Limitations on Performance in SISO Systems

- In this chapter, we will discuss the fundamental limitations on the ability to achieve acceptable control performance in SISO systems
- Central to this treatment is the notion of *input-output controllability*

Input-output controllability

- A comprehensive view of control design should address the following three questions:
  - How well can the plant be controlled?
  - What control structure should be used?
  - How might the process be changed to improve control?

Definition. *Input-output controllability* is the ability to achieve acceptable control performance; that is, to keep the outputs within specified bounds from their references, in spite of unknown but bounded disturbances and plant uncertainty, using available inputs and measurements.

- A plant is controllable if there exists a controller that gives acceptable performance for all expected plant variations
  - NOTE: controllability is a property of the *plant* (not the controller)

Perfect control and plant inversion

- Recall the input-output relationship
$$\boldsymbol{y} = G\boldsymbol{u} + G_d\boldsymbol{d}$$
- For "perfect control" ($\boldsymbol{y} = \boldsymbol{r}$) we have the feedforward controller
$$\boldsymbol{u} = G^{-1}\boldsymbol{r} - G^{-1}G_d\boldsymbol{d}$$
which is not realizable for strictly proper plants.
- With feedback control $\boldsymbol{u} = K(\boldsymbol{r} - \boldsymbol{y})$ we have
$$\boldsymbol{u} = KS\boldsymbol{r} - KSG_d\boldsymbol{d}$$
or since $T = GKS$,
$$\boldsymbol{u} = G^{-1}T\boldsymbol{r} - G^{-1}TG_d\boldsymbol{d}$$
- Where $T \approx I$, feedback input is the same as perfect control feedforward input $\Rightarrow$ high gain feedback generates an inverse of G !
- Consequently, perfect control *cannot* be achieved if:
  - G contains RHP zeros (since G^{-1} is unstable)
  - G contains time delays (since G^{-1} is anti-causal)
  - G has more poles than zeros (since G^{-1} is unrealizable)
- For feedforward control, perfect control *cannot* be achieved if:
  - G is uncertain (since G^{-1} cannot be obtained exactly)
- Practically speaking, because of input constraints, perfect control *cannot* be achieved if:
  - $\left|G^{-1}G_d\right|$ is large
  - $\left|G^{-1}R\right|$ is large

Fundamental limitations on sensitivity

$S + T = 1$

- From the basic definitions $S = (I + L)^{-1}$ and $T = L\,(I + L)^{-1}$ we found

$$S + T = I$$

  or $S + T = 1$ for SISO systems.

- Ideally, we want S small to achieve small control errors, and T small to avoid measurement noise
- However, these requirements are not simultaneously possible at any frequency
  - Either $|S(j\omega)| \geq 0.5$ or $|T(j\omega)| \geq 0.5$

Interpolation constraints

- If p is a RHP pole of $G(s)$, then

$$T\,(p) = 1, \quad S\,(p) = 0$$

- Similarly, if z is a RHP zero of $G(s)$, then

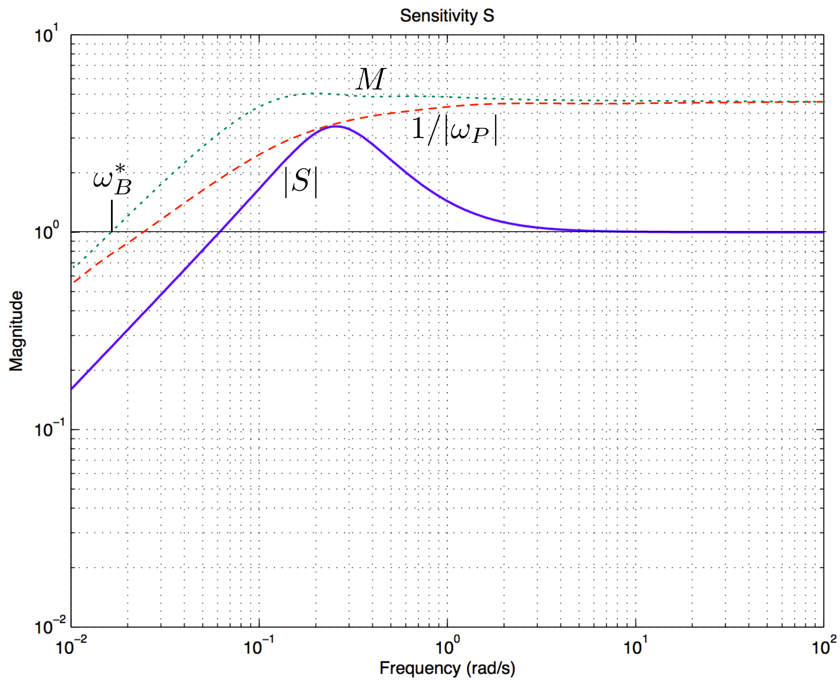
$$T\,(z) = 0, \quad S\,(z) = 1$$

- These interpolation constraints are necessary for internal stability

Sensitivity integrals

- Often termed the "waterbed effect", this condition essentially says that if we push the sensitivity down at some frequencies then it will have to increase at others.

- In general, a trade-off between sensitivity reduction and sensitivity increase must be performed whenever:
  - $L(s)$ has at least two more poles than zeros
  - $L(s)$ has a RHP-zero



  - The figure above shows a typical sensitivity function depicted by the solid blue line
  - Note that $|S|$ has a peak value greater than 1 ; it turns out this is unavoidable in practice

Two more poles than zeros

Theorem 5.1 BODE SENSITIVITY INTEGRAL

Suppose that the open-loop transfer function $L(s)$ is rational and has at least two more poles than zeros. Suppose also that $L(s)$ has N_p RHP poles at locations p_i. Then for closed-loop stability the sensitivity function must satisfy

$$\int_0^\infty \ln|S(j\omega)|\,d\omega = \pi \cdot \sum_{i=1}^{N_p} \mathrm{Re}\,(p_i)$$

- Stable Plant.

$$\int_0^\infty \ln|S(j\omega)|\,d\omega = 0$$

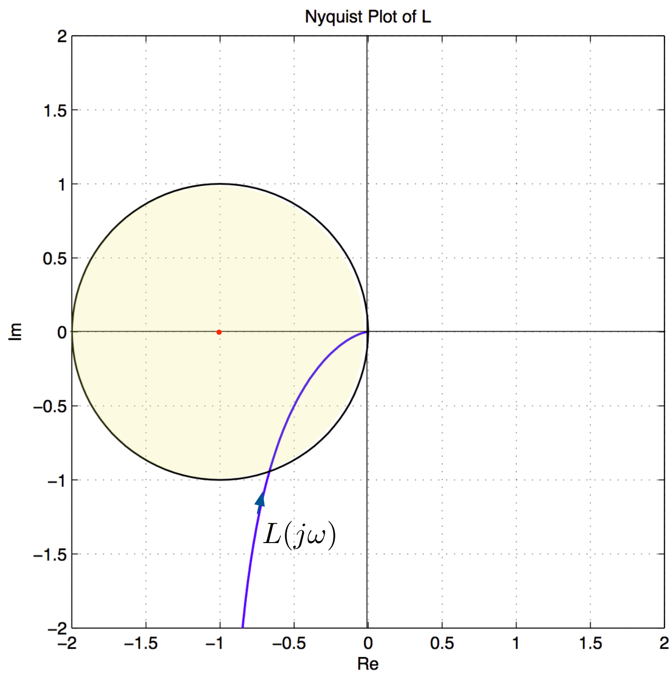
  - Thus the area of sensitivity reduction must equal the area of sensitivity increase – meaning an increase in bandwidth must come at the expense of a larger peak in $|S|$
- Unstable Plant.
  - Unstable poles usually increase the peak of $|S|$ as seen from the positive contribution $\pi \cdot \sum_{i=1}^{N_p} \mathrm{Re}(p_i)$
  - The area of sensitivity increase ($|S| > 1$) exceeds that of sensitivey reduction by an amount proportional to the sum of the distance from the unstable poles to the LHP

Example 5.1

- Consider the system with loop transfer function

$$L(s) = \frac{1}{s(s+1)}$$

- Examining the Nyquist plot, we see there exists a frequency range over which $L(j\omega)$ is inside the unit circle centered at -1
  - This is due to the phase contributions of the two poles



– As a result, $|S| = |1 + L|^{-1}$ is greater than one in this region; indeed $|S| > 1$ whenever L is inside the unit circle

RHP zeros

Theorem 5.2 WEIGHTED SENSITIVITY INTEGRAL

Suppose that $L(s)$ has a single real RHP zero z or a complex conjugate pair of zeros $z = x \pm jy$, and has N_p RHP poles, p_i . Let $\bar{p}_i$ denote the complex conjugate of p_i . Then for closed-loop stability the sensitivity function must satisfy

$$\int_0^\infty \ln |S\,(j\omega)| \cdot w\,(z,\,\omega)\,d\omega = \pi \cdot \ln \prod_{i=1}^{N_p} \left| \frac{p_i + z}{\bar{p}_i - z} \right|$$

where if the zero is real,

$$w\,(z,\,\omega) = \frac{2z}{z^2 + \omega^2} = \frac{2}{z} \cdot \frac{1}{1 + \left(\frac{\omega}{z}\right)^2}$$

and if the zero is complex,

$$w(z, \omega) = \frac{x}{x^2 + (y-\omega)^2} + \frac{x}{x^2 + (y+\omega)^2}$$

- Note that the weighting $w(z, \omega)$ effectively attenuates the contribution from $\ln |S|$ to the sensitivity integral for $\omega > z$
- Thus, for a stable plant (with $|S| \approx 1$ for ω large) we have approximately

$$\int_0^z \ln |S(j\omega)| \, d\omega \approx 0$$

Example 5.2

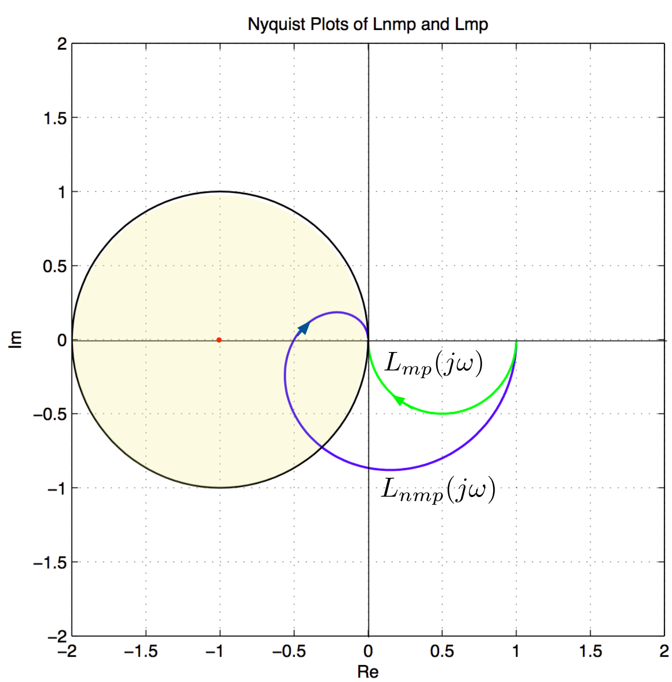
- Consider a non-minimum phase loop transfer function

$$L_{nmp}(s) = \frac{1}{s+1} \cdot \frac{-s+1}{s+1}$$

and its minimum-phase version

$$L_{mp}(s) = \frac{1}{s+1}$$

- Generating Nyquist plots of these two transfer functions gives



– The additional phase lag contributed by the RHP zero and extra pole of $L_{nmp}(s)$ causes the Nyquist plot to penetrate the unit circle and $|S| > 1$ in this region

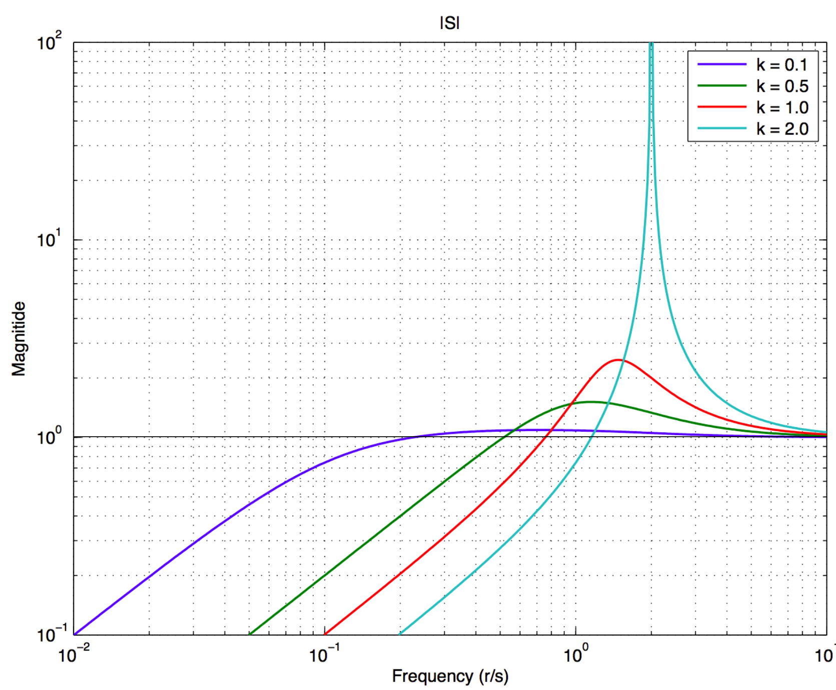
Example 5.3

- Consider the non-minimum phase loop transfer function given by

$$\frac{k}{s} \cdot \frac{-s+2}{s+2}$$

where $k = \ 0.1, \ 0.5, \ 1.0, \ 2.0$ and we note the plant has a RHP zero at $z = 2$

- Examining the magnitude plots of the sensitivity function, we see that an increase in the controller gain k produces a higher bandwidth and a corresponding larger peak for S



– For the case $k = 2$, the closed-loop system becomes unstable with a pair of complex conjugate poles on the imaginary axis and the peak of $|S|$ is infinite

Fundamental limitations: bounds on peaks

- So far, we've seen that a RHP zero implies that a peak in $|S|$ is unavoidable, and that the peak will increase as we decrease $|S|$ at other frequencies
- This section explores computing bounds on these peaks – which will be ultimately more important for control design

Minimum peaks for S and T

- We first state an important result from complex analysis known as the *maximum modulus principle*
- MAXIMUM MODULUS PRINCIPLE. Suppose $f(s)$ is stable (i.e., analytic in the RHP). Then the maximum value of $|f(s)|$ over all s in the RHP is attained on the region's boundary, i.e., somewhere along the imaginary axis. Hence we may state for stable $f(s)$,

$$\|f\ (j\omega)\|_{\infty} = \max_{\omega} |f\ (j\omega)| \geq |f\ (s_0)| \quad \forall\, s_0 \in \mathrm{RHP}$$

- One can think of $|f(s)|$ as a 3D plot over the complex plane where $|f(s)|$ has high peaks at its poles and wells at its zeros
  – So if $f(s)$ has LHP poles only $\Rightarrow |f(s)|$ slopes downward from the LHP to the RHP
- The following results follow from the maximum modulus principle

Theorem 5.3 WEIGHTED SENSITIVITY PEAK

Suppose that $G(s)$ has a RHP zero z and let $w_P(s)$ be any stable weighting function. Then for closed-loop stability the weighted sensitivity function must satisfy

$$\|w_P S\|_\infty \geq |w_P(z)S(z)| = |w_P(z)|$$

Theorem 5.4 WEIGHTED COMPLIMENTARY SENSITIVITY PEAK

Suppose that $G(s)$ has a RHP pole p and let $w_T(s)$ be any stable weighting function. Then for closed-loop stability the weighted complimentary sensitivity function must satisfy

$$\|w_T T\|_\infty \geq |w_T(p)T(p)| = |w_T(p)|$$

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