# *Mathematical Models of Dynamic Systems*

## Transfer Function Models

- Classical control theory is based on input-output relationships for linear, time-invariant dynamic systems represented by transfer functions:

$$G(s) = \frac{Y(s)}{U(s)}$$

where $Y(s)$ and $U(s)$ are the Laplace transforms of the output and input constant coefficient ordinary differential equations, respectively, assuming all initial conditions are zero.

### Example: Second-Order System

$$\ddot{y}(t) + 2\zeta\omega_n\dot{y}(t) + \omega_n^2 y(t) = \omega_n^2 u(t)$$

- Taking the Laplace transform, we get

$$s^2Y(s) + s\zeta\omega_n Y(s) + \omega_n^2 Y(s) = \omega_n^2 U(s)$$

- Re-arranging, we obtain the system transfer function:

$$G(s) = \frac{Y(s)}{U(s)} = \frac{\omega_n^2}{s^2 + \zeta\omega_n s + \omega_n^2}$$

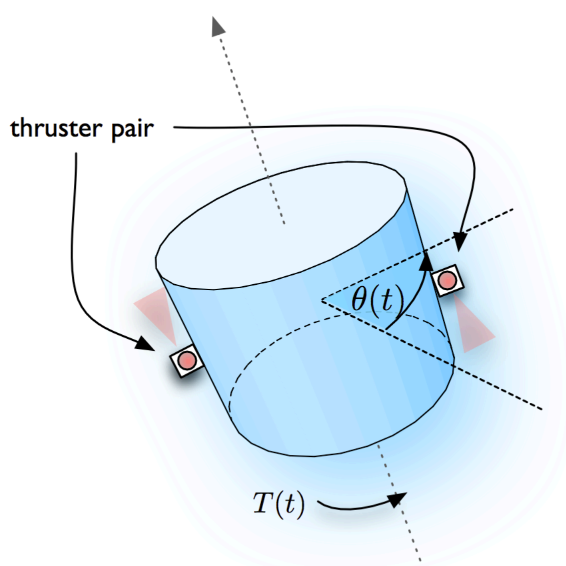
### Comments on Transfer Functions:

- A transfer function harbors all the dynamic information contained in the mathematical model of a system.
- Points in the $s$-plane at which $G(s)$ is analytic are called *ordinary points*

- Points in the $s$-plane at which $G(s)$ is not analytic are called *singular points*
  - Singular points at which $G(s)$ or its derivatives approach infinity are called *poles*
- Points at which the function $G(s)$ equals zero are called *zeros*
- The poles determine the form of the system dynamic response, while the zeros affect the relative contributions of poles to the overall response

## Example: Satellite Attitude Control

Determine the transfer function for a satellite attitude control system where torque $T(t)$ is the input, and the angular displacement of the satellite $\theta(t)$ is the output.




- A thruster pair is located on the circumference of the satellite to impart a torque $T(t)$ about the principal axis of rotation.

- The moment of interia about the prinicpal axis of rotation at the center of mass is $J$.
- To find the transfer function, we first write the governing differential equation by Newtons second law:

$$J\frac{d^2\theta}{dt^2} = T$$

- Taking Laplace transform of both sides,

$$Js^2\Theta(s) = T(s)$$

- Re-arranging, we obtain the result,

$$G(s) = \frac{\Theta(s)}{T(s)} = \frac{1}{Js^2}$$

showing that the dynamic system is a double-integrator.

## State-Space Models

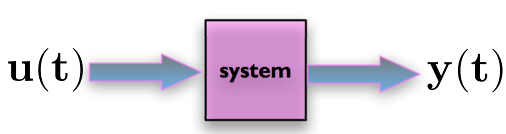
- Many dynamic systems of practical interest admit non-linear models and may have many inputs and many outputs
- Generalizations of classical transfer function methods to these more complicated systems are difficult to apply
- It is instead advantageous to describe a system's possibly high order equations in terms of a set of $n$ first-order differential equations
- This is the basis of the *state-space* approach to system analysis
- Use of vector-matrix notation greatly simplifies the mathematical representation of systems of equations

## *Key Ideas of State-Space Analysis*

- STATE. The state of a system at time $t_k$ is the minimum set of information at $t_k$ that, together with the input $u(t)$, $t \geq t_k$ uniquely determines the behavior of the system for all $t \geq t_k$.
  - The state of a dynamic system at time $t$ is uniquely determined by the state at time $t_k$ and the input for $t \geq t_k$ ; it is not dependent on the state and input before $t_k$ (why?)
- STATE VARIABLES. The *state variables* for a dynamic system are the variables making up the smallest set that determine the state of the dynamic system.
  - State variables don't need to be physically measurable or observable quantities.
  - State variables provide access to what is going on inside a dynamic process.
  - A set of state variables is not unique for a given system.
- STATE VECTOR. If $n$ state variables are needed to completely describe the behavior of a system, then these $n$ state variables can be consisered the $n$ components of a *state vector*, $\boldsymbol{x}$.

## *State Equations*

- A helpful way to think about the action of state equations is to consider an input-output model of a general dynamic system,



- In this sense, the output $\boldsymbol{y}(t)$ for $t \geq t_k$ depends on the value $\boldsymbol{y}(t_k)$ and the input $\boldsymbol{u}(t)$ for $t \geq t_k$
- It seems the system must somehow retain - or memorize - values of the input for $t \geq t_k$
- This memorization role is played in dynamic systems by *integrators*
- Output of integrators can be considered as the variables that define the internal state of the dynamic system $\Rightarrow$ state variables!
  - Number of state variables needed to define system dynamics $\Rightarrow$ number of integrators in the system!
- So, for a dynamic system with $n$ integrators, $r$ inputs, and $m$ outputs, we can write

$$
\begin{aligned}
\dot{x}_1(t) &= f_1(x_1,\, x_2,\, \ldots,\, x_n;\, u_1,\, u_2,\, \ldots,\, u_r;\, t) \\
\dot{x}_2(t) &= f_2(x_1,\, x_2,\, \ldots,\, x_n;\, u_1,\, u_2,\, \ldots,\, u_r;\, t) \\
&\vdots \\
\dot{x}_n(t) &= f_n(x_1,\, x_2,\, \ldots,\, x_n;\, u_1,\, u_2,\, \ldots,\, u_r;\, t)
\end{aligned}
$$

- Likewise, for the $m$ outputs,

$$
\begin{aligned}
y_1(t) &= g_1(x_1,\, x_2,\, \ldots,\, x_n;\, u_1,\, u_2,\, \ldots,\, u_r;\, t) \\
y_2(t) &= g_2(x_1,\, x_2,\, \ldots,\, x_n;\, u_1,\, u_2,\, \ldots,\, u_r;\, t) \\
&\vdots \\
y_m(t) &= g_m(x_1,\, x_2,\, \ldots,\, x_n;\, u_1,\, u_2,\, \ldots,\, u_r;\, t)
\end{aligned}
$$

- Gathering these up into corresponding vectors and linearizing about the operating state, we can write

$$
\begin{aligned}
\dot{\boldsymbol{x}}(t) &= A(t)\boldsymbol{x}(t) + B(t)\boldsymbol{u}(t) \\
\boldsymbol{y}(t) &= C(t)\boldsymbol{x}(t) + D(t)\boldsymbol{u}(t)
\end{aligned}
$$

where $A(t)$ is the state matrix, $B(t)$ is the input matrix, $C(t)$ is the output matrix, and $D(t)$ is the direct transmission or feed-through matrix.

- For *time-invariant* systems, we drop the dependence on $t$ in the matrix functions
- Equivalent state-space representations are related by change of basis through similarity transformations as follows:
  - Let $\boldsymbol{x} = T\boldsymbol{\xi}$, where $T$ is a square non-singular transformation matrix. Then we can write

$$\begin{aligned}\dot{\boldsymbol{\xi}} &= T^{-1}\dot{\boldsymbol{x}} \\ &= T^{-1}\left[A\boldsymbol{x} + B\boldsymbol{u}\right] \\ &= T^{-1}\left[AT\boldsymbol{\xi} + B\boldsymbol{u}\right] \\ &= T^{-1}AT\boldsymbol{\xi} + T^{-1}B\boldsymbol{u} \\ \boldsymbol{y} &= CT\boldsymbol{\xi} + D\boldsymbol{u}\end{aligned}$$

Thus,

$$\begin{aligned}\dot{\boldsymbol{\xi}} &= \widetilde{A}\boldsymbol{\xi} + \widetilde{B}\boldsymbol{u} \\ \boldsymbol{y} &= \widetilde{C}\boldsymbol{\xi} + \widetilde{D}\boldsymbol{u}\end{aligned}$$

*Relationship to Transfer Function*

- Given the state equations for a linear, time-invariant system:

$$\begin{aligned}\dot{\boldsymbol{x}} &= A\boldsymbol{x} + B\boldsymbol{u} \\ \boldsymbol{y} &= C\boldsymbol{x} + D\boldsymbol{u}\end{aligned}$$

where we have let $\boldsymbol{x} \equiv \boldsymbol{x}(t)$ for simplicity of notation, we take the Laplace transform to obtain

$$sX(s) - \boldsymbol{x}(0) = AX(s) + BU(s)$$

$$Y(s) = CX(s) + DU(s)$$

- Rearranging,

$$\begin{aligned}(sI - A)X(s) &= BU(s) + \boldsymbol{x}(0) \\ X(s) &= (sI - A)^{-1}BU(s) + (sI - A)^{-1}\boldsymbol{x}(0)\end{aligned}$$

and

$$Y(s) = \left[C(sI - A)^{-1}B + D\right]U(s) + C(sI - A)^{-1}x(0)$$

- Therefore,

$$G(s) = \frac{Y(s)}{U(s)} = C(sI - A)^{-1}B + D$$

- Since

$$(sI - A)^{-1} = \frac{1}{\det(sI - A)} \cdot \mathrm{adj}(sI - A)$$

we see that $\det(sI - A)^{-1} = 0$ is the characteristic polynomial for $G(s)$, and the eigenvalues of $A$ are identically the poles of $G(s)$.

- Also, if we introduce transformed state matrices from above, we obtatin for the transfer function $\tilde{G}(s)$

$$\begin{aligned}\tilde{G}(s) &= \tilde{C}(sI - \tilde{A})^{-1}\tilde{B} + \tilde{D} \\ &= CT(sI - T^{-1}AT)^{-1}T^{-1}B + D \\ &= C(sI - A)^{-1}B + D \\ &= G(s) \\ &\Rightarrow \text{Invariant under similarity transformation!}\end{aligned}$$

*Dynamic Response*

- The time response $y(t)$ of a linear time-invariant system represented by transfer function $G(s)$ is found by inverse Laplace transform of $Y(s) = G(s)U(s)$ for a given input $U(s) = \mathscr{L}\{u(t)\}$.

- Since multiplication in the complex domain yields convolution in the time domain, we can write

$$y(t) = \int_0^t u(\tau)g(t-\tau)d\tau$$
$$= \int_0^t g(\tau)u(t-\tau)d\tau$$

- The function $g(t)$ is the *impulse response* function of the dynamic system, where $g(t) = 0$ for $t < 0$.

NOTE: The transfer function $G(s)$ and the impulse response $g(t)$ contain the same dynamic information about the system. Recall that by definition of a transfer function, all initial conditions are assumed to be zero.

- Let's now examine the form of response for a dynamic system in the state-space representation.
- Consider the simple single variable system given by

$$\dot{x}(t) = ax(t), \quad \text{where } x(0) = x_0$$

- Taking Laplace transform,

$$sX(s) - x_0 = aX(s)$$
$$(s-a)X(s) = x_0$$
$$X(s) = (s-a)^{-1}x_0$$

- By inverse Laplace, the unforced response is given by

$$x(t) = e^{at}x_0$$

- By analogy, for the matrix case we have,

$$\dot{\boldsymbol{x}}(t) = A\boldsymbol{x}(t)$$

where $\boldsymbol{x}(0) = \boldsymbol{x}_0$

- And after Laplace transformation,

$$\boldsymbol{x}(t) = \mathscr{L}^{-1}[(sI - A)^{-1}]\boldsymbol{x}_0$$

- Now, since

$$(sI - A)^{-1} = \frac{I}{s} + \frac{A}{s^2} + \frac{A^2}{s^3} + \cdots$$

$$\begin{aligned} \mathscr{L}^{-1}\left[(sI - A)^{-1}\right] &= I + At + \frac{A^2t^2}{2!} + \frac{A^3t^3}{3!} + \cdots \\ &= e^{At} \\ \boldsymbol{x}(t) &= e^{At}\boldsymbol{x}_0 \end{aligned}$$

- In this development, the matrix exponential $e^{At}$ is termed the *state transition matrix*
- The forced part of the solution is found from the state equation

$$\dot{\boldsymbol{x}}(t) = A\boldsymbol{x}(t) + B\boldsymbol{u}(t)$$

- Following the earlier development, we get

$$\boldsymbol{x}(t) = e^{At}\boldsymbol{x}_0 + \int_0^t e^{A(t-\tau)}B\boldsymbol{u}(\tau)d\tau$$

- Incorporating the output equation $\boldsymbol{y}(t) = C\boldsymbol{x}(t) + D\boldsymbol{u}(t)$ gives

$$\boldsymbol{y}(t) = Ce^{At}\boldsymbol{x}_0 + \int_0^t Ce^{A(t-\tau)}B\boldsymbol{u}(\tau)d\tau + D\boldsymbol{u}(t)$$

*Dyadic Expansion of Full Response*

- Let's now take a look at the form of response under a diagonalizing transformation
- In particular, apply the eigen-decomposition to write

$$A = W\Lambda W^{-1}$$

where $\Lambda$ is a diagonal matrix containing the eigenvalues $\lambda_i$ of $A$.

- In this case, the state transition matrix becomes

$$e^{\Lambda t} = \begin{bmatrix} e^{\lambda_1 t} & & & 0 \\ & e^{\lambda_2 t} & & \\ & & \ddots & \\ 0 & & & e^{\lambda_n t} \end{bmatrix}$$

- We have that

$$W^{-1} = V = \begin{bmatrix} \boldsymbol{v}_1^t \\ \boldsymbol{v}_2^t \\ \vdots \\ \boldsymbol{v}_n^t \end{bmatrix} \quad \text{and} \quad W = \begin{bmatrix} \boldsymbol{w}_1 & \boldsymbol{w}_2 & \cdots & \boldsymbol{w}_n \end{bmatrix}$$

represent matrices containing the left and right eigenvectors of $A$, respectively.

- We can express the *dyadic decomposition* of $e^{At}$ as

$$e^{At} = \sum_{i=1}^{n} e^{\lambda_i t} \boldsymbol{w}_i \boldsymbol{v}_i^t$$

where the $\boldsymbol{w}_i \boldsymbol{v}_i^t$ form a vector dyad.

- This form of decomposition shows that the fundamental forms of response represented by the eigenvalues are expressed geometrically through the corresponding matrix dyad.

## Discrete-Time Systems

- Practical implementations of modern control methodologies rely heavily on digital computers and discrete-time representations of dynamic systems.
- Fortunately, much of the forgoing development introduced for continuous-systems can be applied to discrete-time systems.
  - The key to this analogy is the treatment of continuous processes as sampled-data systems and the use of *linear difference equations.*
- Assume an analog-to-digital (A/D) converter takes samples of the constant signal $y(t)$ at discrete times so that the samples $\hat{y}(kT) = y(kT)$
  - Here, $T$ is a constant time interval and $k$ is a counter or index.
- Consider a linear discrete-time function $y(k)$ computed using past outputs, $y(k-1)$, $y(k-2)$, ... and current and past inputs, $u(k)$, $u(k-1)$, $u(k-2)$,...
- Such a function can be written as (where we've implied dependence on $T$):

$$y(k) = -a_1 y(k-1) - a_2 y(k-2) - a_3 y(k-3) - \cdots - a_n y(k-n) + b_0 u(k) + b_1 u(k-1) +$$

- This is a *linear recurrence equation* or *difference equation*.
- We can define the differences in the $y(k)'s$ as

$$\nabla y(k) = y(k) - y(k-1)$$
$$\nabla^2 y(k) = \nabla y(k) - \nabla y(k-1)$$
$$\nabla^n y(k) = \nabla^{n-1} y(k) - \nabla^{n-1} y(k-1)$$

- Substituting in our difference equation above gives

$$\begin{aligned} y(k) &= y(k) \\ y(k-1) &= y(k) - \nabla y(k) \\ y(k-2) &= y(k) - 2\nabla y(k) + \nabla^2 y(k) \end{aligned}$$

- So, an equivalent difference equation can be written as

$$a_2 \nabla^2 y(k) - (a_1 + 2a_2)\nabla y(k) + (a_2 + a_1 + 1) y(k) = b_0 u(k)$$

where $b_1 = b_2 = 0$ for simplicity. It's easy to see there is a certain appealing similarity with continuous time linear differential equations.

**Example**

$$y(k) = y(k-1) + y(k-2)$$

$$y(0) = y(1) = 1$$

*The z-Transform*

- Given a signal with discrete values $y(0),\ y(1),\ y(2), \ldots, y(k), \ldots$ define the z-transform as

$$Y(z) = \mathbb{Z}\{y(\mathrm{k})\} \triangleq \sum_{k=-\infty}^{\infty} y(\mathrm{k}) z^{-k}, \quad r_0 < |z| < R_0$$

- The $z$-transform has the same role in discrete systems that the Laplace transform has in continuous systems.
- In similar fashion to Laplace, we define the ratio of the $z$-transform of the output to the $z$-transform of the input as the transfer function,

$$G(z) = \frac{b(z)}{a(z)} = \frac{b_0 + b_1 z^{-1} + b_2 z^{-2} + \cdots + b_m z^{-m}}{1 + a_1 z^{-1} + a_2 z^{-2} + \cdots + a_n}$$

Comments on discrete-time transfer functions:

- $G(z)$ is a rational function of the complex variable $z$
- The transfer function $G(z) = z^{-1}$ is a delay of one time unit → *unit delay*
- Roots of $b(z) = 0$ are the zeros of $G(z)$
- Roots of $a(z) = 0$ are the poles of $G(z)$

Define the *discrete pulse* as

$$u(k) = \begin{cases} 1, & (k = 0) \\ 0, & (k \neq 0) \end{cases} = \delta(k)$$

Therefore, $U(z) = 1$ and,

$$Y(z) = G(z)$$

- As for continuous-time systems, $G(z)$ is the transform of the response to a unit-pulse input:

$$G(z) = \sum_{-\infty}^{\infty} y(k) z^{-k} \triangleq \sum_{-\infty}^{\infty} g(k) z^{-k}$$

- The values $[g(0),\ g(1),\ g(2),\ \ldots]$ comprise the *unit pulse response* or *weighting sequence* of $G(z)$.
- Negative values in the summation correspond to inputs applied before time equals zero.
  - A system which responds before the input is applied is termed *noncausal*.

## *State-Space Representation*

- The state equations for a discrete time system may be written

$$\begin{aligned} \boldsymbol{x}(k+1) &= A\boldsymbol{x}(k) + B\boldsymbol{u}(k) \\ \boldsymbol{y}(k) &= C\boldsymbol{x}(k) + D\boldsymbol{u}(k) \end{aligned}$$

## Controllability and Observability

- Recall from your study of modern control, the important concepts of *controllability* and *observability* – these will play an important role in model predictive control as well.
- Let's review the main ideas as they apply to discrete-time systems.

### *Controllability*

- A discrete-time system is *controllable* if for any initial state $\boldsymbol{x}_0$ and some final time $k$ there exists a control that transfers the state to any desired value at time $k$.

#### *Tests for Controllability*

TEST 1. The n-state discrete linear time-invariant system

$$\boldsymbol{x}(k+1) = A\boldsymbol{x}(k) + B\boldsymbol{u}(k)$$

has the controllability matrix $\mathcal{C}$ defined by

$$\mathcal{C} = \left[\begin{array}{cccc} B & AB & \cdots & A^{n-1}B \end{array}\right]$$

$\Rightarrow$ The system is controllable if and only if $rank(P) = n$.

TEST 2. The n-state discrete linear time-invariant system

$$\boldsymbol{x}(k+1) = A\boldsymbol{x}(k) + B\boldsymbol{u}(k)$$

is controllable if and only if the *discrete-time controllability Grammian* defined by

$$W_{dc} = \sum_{i=0}^{\infty} A^i BB^T (A^T)^i$$

is positive definite.

NOTE: $W_{dc}$ can be found as the unique positive definite solution to the Lyapunov equation

$$W_{dc} - AW_{dc}A^T = BB^T$$

- The notion of controllability is easy to see using the eigen-decomposition of the state-space representation.
- Given the system $\boldsymbol{x}(k+1) = A\boldsymbol{x}(k) + B\boldsymbol{u}(k)$, perform the eigen-decompostion of $A$ to write:

$$\begin{aligned} \boldsymbol{x}(k+1) &= A\boldsymbol{x}(k) + B\boldsymbol{u}(k) \\ &= W\Lambda W^{-1}\boldsymbol{x}(k) + B\boldsymbol{u}(k) \\ W^{-1}\boldsymbol{x}(k+1) &= \Lambda W^{-1}\boldsymbol{x}(k) + W^{-1}B\boldsymbol{u}(k) \\ \tilde{\boldsymbol{x}}(k+1) &= \Lambda\tilde{\boldsymbol{x}}(k) + \tilde{B}\boldsymbol{u}(k) \end{aligned}$$

- Since $\Lambda$ is a diagonal matrix, it is obvious that if any row $\tilde{B}$ is zero, the corresponding mode, $\lambda_i$ is uncontrollable.

*Observability*

- A discrete-time system is observable if for any initial state $x_0$ and some final time $k$ the initial state $x_0$can be uniquely determined by knowlede of the input $u_i$ and output $y_i$ for all $i \in \left[\ 0,\ \ k\ \right]$.
- In other words, a system is completely observable if every transition of the state $\boldsymbol{x}(k)$ eventually affects every element of the output vector $\boldsymbol{y}(k)$.
- Observability is *dual* to controllability (Kalman).

**Example**

- Given the discrete-time system with

$$A = \begin{bmatrix} 0.5 & 1 \\ 0 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 1 \\ 0.5 \end{bmatrix}$$

- The controllability matrix is

$$\mathcal{C} = \begin{bmatrix} B & AB \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 0.5 & 0.5 \end{bmatrix}$$

which is clearly singular.

- Computing the transfer function from $U(z)$ to $X(z)$, we obtain

$$\begin{aligned} G(z) = C(zI - A)^{-1}B &= \frac{(z-0.5)}{(z-1)}\frac{1}{(z-0.5)} \\ &= \frac{1}{z-1} \end{aligned}$$

- Since the mode at $z = 0.5$ cancelled, it is not connected to the input.
- Alternatively, performing the eigen-decomposition of $A$ yields,

$$\begin{aligned} A &= W\Lambda W^{-1} \\ &= \begin{bmatrix} 1 & 0.8944 \\ 0 & 0.4472 \end{bmatrix} \begin{bmatrix} 0.5 & 0 \\ 0 & 1.0 \end{bmatrix} \begin{bmatrix} 1 & -2.0 \\ 0 & 2.2361 \end{bmatrix} \end{aligned}$$

and

$$\tilde{B} = W^{-1}B = \begin{bmatrix} 1 & -2.0 \\ 0 & 2.2361 \end{bmatrix} \begin{bmatrix} 1 \\ 0.5 \end{bmatrix} = \begin{bmatrix} 0 \\ 1.118 \end{bmatrix}$$

- Again we see that the mode described by $\lambda = 0.5$ is uncontrollable.

*Tests for Observability*

TEST 1. The n-state discrete linear time-invariant system

$$\begin{aligned} \boldsymbol{x}(k+1) &= A\boldsymbol{x}(k) + B\boldsymbol{u}(k) \\ \boldsymbol{y}(k) &= C\boldsymbol{x}(k) \end{aligned}$$

has the observability matrix $\mathcal{O}$ defined by

$$\mathcal{O} = \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix}$$

$\Rightarrow$ The system is observable if and only if $rank(\mathcal{O}) = n$.

TEST 2. The n-state discrete linear time-invariant system

$$\begin{aligned} \boldsymbol{x}(k+1) &= A\boldsymbol{x}(k) + B\mathrm{u}(k) \\ \boldsymbol{y}(k) &= C\boldsymbol{x}(k) \end{aligned}$$

is observable if and only if the *observability Grammian*, defined by

$$W_{do} = \sum_{i=0}^{\infty} (A^T)^i C C^T A^i$$

is positive definite.

*Stabilizability and Detectability*

- The concepts of *stabilizabilty* and *detectability* are closely related to controllability and observability as discussed above:
  - A system is stabilizable if its uncontrollable modes, if any, are stable. Its controllable modes may be stable or unstable.
  - A system is detectable if its unobservable modes, if any, are stable. Its unobservable modes may be stable or unstable.

## *Minimal Realization*

- A realization of the transfer function $G(z) = C(zI - A)^{-1}B$ is a minimal realization if no other smaller dimension realization of the triplet $\begin{pmatrix} A, & B, & C \end{pmatrix}$ exists.
- It is then known that:
  - *A minimal realization is both controllable and observable.* [Kailath, 1980]
- In terms of a SISO transfer function representation, this condition is equivalent to there being no pole-zero cancellations between the numerator and denominator.

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