Limitations on Performance in MIMO Systems

- In this chapter, we generalize the results of Chapter 5 to MIMO systems
- Most of the results on fundamental limitations and controllability analysis for SISO systems also hold for MIMO systems – with the important consideration of *directions*

Introduction

- In a MIMO system, the plant gain, RHP-zeros, delays, RHP-poles and disturbances have directions associated with them due to their matrix structure
  - A multivariable plant may have a RHP-zero and a RHP-pole at the same location, but their effects may not interact!
- We shall quantify directionality by their *output* directions:
  - $\boldsymbol{y}_z$: output direction of a RHP-zero, i.e., $G(z)\boldsymbol{u}_z = 0 \cdot \boldsymbol{y}_z$
  - $\boldsymbol{y}_p$: output direction of a RHP-pole, i.e., $G(p)\boldsymbol{u}_p = \infty \cdot \boldsymbol{y}_p$
  - $\boldsymbol{y}_d$: output direction of a disturbance, i.e., $\boldsymbol{y}_d = \dfrac{1}{\|\boldsymbol{g}_d\|_2}\boldsymbol{g}_d$
  - $\boldsymbol{u}_i$: i^{th} output direction (singular vector) of the plant, i.e., $G\boldsymbol{v}_i = \sigma_i \boldsymbol{u}_i$
- Vectors are usually normalized so that $\|\boldsymbol{x}\|_2 = 1$

- Angles between vector directions may be quantified via the inner product, e.g.,

$$\phi = \cos^{-1}\left|\boldsymbol{y}_z^*\boldsymbol{y}_p\right|$$

Fundamental limitations on sensitivity

- From the identity $S + T + I$ we get

$$|1-\bar{\sigma}(S)| \le \bar{\sigma}(T) \le 1+\bar{\sigma}(S)$$

$$|1-\bar{\sigma}(T)| \le \bar{\sigma}(S) \le 1+\bar{\sigma}(T)$$

- $\Rightarrow$ S and T cannot be small simultaneously; $\bar{\sigma}(S)$ is small if and only if $\bar{\sigma}(T)$ is large

Interpolation constraints

- RPH ZERO. If $G(s)$ has a RHP-zero at z with output direction $\boldsymbol{y}_z$, then for internal stability,

$$\boldsymbol{y}_z^*T(z) = 0; \quad \boldsymbol{y}_z^*S(z) = \boldsymbol{y}_z^*$$

- RHP pole. If $G(s)$ has a RHP-pole at p with output direction $\boldsymbol{y}_p$,then for internal stability,

$$S(p)\boldsymbol{y}_p = 0; \quad T(p)\boldsymbol{y}_p = \boldsymbol{y}_p$$

Sensitivity integrals

- Integral constraints on sensitivity ("waterbed effect") may be generalized to MIMO via determinants,

$$\int_0^\infty \ln|\det S\,(j\omega)|\,d\omega = \sum_j \int_0^\infty \ln\sigma_j\,(S\,(j\omega))\,d\omega = \pi\cdot\sum_{i=1}^{N_p}\mathrm{Re}\,(p_i)$$

Minimum peaks for S and T

- THEOREM 7.1 SENSITIVITY AND COMPLEMENTARY SENSITIVITY PEAKS. Consider a rational plant $G(s)$. Let z_i be the N_z RHP-zeros of $G(s)$ with output zero direction vectors $\boldsymbol{y}_{z,i}$. Let p_i be the N_p RHP-poles of $G(s)$ with output pole direction vectors $\boldsymbol{y}_{p,i}$. Then we have the following tight lower bound on $\|S\|_\infty$ and $\|T\|_\infty$:

$$M_{S,\min} = M_{T,\min} = \sqrt{1 + \sigma^2\left(Q_z^{-1/2} Q_{zp} Q_p^{-1/2}\right)}$$

where

$$[Q_z]_{ij} = \frac{\boldsymbol{y}_{z,i}^* \boldsymbol{y}_{z,j}}{z_i + \bar{z}_j}, \quad [Q_p]_{ij} = \frac{\boldsymbol{y}_{p,i}^* \boldsymbol{y}_{p,j}}{\bar{p}_i + p_j}, \quad [Q_{zp}]_{ij} = \frac{\boldsymbol{y}_{z,i}^* \boldsymbol{y}_{p,j}}{z_i - p_j}$$

One RHP-pole and one RHP-zero

- THEOREM 7.2 WEIGHTED SENSITIVITY. Suppose the plant $G(s)$ has a RHP-zero at $s = z$. Let $w_p(s)$ be any stable scalar weight. Then for closed-loop stability the weighted sensitivity function must satisfy

$$\left\| w_p(s) S(s) \right\|_\infty \geq \left| w_p(z) \right|$$

  - In MIMO systems we generally have the freedom to move the effect of RHP zeros to different outputs by appropriate control

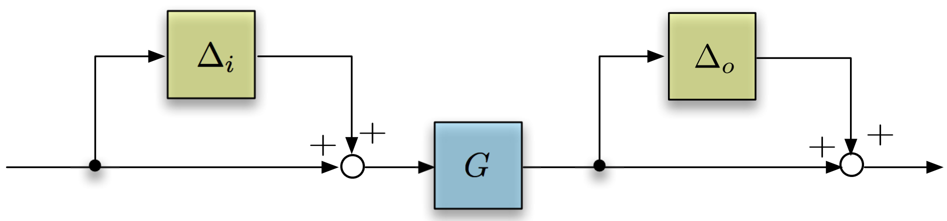
- Theorem 7.3 WEIGHTED COMPLEMENTARY SENSITIVITY. Suppose the plant $G(s)$ has a RHP-pole at $s = p$. Let $w_T(s)$ be any stable scalar weight. Then for closed-loop stability the weighted complementaty sensitivity function must satisfy

$$\|w_T(s) T(s)\|_\infty \geq |w_T(p)|$$

Limitations imposed by uncertainty

Input and output uncertainty

- In a multiplicative (relative) form, the output and input uncertainties are given by:
  - OUTPUT UNCERTAINTY: $G' = (I + \Delta_o)\, G$ or $\Delta_o = (G' - G)\, G^{-1}$
  - INPUT UNCERTAINTY: $G' = G\,(I + \Delta_i)$ or $\Delta_i = G^{-1}\,(G' - G)$



Uncertainty and the benefits of feedback

- We've shown previously that feedback can reduce the effects of uncertainty
  - But, uncertainty can also pose limitations on achievable performance – especially near crossover frequencies
  - We now examine how the magnitude of the sensitivity, $\bar{\sigma}(S')$, is affected by multiplicative uncertainty as given above
- FEEDBACK CONTROL. With one degree-of-freedom feedback control the nominal transfer function is $\boldsymbol{y} = T\boldsymbol{r}$, where $T = L(I + L)^{-1}$ is the complementary sensitivity function.
  - Ideally, we want $T = I$
  - The change in response with model error is $\boldsymbol{y}' - \boldsymbol{y} = (T' - T)\boldsymbol{r}$ where

$$T' - T = S'\Delta_o T$$

- Thus, $\boldsymbol{y}' - \boldsymbol{y} = S'\triangle_o T\boldsymbol{r} = S'\triangle_o\boldsymbol{y}$, and we see that
  - With feedback control the effect of the uncertainty is reduced by a factor of S' relative to that with feedforward control

Effect of uncertainty on feedback sensitivity peak

- We will derive upper bounds on $\bar{\sigma}(S')$ which involve the plant and controller condition numbers:

$$\gamma(G) = \frac{\bar{\sigma}(G)}{\underline{\sigma}(G)}, \quad \gamma(K) = \frac{\bar{\sigma}(K)}{\underline{\sigma}(K)}$$

- Factorization of S' in terms of the nominal sensitivity S gives,
  - OUTPUT UNCERTAINTY:

$$S' = S(I + \triangle_o T)^{-1}$$

  - INPUT UNCERTAINTY:

$$\begin{aligned} S' &= S\left(I + G\triangle_i G^{-1}T\right)^{-1} \\ &= SG(I + \triangle_i T_i)^{-1} G^{-1} \end{aligned}$$

$$\begin{aligned} S' &= \left(I + TK^{-1}\triangle_i K\right)^{-1} S \\ &= K^{-1}(I + T_i\triangle_i)^{-1} KS \end{aligned}$$

- We assume that :
  - G' and G are both stable
  - System is closed-loop stable
- In that case we get that $(I + \triangle_o T)^{-1}$ and $(I + \triangle_i T_i)^{-1}$ are stable
- The magnitude of the multiplicative (relative) uncertainty at each frequency can be bounded in terms of it singular value,

$$\bar{\sigma}\left(\triangle_i\right) \leq |w_i|, \quad \bar{\sigma}\left(\triangle o\right) \leq |w_o|$$

where $w_i(s)$ and $w_o(s)$ are scalar weights.

– Typically, the uncertainty bound $|w_i|$ or $|w_o|$ is 0.2 at low frequencies and exceeds 1 at higher frequencies

Upper bound on $\bar{\sigma}\left(S'\right)$ for output uncertainty

- From the output uncertainty expression above we derive

$$\bar{\sigma}\left(S'\right) \leq \bar{\sigma}\left(S\right)\bar{\sigma}\left(\left(I + \triangle_o T\right)^{-1}\right) \leq \frac{\bar{\sigma}\left(S\right)}{1 - |w_o|\,\bar{\sigma}\left(T\right)}$$

Upper bound on $\bar{\sigma}\left(S'\right)$ for input uncertainty

- The sensitivity function can be much more sensitive to input uncertainty than output uncertainty – from above we derive

$$\begin{aligned}\bar{\sigma}\left(S'\right) &\leq \gamma\left(G\right)\bar{\sigma}\left(S\right)\bar{\sigma}\left(\left(I + \triangle_i T_i\right)^{-1}\right)\\ &\leq \gamma\left(G\right)\frac{\bar{\sigma}\left(S\right)}{1 - |w_i|\,\bar{\sigma}\left(T_i\right)}\end{aligned}$$

$$\begin{aligned}\bar{\sigma}\left(S'\right) &\leq \gamma\left(K\right)\bar{\sigma}\left(S\right)\bar{\sigma}\left(\left(I + T_i\triangle_i\right)^{-1}\right)\\ &\leq \gamma\left(K\right)\frac{\bar{\sigma}\left(S\right)}{1 - |w_i|\,\bar{\sigma}\left(T_i\right)}\end{aligned}$$

- The first of these implies if $\gamma(G) \approx 1$ then the system is insensitive to input uncertainties, irrespective of the controller
- The second implies that if we use a “round” controller, i.e., $\gamma(K) \approx 1$, then the sensitivity function is *not* sensitive to input uncertainty

MIMO Input-Output controllability

- Let's now summarize the main findings in an analysis procedure for input-output controllability for MIMO systems
- As expected, the presence of directions in MIMO plants makes things more difficult than for the SISO case

Controllability analysis procedure

The procedure assumes decisions have already been made on plant inputs and outputs.

1. SCALE ALL VARIABLES. e.g., scale inputs, outputs, disturbances and references to obtain a scaled system model
2. OBTAIN A MINIMAL REALIZATION. i.e., all possible pole-zero cancellations have been made
3. CHECK FUNCTIONAL CONTROLLABILITY.
   (a) To control outputs independently, need as many inputs u_i as outputs y_i
   (b) Need the rank of $G(s)$ to be equal to the number of outputs
4. COMPUTE THE POLES.
5. COMPUTE THE ZEROS.
6. CALCULATE BOUNDS ON CLOSED-LOOP TRANSFER FUNCTIONS.
   (a) Large peaks ($\gg 1$) for any of S, T, KS, SG_d, KSG_d, S_i, and T_i indicates poor closed-loop performance or poor robustness against uncertainty.

7. OBTAIN FREQUENCY RESPONSE OF $G(j\omega)$.
   (a) Compute RGA matrix $\Lambda = G \times (G^{\dagger})^T$
   (b) Plants with large RGA elements at crossover frequencies are difficult to control.

8. COMPUTE SINGULAR VALUES (PRINCIPAL GAINS) OF $G(j\omega)$
   (a) Pay attention to signal scaling
   (b) Plot as functions of frequency (Bode plots)

9. EXAMINE $\underline{\sigma}(G(j\omega))$.
   (a) $\underline{\sigma}(G(j\omega))$ should be as large as possible at frequencies where control is needed.
   (b) If $\underline{\sigma}(G(j\omega)) < 1$ then we cannot (at frequency ω) make independent output changes of unit magnitude by using inputs of unit magnitude.

10. EXAMINE ELEMENTS OF G_d WITH RESPECT TO DISTURBANCES.
    (a) At frequencies where elements are larger than 1, we need control.
    (b) Require for each disturbance that S is less than $1/\|\boldsymbol{g}_d\|_2$ in the disturbance direction $\boldsymbol{y}_d$. The $\boldsymbol{g}_d$ are the columns of G_d.
    (c) Must at least require $\underline{\sigma}(S) \le 1/\|\boldsymbol{g}_d\|_2$ and may have to require $\bar{\sigma}(S) \le 1/\|\boldsymbol{g}_d\|_2$.

11. INVESTIGATE DISTURBANCES AND INPUT SATURATION.
    (a) Consider the input magnitudes needed for perfect control by computing $G^{\dagger}G_d$

(b) If all elements less than one at all frequencies, input saturation should not be a problem.

(c) If some elements of $G^{\dagger}G_d$ are larger than one, then perfect control cannot be achieved at this frequency.

12. CHECK REQUIREMENT COMPATIBILITY.

(a) Look at disturbances, RHP-poles, RHP-zeros, and their associated directions.

(b) For example, must require for each disturbance and RHP-zero that $|\boldsymbol{y}_z^* \boldsymbol{g}_d(z)| \le 1$

13. CONSIDER UNCERTAINTY.

(a) If the condition number $\gamma(G)$ is small, then should expect no problems with uncertainty.

(b) If RGA elements are large, expect strong sensitivity to uncertainty.

14. CONSIDER PLANT DIAGONALIZATION.

(a) If plant is unstable, then require a lower-level stabilizing controller first.

15. EXAMINE PLANT CONDITION NUMBER.

(a) A system's gain may vary widely with input direction – termed *strong directionality*.

(b) Can check this via: i) condition number, and ii) RGA

(c) Large condition number implies plant is ill-conditioned – sensitive to unstructured input uncertainty.

(d) Condition number may be reduced by scaling the plant.

Example 7.1

- Perform a controllability analysis for the system whose transfer function matrix is given by

$$G(s) = \begin{bmatrix} \dfrac{10}{s+10} & \dfrac{1.1}{s+1} \\ \\ \dfrac{10}{s+10} & 1 \end{bmatrix}$$

- Computing the determinant,

$$\begin{aligned} \det\{G(s)\} &= \left(\frac{10}{s+10}\right)\cdot 1 - \left(\frac{10}{s+10}\right)\cdot\left(\frac{1.1}{s+1}\right) \\ &= \left(\frac{10}{s+10}\right)\cdot\left(1-\frac{1.1}{s+1}\right) \\ &= \frac{10\cdot(s-.1)}{(s+10)(s+1)} \end{aligned}$$

  - Poles: $p_1 = -10$, $p_2 = -1$
  - Zeros: $z_1 = +0.1$ (RHP)

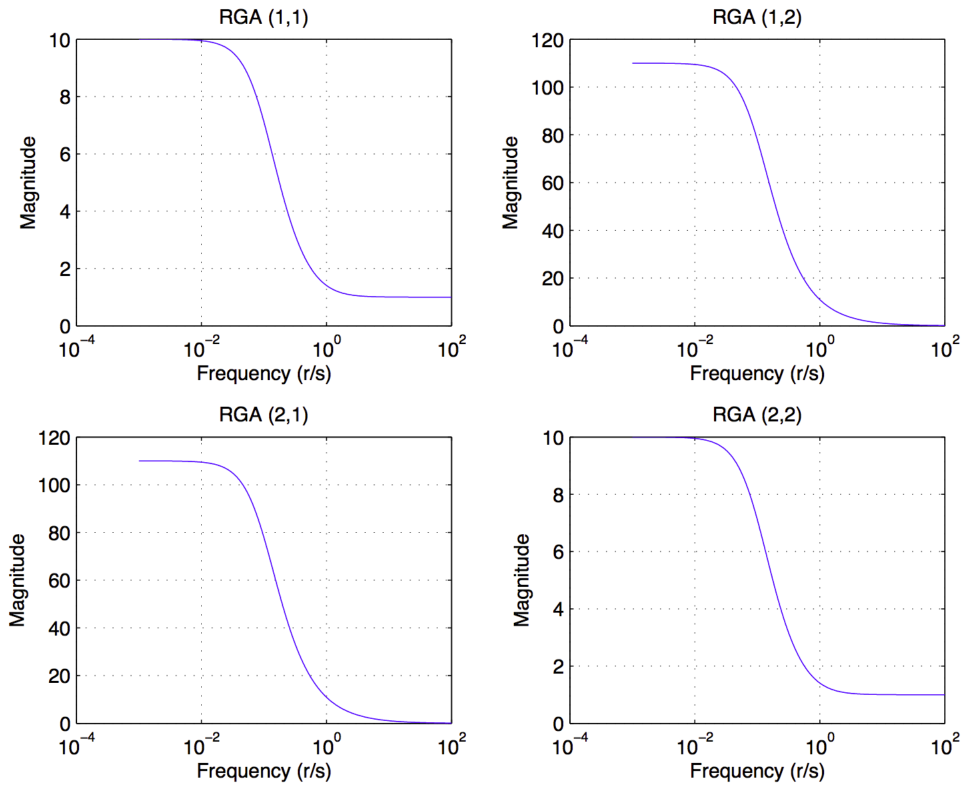
- Computing the RGA,

$$\Lambda(G) = G \otimes \left(G^{-1}\right)^T$$

  - After cancelling common poles-zero pairs,

$$\Lambda(G) = \begin{bmatrix} \dfrac{s+1}{s-0.1} & \dfrac{-1.1}{s-0.1} \\ \\ \dfrac{-1.1}{s-0.1} & \dfrac{s+1}{s-0.1} \end{bmatrix}$$

– Plotting the frequency-dependent RGA magnitudes,



– Here we see that the RGA values are all large for the low frequency range, but off-diagonals drop to small levels for $\omega > 1 \, {}^{\text{rad}}/_{\text{sec}}$ indicating high interaction at low frequencies and $\Lambda(G) \to I$ at high frequencies

- However, the non-interacting "high-frequency" range is beyond the RHP zero value where bandwidth is limited

– Computing the steady-state value of the RGA,

$$\Lambda(G(0)) = \begin{bmatrix} -10 & 11 \\ 11 & -10 \end{bmatrix}$$

- Thus the best pairing for decoupled control is: $\begin{pmatrix} y_1 & u_2 \end{pmatrix}$ and $\begin{pmatrix} y_2 & u_1 \end{pmatrix}$

• Computing the steady-state value of $G(s)$,

$$G(0) = \begin{bmatrix} 1 & 1.1 \\ 1 & 1 \end{bmatrix}$$

– Evaluating its SVD,

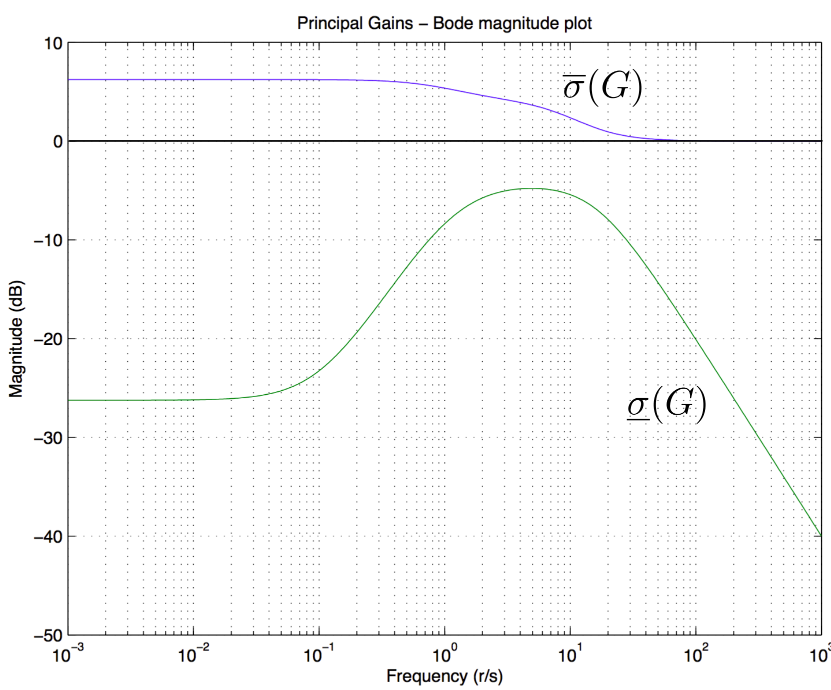
$$G(0) = \begin{bmatrix} -.7245 & -.6892 \\ -.6892 & .7245 \end{bmatrix} \begin{bmatrix} 2.0512 & 0 \\ 0 & 0.0488 \end{bmatrix} \begin{bmatrix} -.6892 & .7245 \\ -.7245 & -.6892 \end{bmatrix}^T$$

– Hence we can see that the most difficult output direction at steady-state is

$$\underline{\boldsymbol{u}}(0) = \begin{bmatrix} -.6892 \\ .7245 \end{bmatrix}$$

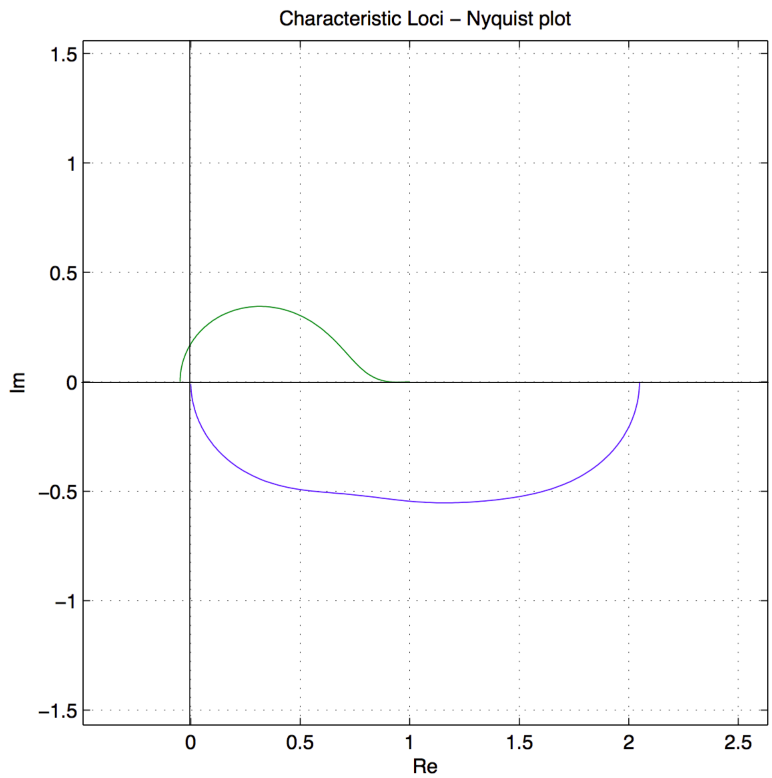
as it is associated with the very small gain $\underline{\sigma}(G) = 0.0488$.

- Computing and plotting the principal gains,



– Since the maximum singular value is less than about 2 at all frequencies, there may be some difficulties with control

- Computing and plotting the characteristic loci,



  – CL show no net encirclements of the critical point – since no open-loop RHP-poles, indicates stability under unity feedback

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