Model Predictive Control with Constraints

- We begin this section with a straightforward example:

Example 5.1

A continuous-time model of an undamped oscillator is given by

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -4 & 0 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u(t)$$

$$y(t) = \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$$

- The eigenvalues of the system matrix are located at $\lambda_{1,2} = \pm j2$ (purely imaginary), so the natural frequency is $\omega = 2\,{}^{\text{rad}}/_{\text{sec}}$
- Assuming a sampling interval $\triangle t = 0.1$, we compute the corresponding discrete-time model:

$$\begin{bmatrix} x_1(k+1) \\ x_2(k+1) \end{bmatrix} = \begin{bmatrix} 0.9801 & .0993 \\ -.3973 & 0.9801 \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix} + \begin{bmatrix} 0.0993 \\ -0.0199 \end{bmatrix} u(k)$$

$$y(k) = \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix}$$

- The eigenvalues of the discrete-time system are located at $\lambda_{1,2} = .9801 \pm j0.1986$
  - As we expect, both have modulus 1.0 and lie on the unit circle

- Our objective is to design a predictive control system so that the output of the plant tracks a unit step reference as fast as possible
- Tuning parameters for the problem are selected as:

$$N_c = 3$$
$$N_p = 10$$
$$\bar{R} = 0$$

$\Rightarrow$ What happens if control magnitude *saturates* at ± 20?

- Computing the data matrices for this problem yields,

$$\Phi^T \Phi = \begin{bmatrix} 6.0067 & 4.8853 & 3.8150 \\ 4.8853 & 4.0013 & 3.1475 \\ 3.8150 & 3.1475 & 2.4952 \end{bmatrix}$$

$$\Phi^T F = \begin{bmatrix} 65.5285 & -25.2099 & -6.1768 \\ 53.1281 & -19.6709 & -4.7606 \\ 41.3553 & -14.6974 & -3.5334 \end{bmatrix}$$

$$\Phi^T R_s = \begin{bmatrix} -6.1768 \\ -4.7606 \\ -3.5334 \end{bmatrix}$$

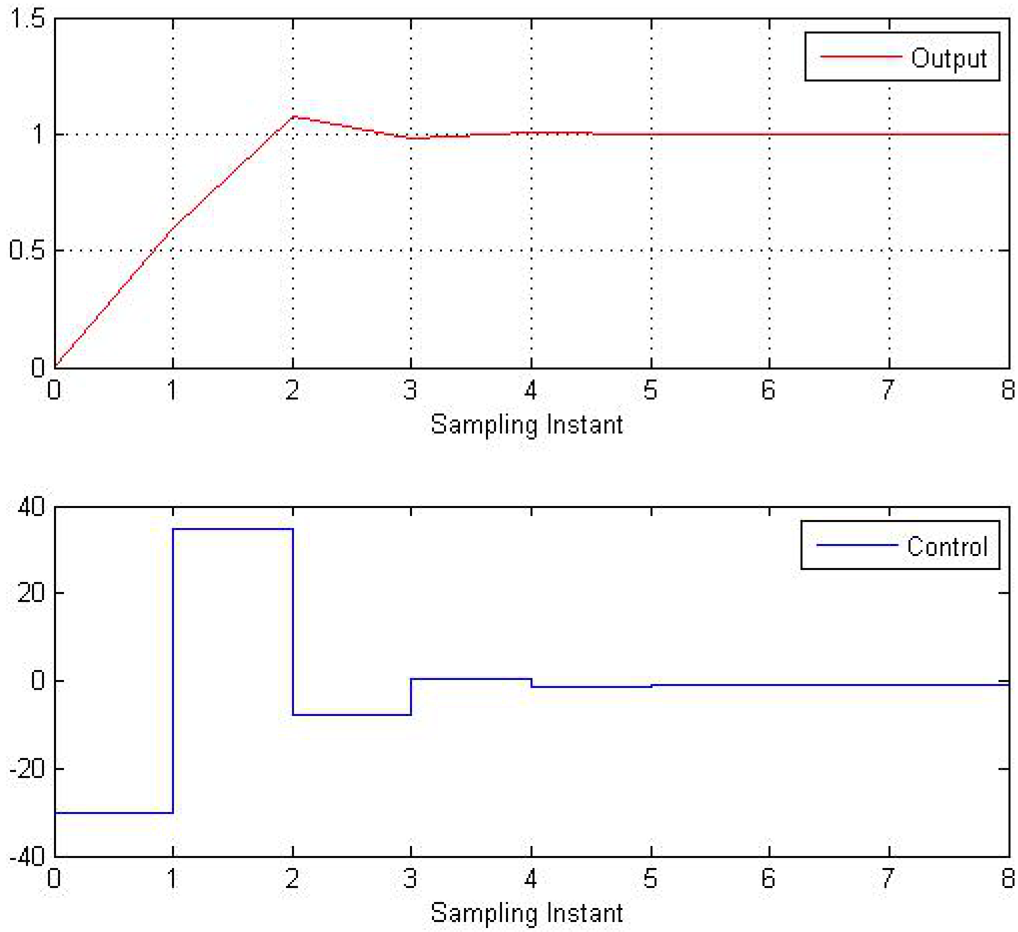
- Computing the state feedback gain matrix we obtain

$$K_{mpc} = \begin{bmatrix} 17.9064 & -39.0664 & -29.9659 \end{bmatrix}$$

  - This gives closed-loop eigenvalues at: $\begin{bmatrix} -0.1946, & 0, & 0 \end{bmatrix}$

- We now design an observer with poles at $\begin{bmatrix} 0, & 0, & 0 \end{bmatrix}$ and examine two cases: (i) no constraint on control; and (ii) with constraint on control.

CASE (I) NO CONSTRAINT.



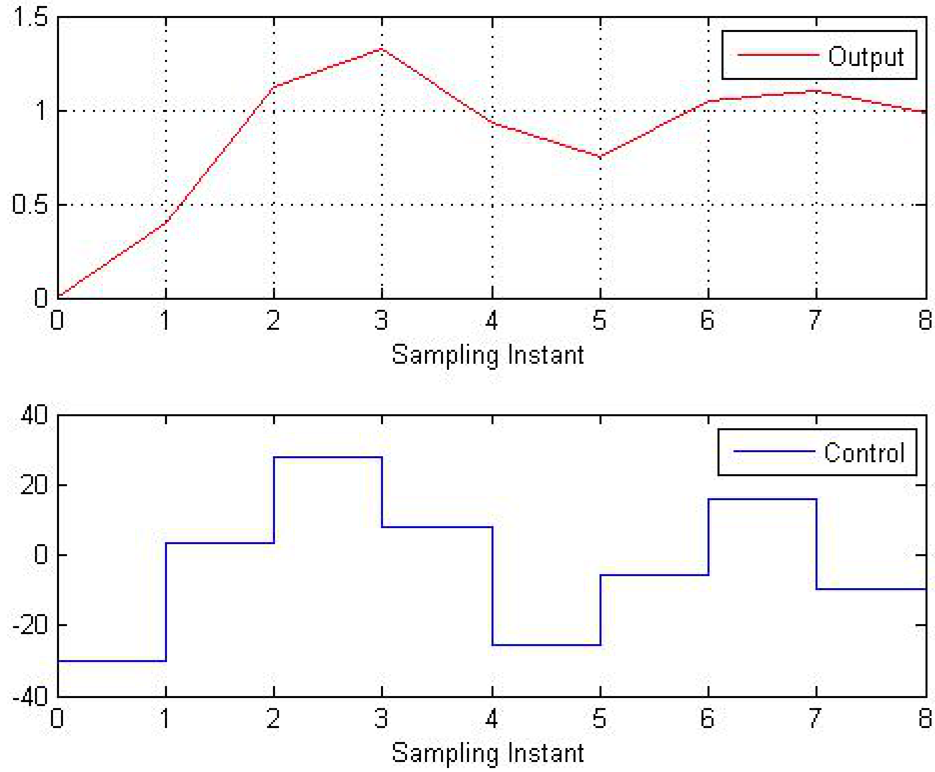
CASE (II) WITH CONSTRAINT.

- Assuming the control amplitude has saturation limits at ± 20 , we write

$$-20 \leq u(k) \leq 20$$

  – So, $u(k) = 20$ when $u(k) \geq 20$ and $u(k) = -20$ when $u(k) \leq 20$

- As we might expect, the closed-loop performance deteriorates when these limits are reached



- So we ask: is it possible to achieve *better* performance by incorporating the constraints *directly* into the model predictive control problem?
- Let's try it...

Formulation of Constrained Control Problems

- The basic idea here is to take the control increment $\triangle u(k)$ obtained for the unconstrained problem and modify it when a constraint becomes active
- This involves formulating an optimization problem directly in the presence of constraints
- Three of the most common types of contraints are placed on:
  – control variable increment

– control variable amplitude

– output variable

- It is also sometimes desirable to constrain state variables internal to the dynamic process; we'll cover this case as well

Numerical Solutions

- The standard optimization problem for constrained model predictive control can be cast as a *quadratic programming* problem
- For consistency with the prevailing literature, we write the objective function J together with a set of constraint equations as:

$$\begin{aligned} J &= \frac{1}{2}\boldsymbol{x}^t E x + \boldsymbol{x}^t F \\ M\boldsymbol{x} &\le \boldsymbol{\gamma} \end{aligned}$$

where we assume E to be symmetric and positive definite.

The Unconstrained Case

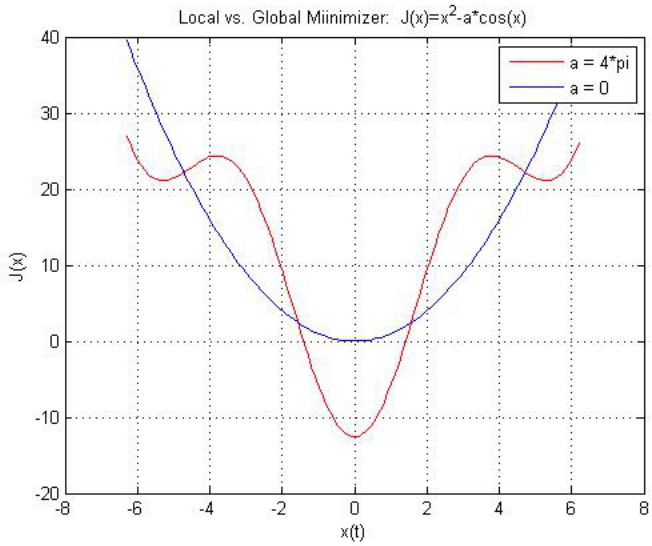
- General optimization problems involve finding a local solution to the problem

$$\min \quad J(\boldsymbol{x}), \qquad \boldsymbol{x} \in \mathbb{R}^n$$

- Here, $J(\boldsymbol{x})$ is the objective function and the minimizing point denoted by $\boldsymbol{x}^\star$ is found among all $\boldsymbol{x}$ vectors in $\mathbb{R}^n$
- We must first make some assumptions on the existence and uniqueness of $\boldsymbol{x}^\star$

– Clearly $\boldsymbol{x}^\star$may not exist if J is unbounded below

– In general, it is only really practical to locate a *local* minimizer, and this may not be a *global* minimizer



- In this treatment we will assume that first and second derivatives exist, are continuous and are given by:

$$g(\boldsymbol{x}) = \nabla J(\boldsymbol{x})$$
$$G(\boldsymbol{x}) = \nabla^2 J(\boldsymbol{x})$$

- At a local minimizer, the following simple stationarity conditions hold:

$$g(\boldsymbol{x}^\star) = 0$$
$$\boldsymbol{s}^t G(\boldsymbol{x}^\star)\boldsymbol{s} \geq 0, \quad \vee \boldsymbol{s}$$

- These are first-order and second-order necessary conditions for a local solution
  – The first condition identifies a *stationary point*
  – The second implies the matrix $G(\boldsymbol{x}^\star)$ is *positive semi-definite* and indicates the presence of non-negative curvature in all directions from the stationary point

- Sufficient conditions for a strict and isolated minimizer $x^\star$ are that the above hold and that $G(x^\star)$ is *positive definite*.
  - How can we determine numerically whether or not $G(x^\star)$ is positive definite?

Quadratic Programming: Equality Constraints

- For unconstrained optimization, the necessary conditions above illustrate the requirement for zero slope and non-negative curvature in any direction at $x^\star$
- In constrained optimization, there is the additional consideration of a feasible region - i.e., a local minimizer must be a *feasible point* that satisfies the constraints
- Additionally, there must be no feasible *descent* directions at $x^\star$.
- The simplest problem involves finding the constrained minimum of a positive definite quadratic function with linear equality constraints.
  - Each linear equality constraint defines a *hyperplane*
  - Positive definite quadratic functions have their level surfaces as *hyperellipsoids*
- The constrained minimum is then found at the point of tangency between the boundary of the feasible set and the minimizing hyperellipsoid

Example 5.2

Minimize

$$J = (x_1 - 2)^2 + (x_2 - 2)^2$$

subject to

$$x_1 + x_2 = 1$$

- By inspection we see that the unconstrained minimum occurs at

$$x_1 = 2$$
$$x_2 = 2$$

- Feasible solutions are the combinations of x_1 and x_2 that satisfy the linear equality $x_1 + x_2 = 1$
- So we have

$$x_2 = 1 - x_1$$

- Substituting into the expression for J gives,

$$\begin{aligned} J &= (x_1 - 2)^2 + (1 - x_1 - 2)^2 \\ &= 2x_1^2 - 2x_1 + 5 \end{aligned}$$

- To minimize J, we satisfy the stationarity condition

$$\frac{\partial J}{\partial x_1} = 4x_1 - 2 = 0$$

giving,

$$x_1 = 0.5$$
$$x_2 = 0.5$$

Lagrange Multipliers

- One approach to minimizing the objective function subject to equality constraints is to augment the original function with the equality constraint multiplied by a *Lagrange multiplier* vector, $\boldsymbol{\lambda}$:

$$J = \frac{1}{2}\boldsymbol{x}^t E\boldsymbol{x} + \boldsymbol{x}^t F + \boldsymbol{\lambda}^t(M\boldsymbol{x} - \boldsymbol{\gamma})$$

  – This creates a new objective function in the $n + m$ variables given by the vectors $\boldsymbol{x}$ $(n \times 1)$ and $\boldsymbol{\lambda}$ $(m \times 1)$

- The solution is found by taking the first partial derivatives with respect to the vectors $\boldsymbol{x}$ and $\boldsymbol{\lambda}$ and then equating the derivatives to zero:

$$\frac{\partial J}{\partial \boldsymbol{x}} = E\boldsymbol{x} + F + M^T\boldsymbol{\lambda} = 0$$
$$\frac{\partial J}{\partial \boldsymbol{\lambda}} = M\boldsymbol{x} - \boldsymbol{\gamma} = 0$$

- From these expressions we can build the *Lagrangian matrix* and write the corresponding linear expression

$$\begin{bmatrix} E & M^T \\ M & \mathbf{0} \end{bmatrix} \begin{bmatrix} \boldsymbol{x} \\ \boldsymbol{\lambda} \end{bmatrix} = \begin{bmatrix} -F \\ \boldsymbol{\gamma} \end{bmatrix}$$

- If the inverse exists and is expressed as

$$\begin{bmatrix} E & M^T \\ M & \mathbf{0} \end{bmatrix}^{-1} = \begin{bmatrix} H & T \\ T & U \end{bmatrix}$$

  then the solution can be written

$$\boldsymbol{x}^\star = -HF + T\boldsymbol{\lambda}$$
$$\boldsymbol{\lambda}^\star = -T^T F - U\boldsymbol{\lambda}$$

- Explicit expressions for H, T and U (when the inverse exists) are given by

$$H = E^{-1} - E^{-1}M^T(ME^{-1}M^T)^{-1}ME^{-1}$$
$$T = E^{-1}M^T(ME^{-1}M^T)^{-1}$$
$$U = (ME^{-1}M^T)^{-1}$$

- This set of equations contains $n + m$ variables in the vectors $\boldsymbol{x}$ and $\boldsymbol{\lambda}$, which form the necessary conditions for minimizing the objective function with equality constraints

- The optimizing $\boldsymbol{\lambda}^\star$ and $\boldsymbol{x}^\star$ can be found directly from the partial derivatives above where

$$\boldsymbol{\lambda}^\star = -(ME^{-1}M^T)^{-1}(\boldsymbol{\gamma} + ME^{-1}F)$$
$$\boldsymbol{x}^\star = -E^{-1}(M^T\boldsymbol{\lambda}^\star + F)$$

- Note that

$$\boldsymbol{x}^\star = -E^{-1}F - E^{-1}M^T\boldsymbol{\lambda}^\star = \boldsymbol{x}^{\boldsymbol{o}} - E^{-1}M^T\boldsymbol{\lambda}^\star$$

where the first term $\boldsymbol{x}^o$ gives the optimal solution in the absence of constraints, and the second term is a correction factor due to the equality constraint.

Example 5.3

Minimize

$$J = \frac{1}{2}\boldsymbol{x}^t E\boldsymbol{x} + \boldsymbol{x}^t F$$

subject to

$$x_1 + x_2 + x_3 = 1$$
$$3x_1 - 2x_2 - 3x_3 = 1$$

where

$$E = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}; \quad F = \begin{bmatrix} -2 \\ -3 \\ -1 \end{bmatrix}$$

- The unconstrained optimizing solution is given by

$$\boldsymbol{x}^o = -E^{-1}F = \begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix}$$

- Note that the cost at the unconstrained minimum is:

$$\begin{aligned} J_o &= \frac{1}{2}\boldsymbol{x}_o^t E\boldsymbol{x}_o + \boldsymbol{x}_o^t F \\ &= \frac{1}{2}\begin{bmatrix} 2 & 3 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix} + \begin{bmatrix} 2 & 3 & 1 \end{bmatrix}\begin{bmatrix} -2 \\ -3 \\ -1 \end{bmatrix} \\ &= \frac{1}{2} \times 14 - 14 = -7 \end{aligned}$$

- From the constraint equations, we write,

$$M = \begin{bmatrix} 1 & 1 & 1 \\ 3 & -2 & -3 \end{bmatrix}; \quad \boldsymbol{\gamma} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

- Computing the minimizing Lagrangian vector $\boldsymbol{\lambda}^\star$ gives,

$$\boldsymbol{\lambda}^\star = -(ME^{-1}M^T)^{-1}(\boldsymbol{\gamma} + ME^{-1}F)$$

$$= -\begin{bmatrix} 3 & -2 \\ -2 & 22 \end{bmatrix}^{-1}\left(\begin{bmatrix} 1 \\ 1 \end{bmatrix} + \begin{bmatrix} -6 \\ 3 \end{bmatrix}\right) = \begin{bmatrix} 1.6452 \\ -0.0323 \end{bmatrix}$$

- Hence, $\boldsymbol{x}^\star$ is given by,

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \boldsymbol{x}^o - E^{-1}M^T\boldsymbol{\lambda}^\star$$

$$= \begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix} - \begin{bmatrix} 1 & 3 \\ 1 & -2 \\ 1 & -3 \end{bmatrix}\begin{bmatrix} 1.6452 \\ -0.0323 \end{bmatrix}$$

$$= \begin{bmatrix} 0.4516 \\ 1.2903 \\ -0.7419 \end{bmatrix}$$

- For the constrained optimizing solution,

$$J^\star = (\boldsymbol{x}^\star)^t E\boldsymbol{x}^\star + (\boldsymbol{x}^\star)^t F = -1.6130$$

indicating the cost function cannot achieve the same value as the unconstrained optimum.

- Alternatively, building the Lagrangian matrix,

$$\mathcal{L} = \begin{bmatrix} E & M^T \\ M & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 1 & 3 \\ 0 & 1 & 0 & 1 & -2 \\ 0 & 0 & 1 & 1 & -3 \\ 1 & 1 & 1 & 0 & 0 \\ 3 & -2 & -3 & 0 & 0 \end{bmatrix}$$

- Computing the eigenvalues of $\mathcal{L}$ we have

$$\lambda\{\mathcal{L}\} = \left[\begin{array}{ccccc} 5.239 & 2.2441 & 1.000 & -1.2441 & -4.2390 \end{array}\right]$$

indicating the matrix is non-singular and invertable.

- Solving the linear equation,

$$\mathcal{L}\left[\begin{array}{c} \boldsymbol{x} \\ \boldsymbol{\lambda} \end{array}\right] = \left[\begin{array}{c} -F \\ \boldsymbol{\gamma} \end{array}\right]$$

we again obtain the solution vector

$$\left[\begin{array}{c} \boldsymbol{x} \\ \boldsymbol{\lambda} \end{array}\right] = \left[\begin{array}{c} .4516 \\ 1.2903 \\ -.7419 \\ 1.6452 \\ -.0323 \end{array}\right]$$

Example 5.4

Consider once again the objective function,

$$J = \frac{1}{2}\boldsymbol{x}^t E \boldsymbol{x} + \boldsymbol{x}^t F$$

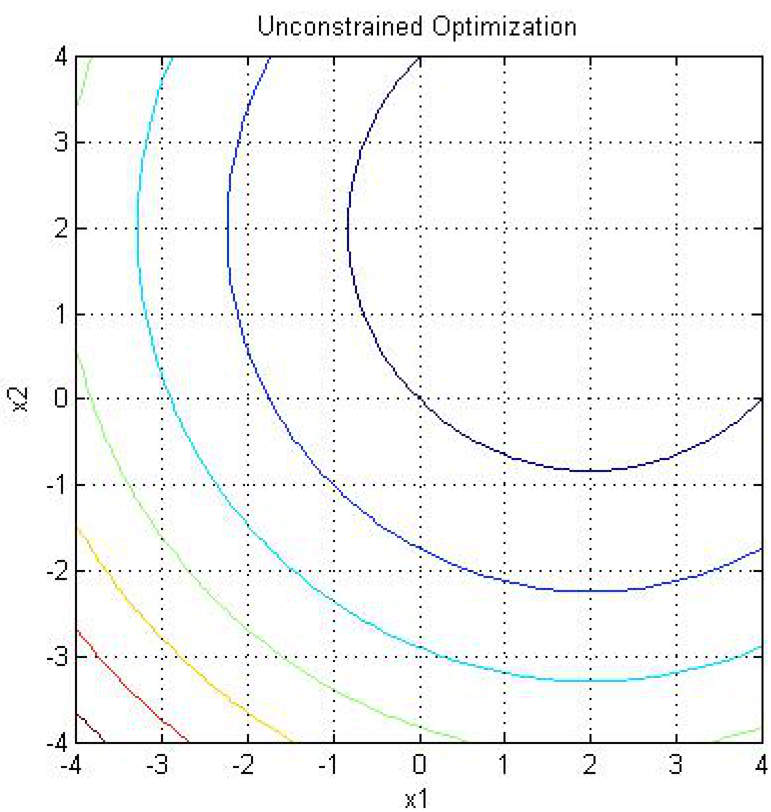
where,

$$E = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}; \quad F = \begin{bmatrix} -2 \\ -2 \end{bmatrix}$$

and the constraints are given by:

$$x_1 + x_2 = 1$$
$$2x_1 + 2x_2 = 6$$

- Find the feasible set of solutions
- The plot below shows contours of equal J for corresponding values of x_1 and x_2



- Note that in general, the number of equality constraints is required to be less than or equal to the number of decision variables.
- What happens when the number of equality constraints exactly *equals* the number of decision variables?
- What happens when the number of equality constraints is *greater than* the number of decision variables?

Quadratic Programming: Inequality Constraints

- In the previous case, it was necessary that all equality constraints be active at the solution
- In this case we have to consider the case where the inequality constraints $M\boldsymbol{x} \leq \boldsymbol{\gamma}$ may comprise both active and *inactive* constraints
  - *active* denotes equality
  - *inactive* denotes strict inequality

Kuhn-Tucker Conditions

- The Kuhn-Tucker conditions outline a set of necessary conditions for the optimization problem:

$$\begin{aligned} E\boldsymbol{x} + F + M^T\boldsymbol{\lambda} &= 0 \\ M\boldsymbol{x} - \boldsymbol{\gamma} &\leq 0 \\ \boldsymbol{\lambda}^t(M\boldsymbol{x} - \boldsymbol{\gamma}) &= 0 \\ \boldsymbol{\lambda} &\geq 0 \end{aligned}$$

- In terms of the active constraints, the conditions become:

$$Ex + F + \sum_{i \in S_{act}} \lambda_i M_i^T = 0$$

$$\begin{aligned} M_i x - \gamma_i &= 0 \quad i \in S_{act} \\ M_i x - \gamma_i &< 0 \quad i \notin S_{act} \\ \lambda_i &\geq 0 \quad i \in S_{act} \\ \lambda_i &= 0 \quad i \notin S_{act} \end{aligned}$$

where M_i is the i^{th} row of the M matrix.

- For an *active* constraint (i.e., $M_i \boldsymbol{x} - \boldsymbol{\gamma}_i = 0$) , the corresponding Lagrange multiplier is *non-negative*
- Otherwise if the constraint is *inactive*, the Lagrange multiplier is *zero*

$\Rightarrow$ So, for the set of active constraints, the problem looks exactly like the equality constraint optimization!

- Assume the previous problem with the equality constraints replaced with inequality constraints,

$$J = \frac{1}{2}\boldsymbol{x}^t E \boldsymbol{x} + \boldsymbol{x}^t F$$

where,

$$E = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}; \quad F = \begin{bmatrix} -2 \\ -2 \end{bmatrix}$$

and the inequality constraints are now given by:

$$\begin{aligned} x_1 + x_2 &\leq 1 \\ 2x_1 + 2x_2 &\leq 6 \end{aligned}$$

- It is clear that the set of variables that satisfy the first inequality will also satisfy the second
  - Thus the first of these is an active constraint, while the second is inactive

- We find the constrained optimum by minimizing J subject to the equality constraint $x_1 + x_2 = 1$, which gives $x_1 = 0.5$ and $x_2 = 0.5$, as found previously

Active Set Methods

- The idea here is to construct an algorithm where we define at each step a set of constraints to be treated as the *active set*
  - Chosen to be a subset of the contraints that are active at the current point
  - Current point is *feasible* for the working set
  - Algorithm moves along a surface defined by the working set of constraints to an improved point
  - If all $\lambda_i \geq 0$, the point is a local solution
  - If some $\lambda_i < 0$, then the objective function value can be further decreased by relaxing the constraint

Example 5.5

Optimize the objective function where

$$E = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}; \; F = \begin{bmatrix} -2 \\ -3 \\ -1 \end{bmatrix}$$

subject to equality constraints

$$\begin{aligned} x_1 + x_2 + x_3 &\le 1 \\ 3x_1 - 2x_2 - 3x_3 &\le 1 \\ x_1 - 3x_2 + 2x_3 &\le 1 \end{aligned}$$

- The feasible solution of the equality constraints exists and is the solution of the linear equations

$$\begin{aligned} x_1 + x_2 + x_3 &= 1 \\ 3x_1 - 2x_2 - 3x_3 &= 1 \\ x_1 - 3x_2 + 2x_3 &= 1 \end{aligned}$$

- The three equality constraints are taken as the first working set
- Calculating the Lagrange multipliers we get

$$\boldsymbol{\lambda} = -(ME^{-1}M^T)^{-1}(\boldsymbol{\gamma} + ME^{-1}F) = \begin{bmatrix} 1.6873 \\ 0.0309 \\ -0.4352 \end{bmatrix}$$

- Since the third element is negative, the third constraint is inactive and is dropped from the constrained equation set
- Thus the problem becomes:

$$J = \frac{1}{2}\boldsymbol{x}^t E \boldsymbol{x} + \boldsymbol{x}^t F$$

subject to

$$x_1 + x_2 + x_3 = 1$$
$$3x_1 - 2x_2 - 3x_3 = 1$$

- Solve this equality constraint problem to get

$$\boldsymbol{\lambda} = \begin{bmatrix} 1.6452 \\ -.0323 \end{bmatrix}$$

  – Here the second element is negative

- The problem collapses once again to

$$J = \frac{1}{2}\boldsymbol{x}^t E \boldsymbol{x} + \boldsymbol{x}^t F$$

now subject only to the single constraint

$$x_1 + x_2 + x_3 = 1$$

- Solving this problem yields $\lambda = 5/3$, leading to

$$\boldsymbol{x}^\star = \begin{bmatrix} 0.3333 \\ 1.3333 \\ -0.6667 \end{bmatrix}$$

  – Clearly, the optimal solution $\boldsymbol{x}^\star$ satisfies the equality constraint

- Checking the others, we obtain

$$M\boldsymbol{x}^\star = \begin{bmatrix} 1 & 1 & 1 \\ 3 & -2 & -3 \\ 1 & -3 & 2 \end{bmatrix} \begin{bmatrix} 0.3333 \\ 1.3333 \\ -0.6667 \end{bmatrix} = \begin{bmatrix} 1.0 \\ 0.3334 \\ -5.0 \end{bmatrix} \le \begin{bmatrix} 1.0 \\ 1.0 \\ 1.0 \end{bmatrix}$$

and indeed the rest of the inequality constraints are also satisfied.

- Some observations:

– The maximum number of equality constraints equals the number of decision variables

– The maximum number of inequality constraints can exceed the number of decision variables if they are not all active

– An iterative procedure is required to solve the optimization problem with inequality constraints

<u>*Primal-Dual Methods*</u>

- A *dual* method can be used to identify the constraints that are not active in the optimization problem so that they can be systematically eliminated in the solution

- Assuming feasibility (i.e., there exists an $\boldsymbol{x}$ such that $M\boldsymbol{x} < \boldsymbol{\gamma}$), the *primal* problem is equivalent to

$$\max_{\lambda \geq 0} \min_{\boldsymbol{x}} \left[\frac{1}{2}\boldsymbol{x}^t E\boldsymbol{x} + \boldsymbol{x}^t F + \boldsymbol{\lambda}^t (M\boldsymbol{x} - \boldsymbol{\gamma}) \right]$$

- The minimum over $\boldsymbol{x}$ is unconstrained and attained by

$$\boldsymbol{x} = -E^{-1}(F + M^T\boldsymbol{\lambda})$$

- Substituting this into the above expression gives the dual problem

$$\max_{\boldsymbol{\lambda} \geq 0} \left(-\frac{1}{2}\boldsymbol{\lambda}^t P\boldsymbol{\lambda} - \boldsymbol{\lambda}^t K - \frac{1}{2}F^T E^{-1} F \right)$$

where

$$P = ME^{-1}M^T$$
$$K = \boldsymbol{\gamma} + ME^{-1}F$$

- Hence the dual is another quadratic programming problem, only now with $\boldsymbol{\lambda}$ as the decision variable instead of $\boldsymbol{x}$

- Note, the dual problem is equivalent to

$$\max_{\boldsymbol{\lambda}\geq 0}\left(\frac{1}{2}\boldsymbol{\lambda}^t P\boldsymbol{\lambda} + \boldsymbol{\lambda}^t K + \frac{1}{2}\boldsymbol{\gamma}^T E^{-1}\boldsymbol{\gamma}\right)$$

Hildreth's Procedure

- Hildreth's quadratic programming procedure is a simple variant of the Gauss-Seidel method for solving a linear set of equations
- Some features:
  - Direction vectors are equal to the standard basis vectors,

$$\boldsymbol{e}_i = \begin{bmatrix} 0 & 0 & \ldots & 1 & \ldots & 0 & 0 \end{bmatrix}^t$$

  - The $\boldsymbol{\lambda}$ vector is varied one component at a time
  - The objective function can be regarded as a quadratic function in each component
  - Adjust $\boldsymbol{\lambda}_i$ to minimize the objective function
  - When $\lambda_i < 0$ then set $\lambda_i = 0$
- The general method can be expressed as:

$$\lambda_i^{m+1} = \max(0,\ w_i^{m+1})$$

where,

$$w_i^{m+1} = -\frac{1}{p_{ii}}[k_i + \sum_{j=1}^{i-1} p_{ij}\lambda_j^{m+1} + \sum_{j=i+1}^{n} p_{ij}\lambda_j^{m}]$$

- p_{ij} is the ij^{th} element of the matrix $P = ME^{-1}M^T$
- k_i is the i^{th} element of the vector $K = \boldsymbol{\gamma} + ME^{-1}F$

Example 5.6

Minimize the cost function $J = \frac{1}{2}\boldsymbol{x}^t E x + F^t \boldsymbol{x}$ where

$$E = \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix}; \quad F = \begin{bmatrix} -1 \\ 0 \end{bmatrix}.$$

and the constraints are

$$0 \le x_1$$
$$0 \le x_2$$
$$3x_1 + 2x_2 \le 4$$

- Forming the linear inequality constraint equation, we have

$$M\boldsymbol{x} \le \boldsymbol{\gamma}$$
$$\begin{bmatrix} -1 & 0 \\ 0 & -1 \\ 3 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \le \begin{bmatrix} 0 \\ 0 \\ 4 \end{bmatrix}$$

- The unconstrained optimal solution is found as

$$\boldsymbol{x}_o = E^{-1}F = \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix}^{-1} \begin{bmatrix} -1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

- Substituting $\boldsymbol{x}_o$ into the expression above shows that the third constraint is violated:

$$\begin{bmatrix} -1 & 0 \\ 0 & -1 \\ 3 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ -1 \\ 5 \end{bmatrix} \not< \begin{bmatrix} 0 \\ 0 \\ 4 \end{bmatrix}$$

- Now, find the optimal $\boldsymbol{\lambda}^\star$:

$$P = ME^{-1}M^T$$

$$= \begin{bmatrix} -1 & 0 \\ 0 & -1 \\ 3 & 2 \end{bmatrix} \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix}^{-1} \begin{bmatrix} -1 & 0 & 3 \\ 0 & -1 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 1 & -5 \\ 1 & 2 & -7 \\ -5 & -7 & 29 \end{bmatrix}$$

$$K = \boldsymbol{\gamma} + ME^{-1}F$$

$$= \begin{bmatrix} 0 \\ 0 \\ 4 \end{bmatrix} + \begin{bmatrix} -1 & 0 \\ 0 & -1 \\ 3 & 2 \end{bmatrix} \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix}^{-1} \begin{bmatrix} -1 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$$

- Iteration:

  k=0

$$\lambda_1^0 = \lambda_2^0 = \lambda_3^0 = 0$$

  k=1

$$w_1^1 + 1 = 0$$

$$\lambda_1^1 + 2w_2^1 + 1 = 0$$

$$-5\lambda_1^1 - 7\lambda_2^1 + 29w_3^1 - 1 = 0$$

$$\lambda_1^1 = \max(0,\ w_1^1) = 0$$

$$\lambda_2^1 = \max(0,\ w_2^1) = 0$$

$$\lambda_3^1 = \max(0,\ w_3^1) = 0.0345$$

k=2

$$w_1^2 + \lambda_2^1 - 5\lambda_3^1 + 1 = 0$$
$$\lambda_1^2 + 2w_2^2 - 7\lambda_3^1 + 1 = 0$$
$$-5\lambda_1^2 - 7\lambda_2^2 + 29w_3^2 - 1 = 0$$

$$\lambda_1^2 = \max(0,\ w_1^2) = 0$$
$$\lambda_2^2 = \max(0,\ w_2^2) = 0$$
$$\lambda_3^2 = \max(0,\ w_3^2) = 0.0345$$

- We see that the procedure has converged giving the optimal Lagrange vector:

$$\boldsymbol{\lambda}^\star = \begin{bmatrix} 0 \\ 0 \\ 0.0345 \end{bmatrix}$$

which generates the optimal solution:

$$\boldsymbol{x}^\star = \boldsymbol{x_o} - E^{-1}M^T\boldsymbol{\lambda}^\star = \begin{bmatrix} 1 \\ 1 \end{bmatrix} - \begin{bmatrix} 0.1724 \\ 0.2414 \end{bmatrix} = \begin{bmatrix} 0.8276 \\ 0.7586 \end{bmatrix}$$

Constraints on Rate of Change

Example 5.7

Consider the continuous-time plant described by

$$G(s) = \frac{10}{s^2 + 0.1s + 3}$$

whose system poles are located at: $-.05 \pm j1.7313$.

Assume a sampling interval of $\triangle T = 0.1$ and design a descrete-time model predictive control system with

$$\begin{aligned} N_c &= 3 \\ N_p &= 20 \\ \bar{R} &= 0.01 \times I \end{aligned}$$

A constraint is imposed on rate-of-change of the control signal as

$$-1.5 \le \triangle u(k) \le 3.0$$

- First, obtain a discrete-time state-space model
  - Continuous-time state-space:

$$A_c = \begin{bmatrix} -0.1 & -3 \\ 1 & 0 \end{bmatrix} \quad B_c = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$C_c = \begin{bmatrix} 0 & 10 \end{bmatrix} \quad D_c = \begin{bmatrix} 0 \end{bmatrix}$$

  - Discrete-time state-space:

$$A_d = \begin{bmatrix} .9752 & -.2970 \\ .0990 & .9851 \end{bmatrix} \quad B_d = \begin{bmatrix} .0990 \\ .005 \end{bmatrix}$$

$$C_d = \begin{bmatrix} 0 & 10 \end{bmatrix} \quad D_d = \begin{bmatrix} 0 \end{bmatrix}$$

- Next, form the augmented state-space model:

$$A = \begin{bmatrix} .9752 & -.2970 & 0 \\ .0990 & .9851 & 0 \\ .990 & 9.8509 & 1 \end{bmatrix} \quad B = \begin{bmatrix} .0990 \\ .005 \\ .0497 \end{bmatrix}$$

$$C = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix} \quad D = \begin{bmatrix} 0 \end{bmatrix}$$

- Now, write the objective function:

$$J = \triangle U^T(\Phi^T\Phi + \bar{R})\triangle U - 2\triangle U^T\Phi^T(R_s - F\boldsymbol{x}(k_i))$$

where,

$$\Phi^T\Phi = \begin{bmatrix} .1760 & .1553 & .1361 \\ .1553 & .1373 & .1204 \\ .1361 & .1204 & .1057 \end{bmatrix}; \quad \Phi^T F = \begin{bmatrix} .1972 & -.1758 & 1.4187 \\ .1740 & -.1552 & 1.2220 \\ .1522 & -.1359 & 1.0443 \end{bmatrix}$$

and

$$R_s = \begin{bmatrix} 1.4187 \\ 1.2220 \\ 1.0443 \end{bmatrix} \times r(k_i)$$

- Select observer poles at $\left\{ 0 \;\; 0 \;\; 0 \right\}$
- Computing the closed-loop compensated eigenvalues for a range of control weighting values (r_w) results in the following values for $K_{mpc}(r_w)$:

$$K_{mpc}(.01) = \begin{bmatrix} 13.1552 & 84.1009 & 3.7417 \end{bmatrix}$$

$$K_{mpc}(.1) = \begin{bmatrix} 11.3704 & 58.4467 & 1.1666 \end{bmatrix}$$

$$K_{mpc}(1.0) = \begin{bmatrix} 9.5534 & 43.5058 & 0.6567 \end{bmatrix}$$

$$K_{mpc}(10) = \left[\ 5.5692\ \ 14.6499\ \ 0.2460\ \right]$$

$$K_{mpc}(100) = \left[\ 3.6823\ \ 3.0537\ \ 0.0868\ \right]$$

- These give rise to the corresponding closed-loop eigenvalues:

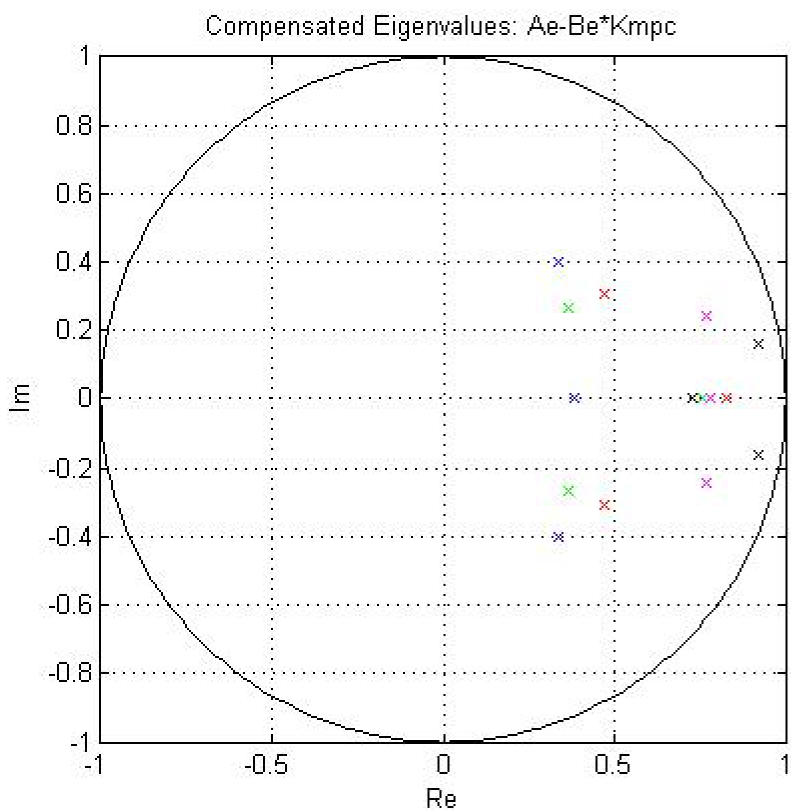
$$\lambda(.01) = \left\{\ .3357 + j0.4\ \ .3357 - j0.4\ \ 0.3824\ \right\}$$

$$\lambda(.1) = \left\{\ .3654 + j0.2650\ \ .3654 - j0.2650\ \ 0.7552\ \right\}$$

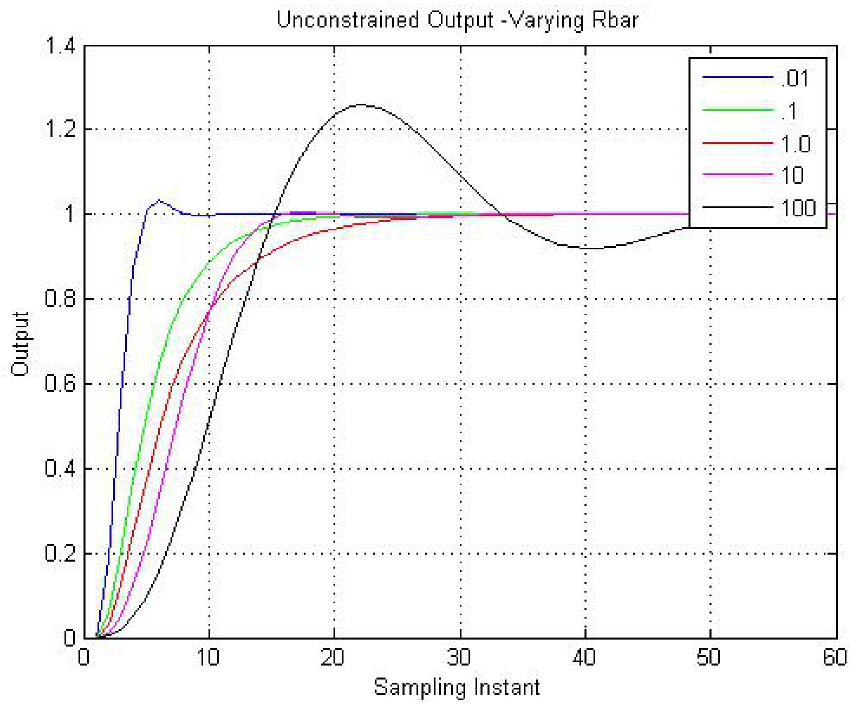
$$\lambda(1.0) = \left\{\ .4697 + j0.3063\ \ .4697 - j0.3063\ \ 0.8262\ \right\}$$

$$\lambda(10) = \left\{\ .7710 + j0.2439\ \ .7710 - j0.2439\ \ .7819\ \right\}$$

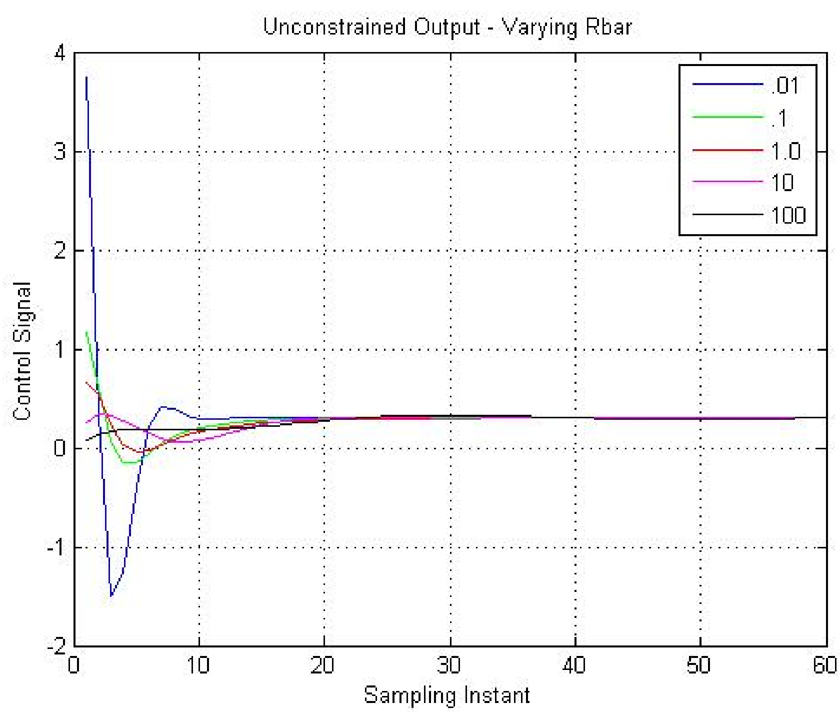
$$\lambda(100) = \left\{\ .9240 + j0.1610\ \ .9249 - j0.1610\ \ .7283\ \right\}$$



- The output response to a unit set-point change is depicted in the following figure for each of the control weighting values.



- The corresponding control inputs are shown in the plot below



- Let us now establish our problem constraint as:

$$\begin{bmatrix} 1 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix} \begin{bmatrix} \triangle u(k_i) \\ \triangle u(k_i+1) \\ \triangle u(k_i+2) \end{bmatrix} \leq \begin{bmatrix} 3.0 \\ -1.5 \end{bmatrix}$$

$$M\boldsymbol{\Delta U} \leq \boldsymbol{\gamma}$$

- Recall that we constructed our original cost function as:

$$J = (\boldsymbol{R}_s - \boldsymbol{Y})^t(\boldsymbol{R}_s - \boldsymbol{Y}) + \boldsymbol{\Delta U^t}\bar{R}\boldsymbol{\Delta U}$$

where

$$\boldsymbol{Y} = F\boldsymbol{x}(k_i) + \Phi\triangle\boldsymbol{U}$$

- Substituting and multiplying out we obtained:

$$J = (\boldsymbol{R}_s - F\boldsymbol{x}(k_i))^t(\boldsymbol{R}_s - F\boldsymbol{x}(k_i)) - 2\boldsymbol{\Delta U^t}\Phi^T(\boldsymbol{R}_s - F\boldsymbol{x}(k_i)) \\ +\boldsymbol{\Delta U^t}(\Phi^T\Phi + \bar{R})\boldsymbol{\Delta U}$$

- We found the optimizing control input sequence from the stationarity condition $\dfrac{\partial J}{\partial \boldsymbol{\Delta U}} = 0$ as:

$$\boldsymbol{\Delta U^*} = (\Phi^T\Phi + \bar{R})^{-1}\Phi^T(\boldsymbol{R_s} - F\boldsymbol{x}(k_i))$$

- Using this expression with the assumption that the initial state $\boldsymbol{x}(0) = 0$, we obtain

$$\boldsymbol{\Delta U} = \begin{bmatrix} 3.7417 & -6.2794 & 2.7112 \end{bmatrix}^t$$

  – Here we see that the constraint is violated since

$$\triangle u(1) = 3.7417 \not< 3.0$$

- Using Hildreth's quadratic programming algorithm we obtain

$$\boldsymbol{\Delta U} = [\ 3.000 \ \ -4.7241 \ \ 1.8843]$$

- Using the first component, $\triangle u(1) = 3.000$ and $y(1) = 0$, the new estimated state variable is

$$\boldsymbol{x}(2) = \begin{bmatrix} .2791 \\ .0149 \\ .1491 \end{bmatrix}$$

- Computing the next incremental input sequence gives

$$\boldsymbol{\Delta U^*} = \begin{bmatrix} -1.9777 & -4.1326 & 3.3456 \end{bmatrix}$$

and the constrained solution via Hildreth's algorithm gives

$$\boldsymbol{\Delta U_c^*} = \begin{bmatrix} -1.5 & -5.1342 & 3.8781 \end{bmatrix}$$

- This generates the new state update

$$\boldsymbol{x}(3) = \begin{bmatrix} .1367 \\ .0366 \\ .5155 \end{bmatrix}$$

- Continuing,

$$\triangle U^* = \begin{bmatrix} -3.0671 & -.6187 & 2.4816 \end{bmatrix}$$

$$\triangle U_c^* = \begin{bmatrix} -1.5 & -3.9045 & 4.2285 \end{bmatrix}$$

and

$$\boldsymbol{x}(4) = \begin{bmatrix} -.0261 \\ .0422 \\ .9372 \end{bmatrix}$$

- And again

$$\triangle U^* = -\begin{bmatrix} -2.9689 & 2.2251 & 1.0865 \end{bmatrix}$$

$$\triangle U_c^* = \begin{bmatrix} -1.5 & -.8547 & 2.7239 \end{bmatrix}$$

and the updated state,

$$\boldsymbol{x}(5) = \begin{bmatrix} -.1865 \\ .0315 \\ 1.2523 \end{bmatrix}$$

- So we obtain the optimal incremental input sequence for the first four time samples as:

$$\triangle U^* = \begin{bmatrix} 3.0 & -1.5 & -1.5 & -1.5 & \ldots \end{bmatrix}$$

giving the output sequence:

$$Y = \begin{bmatrix} 0 & .1491 & .5155 & .9372 & 1.2523 \end{bmatrix}$$

(mostly blank)