MPC Stability

- Predictive control using a receding horizon approach in the presence of constraints constitutes a nonlinear feedback control system
  - This means there is a risk that the resulting behavior might become unstable
- Consequently, standard linear analysis techniques are not sufficient to insure stability analytically
- Nonetheless, the close connection between model predictive control and *infinite horizon* optimal control provides a framework for stability analysis
- Key is the employment of Lyapunov stability theory which seeks to charactarize system behavior via state equilibrium points and their corresponding stability properties

Lyapunov Stability

- In general, we may characterize a (non-linear) system model by the following state-space description:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \vdots \\ \dot{x}_n \end{bmatrix} = \begin{bmatrix} f_1(t,\, x_1,\, \ldots,\, x_n) \\ f_2(t,\, x_1,\, \ldots,\, x_n) \\ \vdots \\ f_n(t,\, x_1,\, \ldots,\, x_n) \end{bmatrix}$$

  or equivalently,

$$\dot{\boldsymbol{x}} = f(t, \dot{\boldsymbol{x}}),\ \boldsymbol{x} \in \mathbb{R}^n$$

- A system model may be represented as

$$\dot{\boldsymbol{x}} = f(t, \dot{\boldsymbol{x}}(t)), \quad \boldsymbol{x}(t_0) = \boldsymbol{x}_0$$

- For example, our familiar linear time-invariant model is

$$\dot{\boldsymbol{x}} = A\boldsymbol{x}, \quad \boldsymbol{x}(t_0) = \boldsymbol{x}_0$$

  with solution

$$\boldsymbol{x}(t) = e^{At}\boldsymbol{x}_0$$

Equilibrium Point

- A vector $\boldsymbol{x}_e$ is an equilibrium point for a dynamic system model if the following holds: once the state vector equals $\boldsymbol{x}_e$ it remains equal to $\boldsymbol{x}_e$ for all future time
- An equilibrium point satisfies

$$f(t, \boldsymbol{x}(t)) = 0$$

DEFINITION: EQUILIBRIUM POINT.
A point $\boldsymbol{x}_e$ is called an equilibrium point of $\dot{\boldsymbol{x}} = f(t, \boldsymbol{x})$ (or simply an equilibrium), at time t_0 if for all $t \geq t_0$, $f(t, \boldsymbol{x}_e) = 0$. Note that if $\boldsymbol{x}_e$ is an equilibrium at t_0, then it is an equilibrium for all $t \geq t_0$.

- For our LTI system above, any equilibrium state $\boldsymbol{x}_e$ must satisfy $A\boldsymbol{x}_e = 0$
- If A^{-1} exists, then a unique equilibrium state exists as $\boldsymbol{x}_e = 0$

Nonlinear Systems

- A nonlinear system may have a number of equilibrium states

- The origin, $\boldsymbol{x} = 0$, may or may not be an equilibrium state of a nonlinear system
- If the origin is not the equilibrium state, it is always possible to translate the origin of the coordinate system to that state – no generality is lost in assuming that the origin is the equilibrium state of interest

Example 6.1

Consider the nonlinear system model,

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} x_2 \\ x_1 - x_2 - x_1^2 \end{bmatrix}$$

- For this particular system, two equilibrium states exist:

$$\boldsymbol{x}_e^{(1)} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}; \quad \boldsymbol{x}_e^{(2)} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

- An equilibrium point $\boldsymbol{x}_e$ in $\mathbb{R}^n$ is an *isolated equilibrium point* if there is an $r > 0$ such that the r-neighborhood of $\boldsymbol{x}_e$ contains no equilibrium points other than $\boldsymbol{x}_e$.

NEIGHBORHOOD OF $\boldsymbol{x}_e$

- The r-neighborhood of $\boldsymbol{x}_e$ can be a set of points of the form

$$B_r(\boldsymbol{x}_e) = \{\boldsymbol{x} : \|\boldsymbol{x} - \boldsymbol{x}_e\| < r\}$$

  where $\|\cdot\|$ can be any p-norm on $\mathbb{R}^n$.

- Stability properties characterize the system behavior if the initial state is close but not *at* the equilibrium point of interest

- When an initial state is *close* to the equilibrium point, the state may remain close, or it may move away from the equilibrium point

INFORMAL DEFINITION: STABILITY.

- An equilibrium state is stable if whenever the initial state is near that point, the state remains near it, perhaps even tending toward the equilibrium point as time increases.

FORMAL DEFINITION: STABILITY.

- An equilibrium state $\boldsymbol{x}_e$ is stable, in the sense of Lyapunov, if for any given t_0 and any positive scalar ϵ there exist a positive scalar

$$\delta = \delta(t_0, \epsilon)$$

such that if

$$\|\boldsymbol{x}(t_0) - \boldsymbol{x}_e\| < \delta$$

then

$$\|\boldsymbol{x}(t;\, t_0,\, x_0) - \boldsymbol{x}_e\| < \epsilon$$

for all $t > t_0$.

COMMENT: Lyapunov stability is a very mild requirement on equilibrium points - it doesn't even imply that $\boldsymbol{x}(t)$ converges to $\boldsymbol{x}_e$ as t approaches infinity. The states are only required to hover around the equilibrium state. The stability condition bounds the amount of wiggle room for $\boldsymbol{x}(t)$.

- *Uniform stability* is a concept which guarantees that the equilibrium point is not losing stability (i.e., the condition holds for all t_0).

Asymptotic Stability in the Sense of Lyapunov

- In general asymptotic stability of an equilibrium state of a differential equation satisfies two conditions:

- Small perturbations in the initial condition produce small perturbations in the solution (Stability)
- There is a domain of attraction such that whenever the initial condition belongs to this domain the solution approaches the equilibrium state at large times (Attractivity of the Equilibrium Point)

DEFINITION: ASYMPTOTIC STABILITY.

- An equilibrium point $x_e = 0$ is asymptotically stable at $t = t_0$ if
  - $x_e = 0$ is stable; and
  - $x_e = 0$ is locally attractive; i.e., there exists $\delta(t_0)$ such that

$$\|x(t_0)\| < \delta \Longrightarrow \lim_{t\to\infty} x(t) = 0$$

  Note that this definition is a local definition in that it describes behavior of a system near an equilibrium point.

- Asymptotic stability does not imply anything about how long it takes to converge to a prescribed neighborhood of $\boldsymbol{x}_e$
- Exponential stability provides a way to express the rate of convergence

DEFINITION: EXPONENTIAL STABILITY; RATE OF CONVERGENCE.

- The equilibrium point $\boldsymbol{x}_e = 0$ is an *exponentially stable* equilibrium point if there exist constants $m,\ \alpha > 0$ and $\epsilon > 0$ such that

$$\|\boldsymbol{x}(t)\| \le m e^{-\alpha(t-t_0)} \|\boldsymbol{x}(t_0)\|$$

  for all $\|\boldsymbol{x}(t_0)\| \le \epsilon$ and $t \ge t_0$. The largest constant α which may be utilized is called the *rate of convergence*.

- Exponential stability is a *strong* form of stability as it implies uniform, asymptotic stability.

- An LTI system is asymptotically stable, meaning, the equilibrium state at the origin is asymptotically stable, if and only if the eigenvalues of A have negative real parts
  - For LTI systems, asymptotic stability is equivalent with convergence (stability condition automatically satisfied)
- For nonlinear systems the state may initially tend away from the equilibrium state of interest but subsequently may return to it

Lyapunov Functions

- The idea behind Lyapunov functions is to seek an aggregate summarizing function that continually decreases toward a minimum
  - For mechanical systems - energy of a free mechanical system with friction always decreases unless the system is at rest (equilibrium)
- A function that allows one to deduce stability is termed a *Lyapunov function*.

THREE PROPERTIES OF A LYAPUNOV FUNCTION, V:

- V is continuous
- V has a unique minimum with respect to all other points in some neighborhood of the equilibrium of interest
- V never increases along any trajectory of the system

Lyapunov Theorem for Continuous Systems

- Given the continous time system,

$$\dot{\boldsymbol{x}}(t) = f(\boldsymbol{x}(t))$$

the equilibrium state of interest is

–

$$\boldsymbol{x}_e = 0$$

- If $\boldsymbol{x}(t)$ is a trajectory, then $V(\boldsymbol{x}(t))$ represents the corresponding values of V along the trajectory.
- In order for $V(\boldsymbol{x}(t))$ not to increase, we need that

$$\dot{V}(\boldsymbol{x}(t)) < 0$$

- Computing the Lyapunov derivative,

$$\dot{V}(\boldsymbol{x}(t)) = \nabla V(\boldsymbol{x})^T \dot{\boldsymbol{x}}$$

- Substituting from the model we get,

$$\dot{V}(\boldsymbol{x}(t)) = \nabla V(\boldsymbol{x})^T f(\boldsymbol{x})$$

- Note that

$$\nabla V(\boldsymbol{x}) = \left[\begin{array}{cccc} \dfrac{\partial V}{\partial x_1} & \dfrac{\partial V}{\partial x_2} & \cdots & \dfrac{\partial V}{\partial x_n} \end{array} \right]^T$$

- For LTI systems, $\dot{\boldsymbol{x}} = A\boldsymbol{x}$ is asymptotically stable
  - That is, the equilibrium state $\boldsymbol{x}_e = 0$ is asymptotically stable if and only if any solution converges to $\boldsymbol{x}_e = 0$ as t tends to infinity for any initial $\boldsymbol{x}_0$
- For the two-dimensional case ($n = 2$), if a trajectory is converging to $x_e = 0$, it should be possible to find a nested set of closed curves $V(x_1, x_2) = c$, $c \geq 0$, such that decreasing values of c yield level curves shrinking in on the equilibrium state $x_e = 0$
- The limiting level curve $V(x_1, x_2) = V(0)$ is 0 at the equilibrium state $\boldsymbol{x}_e = 0$

- The trajectory moves through the level curves by cutting them in the inward direction ultimately ending at $\boldsymbol{x}_e = 0$

- The level curves can be thought of as contours of a cup-shaped surface
- For an asymptotically stable system (i.e., an asymptotically stable equilibrium state $x_e = 0$), each trajectory falls to the bottom of the cup

DEFINITION: POSITIVE DEFINITE FUNCTION.

The function V is positive definite in S , with respect to $\boldsymbol{x}_e$, if V has continuous partials, $V(\boldsymbol{x}_e) = 0$, and $V(\boldsymbol{x}) > 0$ for all $\boldsymbol{x}$ in S, where $\boldsymbol{x} \neq \boldsymbol{x}_e$.

- Assume, for simplicity, $\boldsymbol{x}_e = 0$, then the function V is positive definite in S if V has continuous partials, $V(0) = 0$, and $V(\boldsymbol{x}) > 0$ for all $\boldsymbol{x}$ in S where $\boldsymbol{x} \neq 0$

Example 6.2

A positive definite function of two variables is given by

$$\begin{aligned} V(x_1, x_2) &= 2x_1^2 + 3x_2^2 \\ &= \begin{bmatrix} x_1 & x_2 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \\ &= \boldsymbol{x}^t P \boldsymbol{x} \\ &> 0 \text{ for all } \boldsymbol{x} \neq 0 \end{aligned}$$

DEFINITION: POSITIVE SEMI-DEFINITE FUNCTION.

The function V is positive semi-definite in S , with respect to $\boldsymbol{x}_e$, if V has continuous partials, $V(\boldsymbol{x}_e) = 0$, and $V(\boldsymbol{x}) \geq 0$ for all $\boldsymbol{x}$ in S.

- Assume, for simplicity, $\boldsymbol{x}_e = 0$, then the function V is positive semi-definite in S if V has continuous partials, $V(0) = 0$, and $V(\boldsymbol{x}) \geq 0$ for all $\boldsymbol{x}$ in S where $\boldsymbol{x} \neq 0$.

Example 6.3

A positive semi-definite function of two variables is given by,

$$\begin{aligned} V(x_1, x_2) &= x_1^2 \\ &= \begin{bmatrix} x_1 & x_2 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \\ &= \boldsymbol{x}^t P \boldsymbol{x} \\ &\geq 0 \quad \text{in } \mathbb{R}^2 \end{aligned}$$

Quadratic Forms

- $V = \boldsymbol{x}^t P \boldsymbol{x}$, where $P = P^T$
- If P is not symmetric, we can always make it symmetric
- Observe that because the transposition of a scalar equals itself, we have

$$(\boldsymbol{x}^t P \boldsymbol{x})^T = \boldsymbol{x}^t P^T \boldsymbol{x} = \boldsymbol{x}^t P \boldsymbol{x}$$

Since,

$$\begin{aligned} \boldsymbol{x}^t P \boldsymbol{x} &= \frac{1}{2}\boldsymbol{x}^t P \boldsymbol{x} + \frac{1}{2}\boldsymbol{x}^t P \boldsymbol{x} \\ &= \frac{1}{2}\boldsymbol{x}^t P \boldsymbol{x} + \frac{1}{2}\boldsymbol{x}^t P^T \boldsymbol{x} \\ &= \boldsymbol{x}^t \left(\frac{P + P^T}{2} \right) \boldsymbol{x} \end{aligned}$$

- Note that,

$$\left(\frac{P+P^T}{2}\right)^T = \frac{P+P^T}{2}$$

- Remember that you can test for positive definiteness using the matrix eigenvalues:
  - $V = \boldsymbol{x}^t P \boldsymbol{x}$ where $P = P^T$, is positive definite if and only if all eigenvalues of P are positive
  - $V = \boldsymbol{x}^t P \boldsymbol{x}$ where $P = P^T$,is positive semi-definite if and only if all eigenvalues of P are non-negative
- Note that these tests are only good for the case when $P = P^T$
  - You must symmetrize P before applying the above tests
- Other tests, e.g., the Sylvester's criteria, involve checking the signs of principal minors of P

Example 6.4

Given the matrix P,

$$P = \begin{bmatrix} 2 & -6 \\ 0 & 2 \end{bmatrix}$$

and quadratic form,

$$V = 2x_1^2 - 6x_1x_2 + 2x_2^2$$

is $V = \boldsymbol{x}^t P \boldsymbol{x}$ positive definite, positive semi-definite, negative definite, negative semi-definite, or neither?

STABILITY TEST FOR $\boldsymbol{x}_e = 0$ OF SYSTEM $\dot{\boldsymbol{x}} = A\boldsymbol{x}$.

Let $V = \boldsymbol{x}^t P \boldsymbol{x}$ where $P = P^T$. For V to be a Lyapunov function, i.e., for $\boldsymbol{x}_e = 0$ to be asymptotically stable,

$$\dot{V}(\boldsymbol{x}(t)) < 0$$

- Compute the *Lyapunov derivative*, $\dot{V}(\boldsymbol{x}(t))$
  - Using the chain rule,

$$\begin{aligned}\dot{V}(\boldsymbol{x}(t)) &= \dot{\boldsymbol{x}}^t P \boldsymbol{x} + \boldsymbol{x}^t P \dot{\boldsymbol{x}} \\ &= \boldsymbol{x}^t A^T P \boldsymbol{x} + \boldsymbol{x}^t P A \boldsymbol{x} \\ &= \boldsymbol{x}^t \left(A^T P + PA\right) \boldsymbol{x}\end{aligned}$$

where we have used the relationship,

$$\begin{aligned}\dot{\boldsymbol{x}} &= A\boldsymbol{x} \\ \dot{\boldsymbol{x}}^t &= \boldsymbol{x}^{\boldsymbol{t}} A^T\end{aligned}$$

- Denote,

$$A^T P + PA = -Q$$

then,

$$V = \frac{d}{dt} V = -\boldsymbol{x}^t Q \boldsymbol{x}$$

where

$$Q = Q^T > 0$$

LYAPUNOV THEOREM.

The real matrix A is asymptotically stable; i.e., all eigenvalues of A have negative real parts, if and only if for any $Q = Q^T > 0$, the solution P of the continuous matrix Lyapunov equation

$$A^T P + PA = -Q$$

is (symmetric) positive definite.

Applying the Lyapunov Theorem

- Select an arbitrary symmetric positive definite matrix Q (e.g., $Q = I_n$)
- Solve the Lyapunov equation for $P = P^T$
- If P is positive definite, the matrix A is asymptotically stable; if P is not positive definite, A is not asymptotically stable

NOTE: It would not be sensible to choose P to be positive definite first and then calculate Q. Clearly, unless Q is positive definite, then we can say nothing about the asymptotic stability of A.

Example 6.5

Consider

$$A = \begin{bmatrix} -1 & 3 \\ 0 & -1 \end{bmatrix}$$

- Try

$$P = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

- Compute

$$Q = -\left(A^T P + PA\right)$$

MPC Stability

- In practice, a reasonable selection of tuning parameters and a well-posed optimization problem usually deliver stable performance for model predictive control algorithms
- Nonetheless, scarcity of rigorous stability criteria have been a long-standing criticism of MPC techniques
- A generic solution to this shortcoming is obtained by making use of well-known linear quadratic (LQ) optimal control results studied since the 1960's
- Before presenting this approach we will first cover some preliminary notions.
- Stability analysis for model predictive control involves three key concepts:
  - Infinite Output Horizons
  - Inclusion of the Prediction 'Tail'
  - Terminal Constraints

Prediction Tail

- Inclusion of the *tail* within a class of predictions is important to generating a well-posed MPC problem.
- Optimal predictions at sample instant k are given by

$$\triangle \boldsymbol{u_{k-1}} = \begin{bmatrix} \triangle u_k \\ \triangle u_{k+1|k} \\ \triangle u_{k+2|k} \\ \vdots \end{bmatrix}$$

$$\boldsymbol{y_k} = \begin{bmatrix} y_{k+1|k} \\ y_{k+2|k} \\ y_{k+3|k} \\ \vdots \end{bmatrix}$$

- At the next sampling instant, i.e., $k+1$, the first components of this pair have already occured (and are no longer *predictions*)
- The part that is still a prediction is called the *tail*:

$$\triangle \boldsymbol{u_{k-1\,tail}} = \begin{bmatrix} \triangle u_{k+1|k} \\ \triangle u_{k+2|k} \\ \triangle u_{k+3|k} \\ \vdots \end{bmatrix}$$

$$\boldsymbol{y_{k\,tail}} = \begin{bmatrix} y_{k+2|k} \\ y_{k+3|k} \\ y_{k+4|k} \\ \vdots \end{bmatrix}$$

- Note that the predictions given in the tail at $k+1$ were those computed at the previous sampling instant k
- A stability proof is implied by

$$\triangle u_k = \triangle u_{k-1,tail}; \quad y_{k+1} = y_{k+1,tail}$$

- In other words

> The tail is those parts of the predictions made at the previous sample which have still to take place. These should ideally be part of the current prediction class.

Terminal Constraints

- One perhaps obvious way of ensuring (or enforcing) stability is to have any length horizon, but to add a *terminal constraint* which forces the state to assume a particular value at the end of the prediction horizon

Theorem 6.1 (Maciejowski)

- Suppose predictive control is obtained for the plant

$$\boldsymbol{x}(k+1) = f(\boldsymbol{x}(k),\ u(k))$$

by minimizing the cost function

$$V(k) = \sum_{i=1}^{N_p} \ell(\hat{\boldsymbol{x}}(k+i|k),\ \hat{u}(k+i-1|k))$$

where $\ell(\boldsymbol{x},\ u) \geq 0$ and $\ell(\boldsymbol{x},\ u) = 0$ only if $x = 0$ and $u = 0$, and ℓ is decrescent, subject to terminal constraint

$$\hat{\boldsymbol{x}}(k+N_p|k) = 0$$

- The minimization is carried out over the input signals $\{\hat{u}(k+i|k\ :\ i = 0, 1, \ldots,\ N_u - 1\}$, with $N_u = N_p$ for simplicity and subject to constraints

$$\hat{u}(k+i|k) \in U$$
$$\hat{\boldsymbol{x}}(k+i|k) \in X$$

where U and X are some sets

- We assume an equilibrium condition is defined by $\boldsymbol{x} = 0$, and $u = 0$
- The receding horizon method is applied, with only the first element used from the optimizing input sequence
- Then the equilibrium point is stable providing that the optimization problem is feasible and is solved at each step

Proof

- Let $V^o(t)$ be the optimal value of V which corresponds to the optimal input signal u^o
- Clearly, $V^o(t) \geq 0$ and $V^o(t) = 0$ only if $\boldsymbol{x}(t) = 0$
  - This is so because if $\boldsymbol{x}(t) = 0$ then the optimal strategy is to set $u(t+i) = 0$ for each i
- We must show that $V^o(t+1) \leq V^o(t)$, and hence that $V^o(t)$ is a Lyapunov function
- We write,

$$V^o(t+1) = \min_u \sum_{i=1}^{N_p} \ell(\boldsymbol{x}(t+1+i),\ u(t+i))$$

$$= \min_u \left\{ \sum_{i=1}^{N_p} \ell(\boldsymbol{x}(t+i),\ u(t-1+i)) - \ell(\boldsymbol{x}(t+1),\ u(t)) \right.$$

$$\left. + \ \ell(\boldsymbol{x}(t+1+N_p),\ u(t+N_p)) \right\}$$

$$\leq -\ell(\boldsymbol{x}(t+1),\ u^o(t)) + V^o(t)$$

$$+ \min_{u} \left\{ \ell(\boldsymbol{x}(t+1+N_p),\ u(t+N_p)) \right\}$$

since the optimum is no worse that keeping the optimal solution found at time t.

- But we have assumed that the constraint $\boldsymbol{x}(t+N_p) = 0$ is satisfied, so we can make $u(t+N_p) = 0$ and remain at $\boldsymbol{x} = 0$.
- This gives,

$$\min_{u} \left\{ \ell(\boldsymbol{x}(t+1+N_p),\ u(t+N_p)) \right\} = 0$$

- Since $\ell(\boldsymbol{x}(t),\ u^o(t) \geq 0$, we can conclude that

$$V^o(t+1) \leq V^o(t)$$

- Therefore, $V^o(t)$ is a Lyapunov function. ■

- In general, this is not quite as easy as it may sound, simply adding a terminal constraint may not be feasible for most practical systems

Dual-Mode Control

- A generalization of the terminal constraint idea is to specify a terminal contstraint *set*, $\mathbb{T}$, which contains the origin, rather than just a single point
- Here the idea is to use MPC to drive the state into this set when it is far from equilibrium, and then switch to some other control law, which is guaranteed to stabilize the system from this point forward
  - It is assumed that all constraints are inactive within X_0 so that conventional control may be used

- In the MPC context, dual mode control does not imply a hard switching between modes in real time; it rather describes how the predictions are set up
- Dual mode switching generally applies to predictions only

NOTIONAL DUAL MODE ALGORITHM.

- Define a control law; e.g., $u = -K\boldsymbol{x}$
- Define a terminal set T in the phase plane for state $\boldsymbol{x}$; the region may be ellipsoidal or polyhedral
  - The nominal feedback, $u = -K\boldsymbol{x},\ \boldsymbol{x} \in \mathrm{T}$ implies constraints are satisfied.
- Define N_u and compute the prediction equations
- Define an on-line performance measure as

$$J_k = (\boldsymbol{R}_{s,k+1} - \boldsymbol{Y}_{k+1})^t(\boldsymbol{R}_{s,k+1} - \boldsymbol{Y}_{k+1}) + \triangle\boldsymbol{U}_{k-1}{}^t R \triangle\boldsymbol{U}_{k-1}$$

$$\triangle\boldsymbol{u}_{k-1} = \begin{bmatrix} \triangle u_k & \triangle u_{k+1} & \ldots & \triangle u_{k+N_u-1} \end{bmatrix}^t$$

- Define inequalities to ensure constraint satisfaction during the first N_u steps

$$M\triangle\boldsymbol{u}_{k-1} \leq \boldsymbol{\gamma}$$

- At each sampling instant, minimize the infinite horizon cost as follows:

$$\min_{\triangle\boldsymbol{u}_k} J_k\ s.t.\ M\triangle\boldsymbol{u}_{k-1} \leq \boldsymbol{\gamma};\ u_{k+1} = -K\boldsymbol{x}_{k+1};\ i \geq N_u$$

Infinite Horizon

- It is well-known that making the horizons infinite in unconstrained predictive control leads to guaranteed stability; however it is only recently that this result has been extended to the case with constraints
- The key is to parameterize the problem with a finite number of decision variables so that the optimization can still be performed over a finite-dimensional space
  - By setting $N_p = N_u = \infty$, the cost function J can be used as a potential Lyapunov function
- Important points:
  - Use of infinite horizons guarantees that J is Lyapunov
  - Implicit in the proof that J is Lyapunov is the incorporation of the *tail* into the class of possible predictions
- Let us examine the essential difference between a finite, receding horizon formulation of predictive control, and an infinite horizon formulation
- In the finite case, at time k a particular trajectory is optimal over the prediction horizion of length Np
  - Assuming a perfect model and no disturbances, the plant at time $k+1$ is exactly what was predicted at the previous step
- One might expect that the initial portion of the optimal trajectory over the prediction horizon from time $k+1$ to $k+1+N_p$ would coincide with the previously computed optimal trajectory

– But this is not necessarily the case, since a new time interval has entered the picture which was not considered previously – this might lead to a new optimal trajectory considerably different from that computed at the last time step

- For the case of an infinite horizon, an optimal trajectory is determined over the whole (infinite) horizon
- Since no new information enters at the next time step $k+1$, the optimal trajectory from this instant on is the same as the tail of the previously computed trajectory.

> BELLMAN'S PRINCIPAL OF OPTIMALITY: *From any point on an optimal trajectory, the remaining trajectory is optimal for the corresponding problem initiated at that point.*

- Bellman's principal leads to the cost function decreasing as k increases - which allows it to be used as a Lyapunov function for establishing stability

Infinite Horizon Problem Formulation (Maciejowski)

- Assume a regulator form for our control problem; i.e., we wish to drive a suitably defined output to zero:

$$V(k) = \sum_{i=1}^{\infty} \left\{ \|\hat{z}(k+i|k)\|_Q^2 + \|\Delta\hat{u}(k+i-1|k)\|_R^2 + \|\hat{u}(k+i-1|k)\|_S^2 \right\}$$

- We assume that only the first N_u control moves are non-zero:

$$\Delta u(k+i-1|k) = 0 \quad \text{for } i > N_u$$

- We further assume that $S > 0$ and $Q \geq 0$, $R \geq 0$

  – Also assume full-state measurement, so that $\boldsymbol{x}(k) = \boldsymbol{y}(k)$ and the plant is stable

- Let $V^o(k)$ be the optimal value of the cost function at time k and let u^obe the computed optimal input levels
- Let z^o denote values of the controlled output obtained by applying u^o.
- Since $u^o(k+1) = u^o(k+N_u-1)$ for all $i \geq N_u$, and since the steady-state value of u^o needed to keep z at 0 is 0, we can expect the opimizer to put $u^o(k+i) = 0$ for all $i \geq N_u$ (otherwise the cost would be infinite).
- Then we can write,

$$V^o(k) = \sum_{i=1}^{\infty} \|\hat{z}^o(k+i|k)\|_Q^2 + \sum_{i=1}^{N_u} \left\{ \|\triangle \hat{u}^o(k+i-1|k)\|_R^2 + \|\hat{u}^o(k+i-1|k)\|_S^2 \right\}$$

- If we assume the model is exact and there are no disturbances, then $z^o(k+1) = \hat{z}^o(k+1|k)$.
- Then,

$$V(k+1) = V^o(k) - \|z^o(k+1)\|_Q^2 - \|\triangle \hat{u}^o(k|k)\|_R^2 - \|\hat{u}^o(k|k)\|_S^2$$

- At time $k+1$, the new optimization problem with initial condition $z^o(k+1) = C_z x^o(k+1)$ is solved, so that

$$\begin{aligned} V^o(k+1) &\leq V(k+1) \\ &= V^o(k) - \|z^o(k+1)\|_Q^2 - \|\triangle \hat{u}^o(k|k)\|_R^2 - \|\hat{u}^o(k|k)\|_S^2 \\ &< V^o(k) \end{aligned}$$

- To infer stability, we must show that $V^o(k+1) < V^o(k)$ implies that $\|\boldsymbol{x}(k)\|$ is decreasing
  - If we assume that $S > 0$, then it is clear that $u^o(k)$ is decreasing (since $u^o(k) = \hat{u}^o(k|k)$), which implies that $\boldsymbol{x}^o(k)$ is eventually decreasing (since we have assumed a stable plant)
  - This assumption (there are others) ensures that decreasing V^o implies decreasing $\|\boldsymbol{x}^o\|$ which establishes that V^o is a Lyapunov function for the closed loop, which shows that the closed-loop system is stable

Infinite Horizon Problem Formulation with Constraints (Maciejowski)

- We have already seen that for the regulator problem, the computed input signal $u(k+i-1|k)$ is zero for $i \geq N_u$.
- The cost function can be written as

$$V(k) = \sum_{i=N_u+1}^{\infty} \|\hat{z}(k+i-1|k)\|_Q^2 + \sum_{i=1}^{N_u} \left\{ \|\hat{z}(k+i-1|k)\|_Q^2 + \|\triangle\hat{u}(k+i-1|k)\|_R^2 + \|\hat{u}(k+i-1|k\|_S^2 \right\}$$

- Looking at the first term, we can write

$$\begin{aligned} \hat{z}(k+N_u|k) &= C_z\hat{\boldsymbol{x}}(k+N_u|k) \\ \hat{z}(k+N_u+1) &= C_zA\hat{\boldsymbol{x}}(k+N_u|k) \\ &\vdots \\ \hat{z}(k+N_u+j|k) &= C_zA^j\hat{\boldsymbol{x}}(k+N_u|k) \end{aligned}$$

so that,

$$\sum_{i=N_u+1}^{\infty} \|\hat{z}(k+i-1|k)\|_Q^2 = \hat{\boldsymbol{x}}(k+N_u|k)^t \left[\sum_{i=0}^{\infty} (A^T)^i C_z^T Q C_z A^i \right] \hat{\boldsymbol{x}}(k+N_u|k)$$

- Let's define

$$\bar{Q} = \sum_{i=N_u+1}^{\infty} \|\hat{z}(k+i-1|k)\|_Q^2$$

  – Note that if the plant is stable (i.e., $\lambda_i(A) < 1$, for $i = 1, \ldots n$), then the series converges

- We may then write,

$$A^T \bar{Q} A = \sum_{i=1}^{\infty} (A^T)^i C_z^T Q C_z A^i$$
$$= \bar{Q} - C_z Q C_z$$

- This form of equation is a *matrix Lyapunov equation,* written in standard form as

$$A^T \bar{Q} A + C_z \bar{Q} C_z - Q = 0$$

- Using standard matrix Lyapunov equation solvers (e.g., dlyap in Matlab), we can easily compute the value of $\bar{Q}$
- The matrix Lyapunov equation plays an important role in stability analysis
  – Note that $\bar{Q} \geq 0$ if $Q \geq 0$ and A has all its eigenvalues inside the unit circle, implying the plant is stable
- The cost function can now be re-written as

$$V(k) = \hat{\boldsymbol{x}}(k + N_u|k)^t \bar{Q} \hat{\boldsymbol{x}}(k + N_u|k) + \sum_{i=1}^{N_u} \left\{ \|\hat{z}(k+i-1|k)\|_Q^2 + \|\triangle\hat{u}(k+i-1|k)\|_R^2 + \|\hat{u}(k+i-1|k\|_S^2 \right\}$$

- We have now arrived at a formulation that resembles a predictive control problem over a finite control horizon of length N_u with a terminal cost penalty.

Example 6.6

Consider the plant whose discrete-time transfer function is given by:

$$G(z) = \frac{-0.0539z^{-1} + 0.5775z^{-2} + 0.5188z^{-3}}{1 - 0.6543z^{-1} + 0.5013z^{-2} - 0.2865z^{-3}} = \frac{-0.0539z^2 + 0.577z + 0.5188}{z^3 - 0.6543z^2 + 0.5013z - 0.2865}$$

- A state-space realization is given by

$$A = \begin{bmatrix} 0.6543 & -.5013 & 0.2865 \\ 1.0000 & 0 & 0 \\ 0 & 1.0000 & 0 \end{bmatrix} \quad B = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$$C = \begin{bmatrix} -0.0536 & 0.5775 & 0.5188 \end{bmatrix} \quad D = \begin{bmatrix} 0 \end{bmatrix}$$

- The eigenvalues of A are

$$\lambda(A) = \begin{bmatrix} 0.0239 + j0.6869 & 0.0239 - j0.6869 & 0.6066 \end{bmatrix}$$

and their magnitudes are

$$|\lambda(A)| = \begin{bmatrix} 0.6873 & 0.6873 & 0.6066 \end{bmatrix}$$

$\Longrightarrow$ open loop stable

- If we assume that the weighting matrix $Q = 1$, then we can write

$$\bar{Q} = \sum_{i=0}^{\infty} (A^T)^i C_z^T Q C_z A^i$$

so that we have

$$A^T \bar{Q} A = \bar{Q} - C_z^T Q C_z$$

or equivalently

$$A^T \bar{Q} A - \bar{Q} + C_z^T Q C_z = 0$$

- Substituting values,

$$\begin{bmatrix} 0.6543 & 1.0000 & 0 \\ -0.5013 & 0 & 1.0000 \\ 0.2865 & 0 & 0 \end{bmatrix} \begin{bmatrix} q_{11} & q_{12} & q_{13} \\ q_{12} & q_{22} & q_{23} \\ q_{13} & q_{23} & q_{33} \end{bmatrix} \begin{bmatrix} 0.6543 & -.5013 & 0.2865 \\ 1.0000 & 0 & 0 \\ 0 & 1.0000 & 0 \end{bmatrix}$$

$$- \begin{bmatrix} q_{11} & q_{12} & q_{13} \\ q_{12} & q_{22} & q_{23} \\ q_{13} & q_{23} & q_{33} \end{bmatrix} + \begin{bmatrix} -0.0536 \\ 0.5775 \\ 0.5188 \end{bmatrix} \begin{bmatrix} 1 \end{bmatrix} \begin{bmatrix} -0.0536 & 0.5775 & 0.5188 \end{bmatrix} = 0$$

- This matrix Lyapunov problem may be solved using Matlab's dlyap function -

$$\texttt{X} = \texttt{dlyap}(\texttt{A}', \texttt{C}' * \texttt{Q} * \texttt{C})$$

yielding the solution

$$\bar{Q} = \mathtt{X} = \begin{bmatrix} 1.234 & -0.1011 & 0.1763 \\ -0.1011 & 0.8404 & 0.1716 \\ 0.1763 & 0.1716 & 0.3712 \end{bmatrix}$$

- Checking eigenvalues,

$$\lambda(\bar{Q}) = \begin{bmatrix} 0.2743 & 0.8924 & 1.2881 \end{bmatrix}$$

which follows from our previous analysis.

(mostly blank)

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