \Gamma W\Gamma^T$$

- This gives the stationary Kalman filter gain

$$L'_\infty = P_\infty C^T\left[CP_\infty C^T + V\right]^{-1}$$

- You should recognize that the Kalman filter equations have the form of an observer, with special choice of observer gain

- Optimal predictions are obtained by assuming that future values of the disturbance and noise will be equal to their mean values and hence obtained from the optimal filtered state estimate $\hat{\boldsymbol{x}}(k|k)$ by using the recursion

$$\hat{\boldsymbol{x}}(k+1|k) = A\hat{\boldsymbol{x}}(k|k) + Bu(k)$$
$$\hat{\boldsymbol{x}}(k+\ell|k) = A\hat{\boldsymbol{x}}(k+\ell-1|k) + Bu(k+\ell-1) \quad for\ \ell > 1$$

LQ Optimal Control

- When the optimal estimate $\hat{\boldsymbol{x}}(k|k)$ is obtained, the optimal control $u^*(k)$ is given by

$$u^*(k) = -K_{LQ}\hat{\boldsymbol{x}}(k|k)$$

  where K_{LQ} is computed as the solution to the linear quadratic optimal control problem, i.e.,

$$K_{LQ} = (B^T P_\infty B + R)^{-1} B^T P_\infty A$$

- Here, P_∞ is the positive positive semi-definite symmetric solution to the Algebraic Ricatti Equation

$$P_\infty = A^T P_\infty A - A^T P_\infty B \left[B^T P B + R\right]^{-1} B^T P_\infty A + Q$$

- Note the duality with the Kalman filtering problem.
- With respect to stability, from the infinite horizon LQ problem we know that so long as
  - the algebraic Ricatti equation above has a positive semi-definite solution ($P \geq 0$)
  - a constant state-feedback matrix K_{LQ} is obtained from the solution as above

– then the resulting feedback law will be stable.

- Things are different for the LQ finite horizon problem. In this case we have to use the (much more difficult to solve) Ricatti difference equation:

$$P_{j+1} = A^T P_j A - A^T P_j B(B^T P_j B + R_j)^{-1} B^T P_j A + Q_j$$

- Let us substitute constant weighting matrices for Q_j and R_j

  – Then we can write

$$P_{j+1} = A^T P_j A - A^T P_j B(B^T P_j B + R)^{-1} B^T P_j A + Q$$

- If we now introduce the matrix

$$\mathcal{Q}_j = Q - (P_{j+1} - P_{j)}$$

  then substituting for P_{j+1} above give

$$P_j = A^T P_j A - A^T P_j B(B^T P_j B + R)^{-1} B^T P_j A + \mathcal{Q}_{\subseteqq}$$

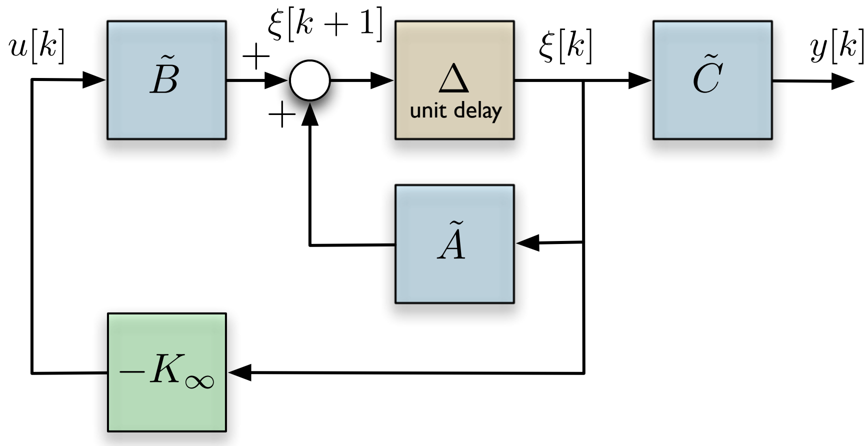
- This is now a (much easier to solve) Algebraic Ricatti Equation

  – This is sometimes called a *Fake Algebraic Ricatti Equation* since it does not arise from an infinite horizon LQ problem

- If this equation has a solution $P_j \geq 0$ and $\mathcal{Q}_j \geq 0$, then applying the contant state feedback gain K_j will give a stable feedback law.

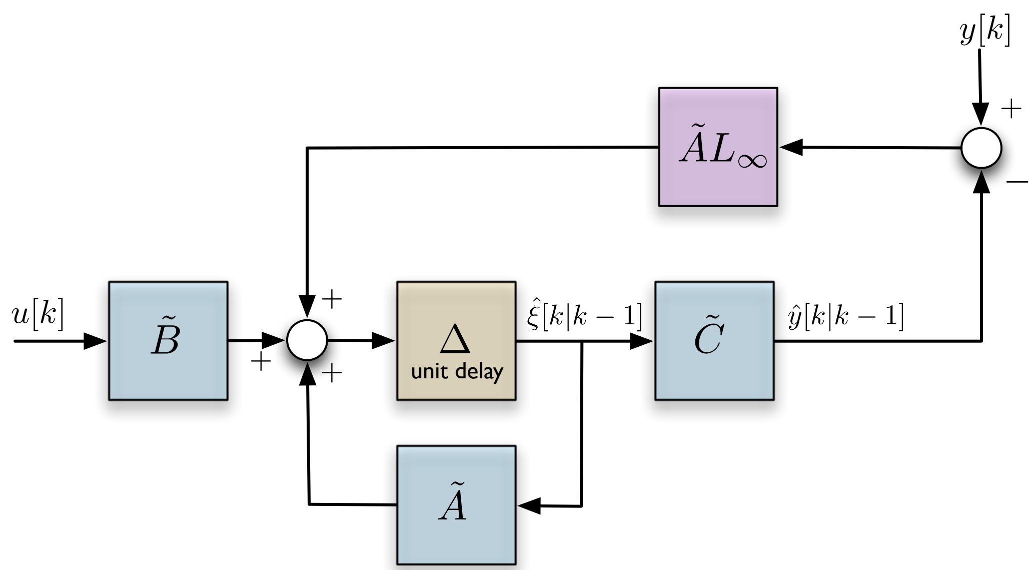
- In order to apply this approach to ensure a stable (unconstrained) predictive control law is to construct the Fake Algebraic Ricatti Equation (with A replaced by $\tilde{A}$, etc.) and ensure that a solution exists with solutions $\mathcal{Q}_{N-1} \geq 0$ and $P_{N-1} \geq 0$.

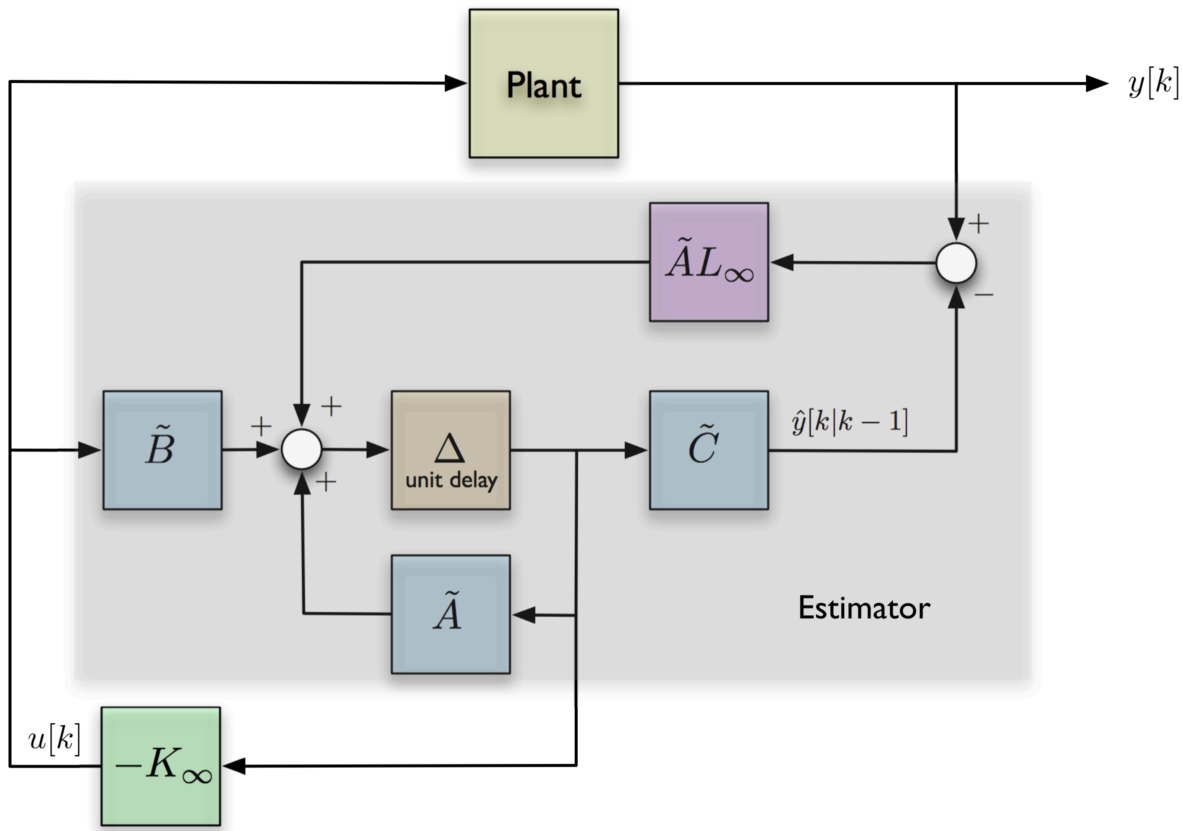
- The LQ state feedback regulator is depicted below.



Loop Transfer Recovery

- The essence of the LQG/LTR approach is the recognition that both the infinite horizon LQ state feedback and the steady-state Kalman filter are themselves feedback systems that each exhibit good feedback properties.





- For continous time systems, it is known that each of the feedback loops exhibits infinite gain margin against gain increases in individual input channels, and at least 60^o phase margin against unmodelled phase lags in individual input channels.
- LQG/LTR theory purports that under certain circumstances, if the state feeback gain K_∞ is obtained by solving the LQ problem with
$$Q = C^T C \quad \text{and} \quad R \to 0$$
then the return ratio for the LQG problem would approach that for the Kalman filtering problem.
- In other words, the return ratio, $\mathcal{L}_{\mathcal{LQG}}$, the sensitivity function, $\mathcal{S}$, and the complementary sensitivity function, $\mathcal{T}$, all evaluated at the output of the plant, converge to the corresponding functions for the loop $\mathcal{L}_{KF}$ for the Kalman filtering problem.

- Note that the convergence is pointwise-in-frequency so that the frequency responses get closer at each frequency as $R \to 0$.
- Kalman filter feedback properties are *recovered* at the plant output by the LTR procedure.

Using the LQG/LTR Procedure to Tune Predictive Controllers

- Instead of the model used previously, we will now use the model:

$$x[k+1] = Ax[k] + Bu[k] + \Gamma_x d[k]$$
$$y[k] = Cx[k] + v[k]$$

- Now, the disturbance $d[k]$ acts on the state (rather than the output) and can be generated by

$$x_w[k+1] = A_w x_w[k] + \Gamma_w w[k]$$
$$d[k] = C_w x_w[k] + D_w w[k]$$

- Our LQG/LTR augmented state vector can now be written

$$\begin{bmatrix} x[k+1] \\ x_w[k+1] \end{bmatrix} = \begin{bmatrix} A & \Gamma_x C_w \\ 0 & A_w \end{bmatrix} \begin{bmatrix} x[k] \\ x_w[k] \end{bmatrix} + \begin{bmatrix} B \\ 0 \end{bmatrix} u[k] + \begin{bmatrix} \Gamma_x D_w \\ \Gamma_w \end{bmatrix} w[k]$$

$$y[k] = \begin{bmatrix} C & 0 \end{bmatrix} \begin{bmatrix} x[k] \\ x_w[k] \end{bmatrix} + v[k]$$

- For generality, we will assume $\triangle w[k]$ is white noise, and construct the equivalent differenced form,

$$\begin{bmatrix} \triangle x[k+1] \\ \Delta x_w[k+1] \\ \eta[k+1] \end{bmatrix} = \begin{bmatrix} A & \Gamma_x C_w & 0 \\ 0 & A_w & 0 \\ CA & C\Gamma_x C_w & I \end{bmatrix} \begin{bmatrix} \Delta x[k] \\ \triangle x_w[k] \\ \eta[k] \end{bmatrix}$$

$$+\begin{bmatrix} B \\ 0 \\ CB \end{bmatrix} \Delta u[k] + \begin{bmatrix} \Gamma_x \\ \Gamma_w \\ C\Gamma_x D_w \end{bmatrix} \Delta w[k]$$

$$y[k] = \begin{bmatrix} 0 & 0 & I \end{bmatrix} \begin{bmatrix} \Delta x[k] \\ \Delta x_w[k] \\ \eta[k] \end{bmatrix} + v[k]$$

- The return-ratio recovered by LTR is that of the Kalman filter shown in the figure above, with the loop broken at the plant model output
- For the simple case where $d[k]$ is integrated white noise, the state x_w is not needed and we can set $D_w = I$
- The model then simplifies to

$$\begin{bmatrix} \Delta x[k+1] \\ \eta[k+1] \end{bmatrix} = \begin{bmatrix} A & 0 \\ CA & I \end{bmatrix} \begin{bmatrix} \Delta x[k] \\ \eta[k] \end{bmatrix}$$

$$+\begin{bmatrix} B \\ CB \end{bmatrix} \Delta u[k] + \begin{bmatrix} \Gamma_x \\ C\Gamma_x \end{bmatrix} \Delta w[k]$$

$$y[k] = \begin{bmatrix} 0 & I \end{bmatrix} \begin{bmatrix} \Delta x[k] \\ \eta[k] \end{bmatrix} + v[k]$$

- For this case, we can compute the Kalman filter return-ratio directly as

$$\tilde{C}\left[zI - \tilde{A}\right]^{-1} \tilde{A} L_\infty = \begin{bmatrix} 0 & I \end{bmatrix} \begin{bmatrix} zI - A & 0 \\ -CA & (z - 1I) \end{bmatrix}^{-1} \begin{bmatrix} L_{\Delta x} \\ L_\eta \end{bmatrix}$$

$$= \begin{bmatrix} 0 & I \end{bmatrix} \begin{bmatrix} (zI - A)^{-1} & 0 \\ \dfrac{CA\,(zI - A)^{-1}}{z - 1} & \dfrac{I}{z - 1} \end{bmatrix} \begin{bmatrix} L_{\Delta x} \\ L_\eta \end{bmatrix}$$

$$= \frac{CA\left(zI - A\right)^{-1} L_{\Delta x} + L_{\eta}}{z - 1}$$

- Observations:
  - The return-ratio has integral action (i.e. poles at $+1$)
  - There are sufficient degrees of freedom in the Kalman filter gains to adjust the gains of the return-ratio
  - There is also freedom to adjust the 'loop shape'
  - Adjustments are made by modifying the parameters of the disturbance/noise model and the covariance matrices
- After disturbance model is adjusted as discussed, then 'recovery step' can be taken
  - Here, we can use the cost function

$$V(k) = \sum_{l=1}^{\infty} \left\{ \left\| \hat{\xi}[k+l|k] \right\|^2_{\tilde{C}^T\tilde{C}} + \rho \left\| \Delta\hat{u}[k+l-1|k] \right\|^2 \right\}$$

$$= \sum_{l=1}^{\infty} \left\{ \left\| \hat{\eta}[k+l|k] \right\|^2 + \rho \left\| \Delta\hat{u}[k+l-1|k] \right\|^2 \right\}$$

$$= \sum_{l=1}^{\infty} \left\{ \left\| \hat{y}[k+l|k] \right\|^2 + \rho \left\| \Delta\hat{u}[k+l-1|k] \right\|^2 \right\}$$

  - We have assumed that the mean value of future measurement noise is zero (i.e., $v[k+l|k] = 0$)
  - Recovery of Kalman filter characteristics is obtained at the plant output as $\rho \to 0$
- How far should we reduce ρ?

– Should monitor sensitivity and complementary sensitivity functions as ρ is reduced
– Stop reducing when feedback properties are adequately good over a wide enough range of frequencies – at least to desired closed-loop bandwith

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