Elements of Linear Systems Theory

- In this chapter, we will summarize important results from linear system theory
- For a thorough treatment of this topic, see for example Kailath (1980)

System descriptions

- Various equivalent representations are used for the mathematical modeling of linear time-invariant (LTI) systems
- Common to all is that they satisfy the property of superposition, i.e.

$$l\left(\alpha_1 u_1 + \alpha_2 u_2\right) = \alpha_1 l(u_1) + \alpha_2 l(u_2)$$

State-space representation

- Consider a system with m inputs and l outputs that has an internal description of n states:

$$\begin{aligned}\dot{\boldsymbol{x}}(t) &= A\boldsymbol{x}(t) + B\boldsymbol{u}(t)\\ \boldsymbol{y}(t) &= C\boldsymbol{x}(t) + D\boldsymbol{u}(t)\end{aligned}$$

  – Here, A, B, C, and D are constant real-valued matrices

- It is usually convenient to rewrite these equations in matrix-vector form as

$$\begin{bmatrix}\dot{\boldsymbol{x}}\\ \boldsymbol{y}\end{bmatrix} = \begin{bmatrix}A & B\\ C & D\end{bmatrix}\begin{bmatrix}\boldsymbol{x}\\ \boldsymbol{y}\end{bmatrix}$$

- Note that by taking Laplace transforms, we obtain an input-output description as:

$$\begin{aligned} sX(s) &= AX(s) + BU(s) \\ (sI - A)\,X(s) &= BU(s) \\ X(s) &= (sI - A)^{-1}\,BU(s) \\ Y(s) &= \left[C\,(sI - A)^{-1}\,B + D\right]U(s) \end{aligned}$$

  - This gives rise to the transfer function matrix,

$$G(s) = C\,(sI - A)^{-1}\,B + D$$

  - We can thus refer to the underlying state-space model by the following notation

$$G \triangleq \left[\begin{array}{cc} A & B \\ C & D \end{array}\right]$$

- It turns out that this representation is *not* unique
  - Define states in a new coordinate frame, $\tilde{\boldsymbol{x}} = S\boldsymbol{x}$, such that $\boldsymbol{x} = S^{-1}\tilde{\boldsymbol{x}}$
  - Then equivalent state-space realizations may be described in terms of the new states,

$$\tilde{A} = SAS^{-1}, \quad \tilde{B} = SB, \quad \tilde{C} = CS^{-1}, \quad \tilde{D} = D$$

- Dynamic response $\boldsymbol{x}(t)$ for the linear system above with initial condition $\boldsymbol{x}(t_0)$ is given by

$$\boldsymbol{x}(t) = e^{A(t-t_0)}\boldsymbol{x}(t_0) + \int_{t_0}^{t} e^{A(t-\tau)}B\boldsymbol{u}(\tau)d\tau$$

– The matrix exponential can be found from

$$\begin{aligned} e^{At} &= I + At + \frac{At^2}{2!} + \frac{At^3}{3!} + \cdots \\ &= \boldsymbol{w}_1 e^{\lambda_1} \boldsymbol{v}_1^T + \boldsymbol{w}_2 e^{\lambda_2} \boldsymbol{v}_2^T + \cdots \boldsymbol{w}_n e^{\lambda_n} \boldsymbol{v}_n^T \end{aligned}$$

– The dyadic expansion in the second form derives from the eigenvalue-eigenvector decomposition of A

Impulse response representation

- The impulse response matrix is

$$g(t) = \begin{cases} 0 & t < 0 \\ Ce^{At}B + D\delta(t) & t \geq 0 \end{cases}$$

- With initial state $\boldsymbol{x}(0) = 0$, the dynamic response to an arbitrary input $\boldsymbol{u}(t)$ is given by the convolution,

$$\boldsymbol{y}(t) = g(t) * \boldsymbol{u}(t) = \int_0^t g(t-\tau)\boldsymbol{u}(\tau)d\tau$$

Transfer function representation – Laplace transforms

- As shown earlier, the Laplace transform of the state-space description (with $\boldsymbol{x}(0) = 0$) gave us the input-output relationship

$$Y(s) = \left[C\,(sI - A)^{-1}\,B + D\right]U(s)$$

and the corresponding transfer function matrix,

$$G(s) = C\,(sI - A)^{-1}\,B + D$$

- Equivalently,

$$G(s) = \frac{1}{\det(SI - A)}\left[C \operatorname{adj}(sI - A)\,B + D\,\det(sI - A)\right]$$

- Using the rule of determinants, we may express the characteristic polynomial as

$$\det(sI - A) = \prod_{i=1}^{n} \lambda_i (sI - A) = \prod_{i=1}^{n} (s - \lambda_i(A))$$

State controllability and state observability

State controllability. The dynamical system $\dot{\boldsymbol{x}} = A\boldsymbol{x} + B\boldsymbol{u}$, or equivalently the pair (A, B), is said to be *state controllable* if, for any initial state $\boldsymbol{x}(0) = \boldsymbol{x}_0$, any time $t_1 > 0$ and any final state $\boldsymbol{x}_1$, there exists an input $\boldsymbol{u}(t)$ such that $\boldsymbol{x}(t_1) = \boldsymbol{x}_1$. Otherwise the system is said to be *state uncontrollable*.

Tests for state controllability

- The pair (A, B) is state controllable if and only if the controllability matrix

$$\mathcal{C} \triangleq \begin{bmatrix} B & AB & A^2B & \cdots & A^{n-1}B \end{bmatrix}$$

has full row rank n .

- Using the dynamical system response $\boldsymbol{x}(t)$ one can verify that for $\boldsymbol{x}(t_1) = \boldsymbol{x}_1$,

$$\boldsymbol{u}(t) = -B^T e^{A^T(t_1 - t)} W_c(t_1)^{-1} \left(e^{At_1}\boldsymbol{x}_0 - \boldsymbol{x}_1 \right)$$

where $W_c(t)$ is the Gramian matrix at time t,

$$W_c(t) \triangleq \int_0^t e^{A\tau} B B^T e^{A^T\tau} d\tau$$

- Thus (A, B) is state controllable if and only if $W_c(t)$ has full rank (and thus is positive definite) for any $t > 0$.
  - For a stable system, we apply the test to the infinite-time Gramian,
$$P \triangleq \int_0^\infty e^{A\tau} B B^T e^{A^T \tau} d\tau$$
  - P may also be obtained as the solution of the Lyapunov equation
$$AP + PA^T = -BB^T$$
- The pair (A, B) is state controllable if for each eigenvalue λ_i of A,
$$\boldsymbol{v_i}^T B \neq 0, \ \forall i$$
where $\boldsymbol{v}_i^T$ is the left eigenvector corresponding to eigenvalue λ_i .

Example 4.1

- Consider the two-state system,
$$A = \begin{bmatrix} -2 & -2 \\ 0 & -4 \end{bmatrix} \; B = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$
$$C = \begin{bmatrix} 1 & 0 \end{bmatrix} \; D = 0$$
- The corresponding transfer function has just *one* state,
$$G(s) = C\,(sI - A)^{-1}\,B = \frac{1}{s+4}$$
- Examining the controllability matrix,
$$\mathcal{C} = \begin{bmatrix} B & AB \end{bmatrix} = \begin{bmatrix} 1 & -4 \\ 1 & -4 \end{bmatrix}$$
we see it has two linearly dependent rows.
- And the controllability Gramian,

$$P = \begin{bmatrix} 0.125 & 0.125 \\ 0.125 & 0.125 \end{bmatrix}$$

is singular.

- Performing the eigen-decomposition,

$$A = \begin{bmatrix} -2 & -2 \\ 0 & -4 \end{bmatrix} = W\Lambda V^T = \begin{bmatrix} 1 & .7071 \\ 0 & .7071 \end{bmatrix} \begin{bmatrix} -2 & 0 \\ 0 & -4 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 0 & 1.4142 \end{bmatrix}$$

  - and examining the product $V^T B$,

$$V^T B = \begin{bmatrix} 1 & -1 \\ 0 & 1.4142 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1.4142 \end{bmatrix}$$

  - we see that the first element is zero, thus the first eigenvalue, $\lambda_1 = -2$, is uncontrollable.

- Note that controllability is a system-theoretic concept – important for computation and realizations – but not always for practical control:
  - It says nothing about how states behave in time
  - Required inputs may be very large with sudden changes
  - Some states may be of no practical importance
  - Existence result provides no “degree” of controllability

State observability. The dynamical system $\dot{\boldsymbol{x}} = A\boldsymbol{x} + B\boldsymbol{u}$, $\boldsymbol{y} = C\boldsymbol{x} + D\boldsymbol{u}$, or the pair (A, C), is said to be *state observable* if, for any time $t_1 > 0$, the initial state $\boldsymbol{x}(0) = \boldsymbol{x}_0$ can be determined from the time history of the input $\boldsymbol{u}(t)$ and the output $\boldsymbol{y}(t)$ in the interval $[0, t_1]$. Otherwise the system is said to be *state unobservable*.

Tests for state observability

- The pair (A, C) is state observable if and only if the observability matrix

$$\mathcal{O} = \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix}$$

has full column rank.

- For a stable system, the infinite-time observability Gramian

$$Q \triangleq \int_0^\infty e^{A^T\tau} C^T C e^{A\tau} d\tau$$

must have full rank n and thus be positive definite.

  – Q may also be found as the solution to the Lyapunov equation,

$$A^T Q + QA = -C^T C$$

- The pair (A, C) is state observable if for each eigenvalue λ_i of A,

$$C\boldsymbol{w}_i \neq 0, \ \forall i$$

where $\boldsymbol{w}_i$ is the right eigenvector corresponding to eigenvalue λ_i.

Stability

Definition

A system is (*internally*) *stable* if none of its components contains hidden unstable modes and the injection of bounded external signals at any place in the system results in bounded output signals measured anywhere in the system.

- A system is termed *bounded* if there exists a constant c such that $|u(t)| < c$ for all t
- The term *internally* emphasizes the requirement for stability from all inputs to all outputs

Definition

A system is *stabilizable* if all unstable modes are state controllable. A system is detectable if all unstable modes are observable. A system with unstabilizable or undetectable modes is said to contain *hidden unstable modes*.

- Any unstable linear system can be stabilized by feedback provided it contains no hidden unstable modes
- Systems with hidden unstable modes must be avoided for practical reasons

Poles

Definition

The poles p_i of a system with state-space description $(A,\ B,\ C,\ D)$ are the eigenvalues $\lambda_i(A),\ i = 1, \ldots, n$ of the matrix A. The characteristic polynomial $\phi(s)$ is defined as

$$\phi(s) \triangleq \det(sI - A) = \prod_{i=1}^{n}(s - p_i)$$

Thus the poles are the roots of the characteristic equation

$$\phi(s) = \det(sI - A) = 0$$

Poles and stability

Theorem 4.1 A linear dyamic system $\dot{x} = Ax + Bu$ is stable if and only if all the poles are in the open left-half plane (LHP); that is,

$$\mathrm{Re}(p_i) = \mathrm{Re}(\lambda_i(A)) < 0,\ \forall i$$

Poles from transfer functions

Theorem 4.2 The pole polynomial $\phi(s)$ corresponding to a minimal realization of a system with transfer function $G(s)$ is the least common denominator of all non-identically zero minors of all orders of $G(s)$

- Recall, a *minor* of a matrix is the determinant of the matrix obtained by deleting certain rows/columns of the matrix

Example 4.2

- Consider the square transfer function matrix,

$$G(s) = \frac{1}{1.25\,(s+1)\,(s+2)} \begin{bmatrix} s-1 & s \\ -6 & s-2 \end{bmatrix}$$

- MINORS OF ORDER 1. These are the four elements of the matrix, which each have $(s+1)\,(s+2)$ in the denominator.
- MINOR OF ORDER 2. This is the determinant,

$$\det\, G(s) = \frac{(s-1)\,(s-2) + 6s}{1.25^2\,(s+1)^2\,(s+2)^2} = \frac{1}{1.25^2\,(s+1)\,(s+2)}$$

- Least common denominator of all the minors is

$$\phi(s) = (s+1)\,(s+2)$$

- Hence the minimal realization has two poles, $p_1 = -1$ and $p_2 = -2$.

Example 4.3

- Consider the two-input, three-output system given by

$$\frac{1}{(s+1)(s+2)(s-1)}\begin{bmatrix} (s-1)(s+2) & 0 & (s-1)^2 \\ -(s-1)(s+2) & (s-1)(s+1) & (s-1)(s+1) \end{bmatrix}$$

- MINORS OF ORDER 1. These are the five non-zero elements of the matrix:

$$\frac{1}{s+1},\ \frac{s-1}{(s+1)(s+2)},\ \frac{-1}{s-1},\ \frac{1}{s+2},\ \frac{1}{s+2}$$

- MINORS OF ORDER 2.
  - Deletion of column 2 gives,

$$\begin{aligned} M_2 &= \frac{(s-1)(s+2)(s-1)(s+1)+(s+1)(s+2)(s-1)^2}{\left((s+1)(s+2)(s-1)^2\right)} \\ &= \frac{2}{(s+1)(s+2)} \end{aligned}$$

  - The other two minors are,

$$M_1 = \frac{-(s-1)}{(s+1)(s+2)^2},\quad M_3 = \frac{1}{(s+1)(s+2)}$$

- Considering all minors, the least common denominator is

$$\phi(s) = (s+1)(s+2)^2(s-1)$$

- The system thus has four poles: $s=-1$, $s=1$, and two at $s=-2$
- NOTE: MIMO poles are essentially the poles of the transfer function matrix elements – but we must further determine their *multiplicity*

Zeros

- In general, zeros z_i are the values of s at which the transfer function matrix $G(s)$ loses rank
  - For SISO systems, zeros are the solutions to $G(s) = 0$

Example 4.4

- Consider the transfer function,

$$G(s) = \frac{Y(s)}{U(s)} = \frac{s+2}{s^2+7s+12}$$

  - Compute the system response to the input

$$u(t) = e^{-2t},\ y(0) = 0,\ \dot{y}(0) = -1$$

- Taking the Laplace transform of the input,

$$\mathcal{L}\{u(t)\} = \frac{1}{s+2}$$

- And solving, we get

$$\begin{aligned} Y(s) &= \frac{s+2}{s^2+7s+12} \cdot \frac{1}{s+2} \\ &= \frac{1}{s^2+7s+12} \end{aligned}$$

$$\begin{aligned} s^2Y(s) - sy(0) - \dot{y}(0) + 7sY(s) - 7y(0) + 12Y(s) &= 1 \\ s^2Y(s) + 7sY(s) + 12Y(s) + 1 &= 1 \\ \Rightarrow\ Y(s) = 0 \end{aligned}$$

- Summarizing...
  - $G(s)$ has a zero at z
  - Introduce input of the form $u(t) = u_0 e^{zt}$
  - Output is $y(t) = 0,\ t > 0$

Zeros from transfer functions

- Define the following terms:

$$\begin{aligned} \boldsymbol{u} &= \boldsymbol{u}_z e^{zt} \\ \boldsymbol{x} &= \boldsymbol{x}_z e^{zt} \\ \boldsymbol{y} &= \boldsymbol{0} \end{aligned}$$

- Then write the state-space equation,

$$\begin{aligned} \dot{\boldsymbol{x}} &= z e^{zt} \boldsymbol{x}_z \\ &= A e^{zt} \boldsymbol{x}_z + B \boldsymbol{u}_z e^{zt} \end{aligned}$$

- Assemble these into a matrix-vector form,

$$\begin{bmatrix} zI - A & -B \end{bmatrix} \begin{bmatrix} \boldsymbol{x}_z \\ \boldsymbol{u}_z \end{bmatrix} = 0$$

- Using the output equation, we have

$$\begin{aligned} \boldsymbol{y} &= C\boldsymbol{x} + D\boldsymbol{u} \\ &= C e^{zt} \boldsymbol{x}_z + D \boldsymbol{u}_z e^{zt} \equiv 0 \end{aligned}$$

- Combining,

$$\begin{bmatrix} zI - A & -B \\ C & D \end{bmatrix} \begin{bmatrix} \boldsymbol{x}_z \\ \boldsymbol{u}_z \end{bmatrix} = 0$$

- Therefore, the zeros are the values of z that satisfy

$$\det \begin{bmatrix} zI - A & -B \\ C & D \end{bmatrix} = 0$$

  – These are known as transmission zeros and can be calculated in Matlab as:

```
zero = tzero(A,B,C,D)
```

Zeros from transfer functions

Definition

A number z_i is a *zero* of $G(s)$ if the rank of $G(z_i)$ is less than the normal rank of $G(s)$. The zero polynomial is defined as

$$z(s) = \prod_{i=1}^{n_z} (s - z_i)$$

where n_z is the number of finite zeros of $G(s)$.

Theorem 4.3 The zero polynomial $z(s)$, corresponding to a minimal realization of the system, is the greatest common divisor of all the numerators of all order-r minors of $G(s)$, where r is the normal rank of $G(s)$, provided that these minors have been adjusted in such a way as to have the pole polynomial $\phi(s)$ as their denominators.

Example 4.5

- Consider the 2×2 transfer function matrix

$$G(s) = \frac{1}{s+2} \begin{bmatrix} s-1 & 4 \\ 4.5 & 2(s-1) \end{bmatrix}$$

- The normal rank of $G(s)$ is 2
- The minor of order 2 is

$$\det\, G(s) = \frac{2\,(s-1)^2 - 18}{(s+2)^2} = 2 \cdot \frac{s-4}{s+2}$$

- Pole polynomial: $\phi(s) = s + 2$
- Zero polynomial: $z(s) = s - 4$
- NOTE: In general, multivariable zeros have no relationship with the zeros of the transfer function elements!

Example 4.6

- Let

$$G(s) = \frac{1}{1.25\,(s+1)\,(s+2)} \begin{bmatrix} s-1 & s \\ -6 & s-2 \end{bmatrix}$$

- The minor of order 2 is

$$\det\, G(s) = \frac{(s-1)\,(s-2) + 6s}{1.25^2\,(s+1)^2\,(s+2)^2} = \frac{1}{1.25^2\,(s+1)\,(s+2)}$$

- Pole polynomial: $\phi(s) = 1.25^2\,(s+1)\,(s+2)$
- Zero polynomial: $z(s) = 1 \quad \Rightarrow \quad$ No multivariable zeros!

Some important remarks on poles and zeros

1. The zeros resulting from a minimal realization are *transmission* zeros.
   (a) If realization is not minimal, may also have *invariant* zeros
   (b) Transmission zeros + invariant zeros = *system* zeros
2. For square systems, the poles and zeros of $G(s)$ are "essentially" the poles and zeros of $\det\, G(s)$.
   (a) This fails when zeros and poles cancel when forming the determinant.
3. Multivariable system poles and zeros exhibit *directions*
   (a) Poles and zeros at the same numerical locations may have different directions, and therefore do not cancel
4. For square systems with a non-singular D-matrix, the number of poles is the same as the number of zeros
5. There are no zeros if the outputs contain direct information about all the states (e.g., $C = I$ and $D = 0$)
6. Zeros usually appear when there are fewer inputs or outputs than states
7. MOVING POLES.
   (a) Feedback control moves the poles.
   (b) Series compensation can cancel poles in $G(s)$ by placing zeros in $K(s)$ (but not move the poles).
   (c) Parallel compensation cannot affect the poles in $G(s)$.
8. MOVING ZEROS.

(a) With feedback, the zeros of the closed-loop transfer function are the zeros of $G(s)$ plus the poles of $K(s)$, i.e., zeros are unaffected by feedback.

(b) Series compensation can counter the effect of zeros in $G(s)$ by placing poles in $K(s)$ to cancel them (but not for RHP zeros)

(c) The only way to move zeros is by parallel compensation $(G(s) + K(s))$

Example 4.7

- Consider the SISO plant and scalar compensator given by

$$G(s) = \frac{z(s)}{\phi(s)}$$
$$K(s) = k$$

- The closed-loop transfer function is given by

$$T(s) = \frac{L(s)}{1 + L(s)} = \frac{kG(s)}{1 + kG(s)} = \frac{kz(s)}{\phi(s) + kz(s)} = k\frac{z_{cl}(s)}{\phi_{cl}(s)}$$

- NOTE:
  - Zero polynomial: $z_{cl}(s) = z(s) \quad \Rightarrow \quad$ zeros are unchanged
  - Poles locations are changed by feedback:

$$k \to 0 \Rightarrow \phi_{cl}(s) \to \phi(s)$$
$$k \to \infty \Rightarrow \phi_{cl}(s) \to kz(s)$$

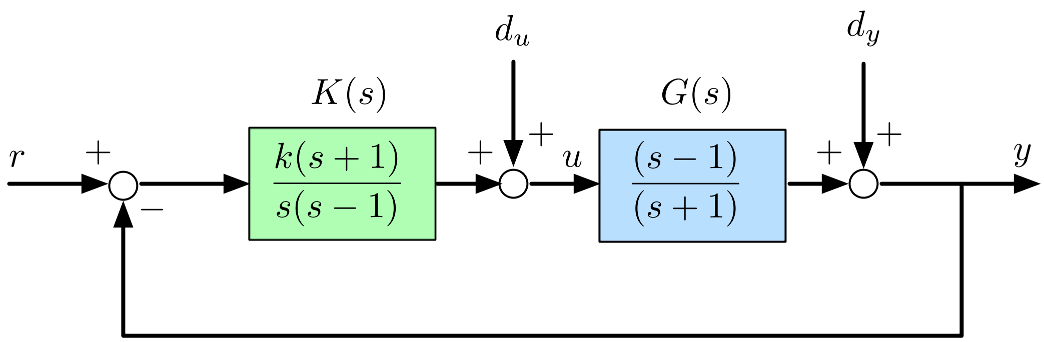
  - This is the classical root locus result

Internal Stability of Feedback Systems

- Classical feedback control suggests it is sufficient to check only the closed-loop transfer function $T(s)$ to ensure closed-loop stabilty
- However, this is not generally enough... consider the following example

Example 4.8

- A unity feedback system is shown below



- In forming the loop transfer function $L = GK$ we cancel the RHP pole $(s-1)$ to obtain

$$L = GK = \frac{k}{s}$$

$$S = (I + L)^{-1} = \frac{s}{s+k}$$

- By inspection, S, the transfer function from d_y to y, is stable (as long as $k \geq 0$)
- However, when we compute the transfer function from d_y to u, we obtain

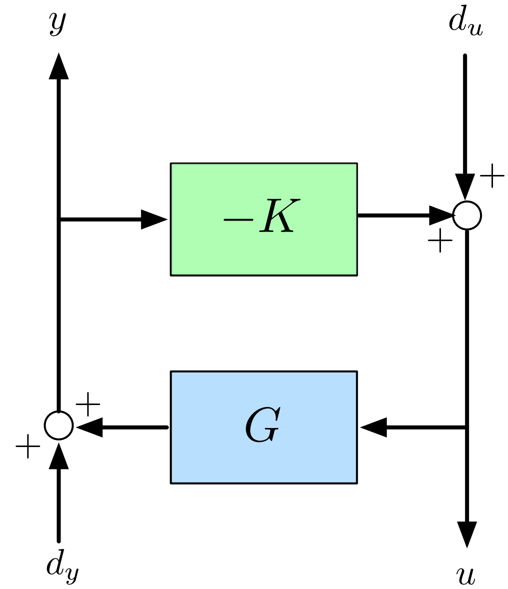
$$u = -K\left(I + GK\right)^{-1} d_y = -\frac{k\left(s+1\right)}{\left(s-1\right)\left(s+k\right)} d_y$$

which is *unstable.*

- Hence, this system is *internally unstable*

General Test for Internal Stability

- Consider the system depicted in the figure below where we introduce and measure signals at both locations between G and K



- The input-output relationships are given by

$$\boldsymbol{u} = (I + KG)^{-1}\boldsymbol{d}_u - K(I + GK)^{-1}\boldsymbol{d}_y$$
$$\boldsymbol{y} = G(I + KG)^{-1}\boldsymbol{d}_u + (I + GK)^{-1}\boldsymbol{d}_y$$

or equivalently in matrix-vector form

$$\begin{bmatrix} \boldsymbol{u} \\ \boldsymbol{y} \end{bmatrix} = \begin{bmatrix} (I + KG)^{-1} & -K(I + GK)^{-1} \\ G(I + KG)^{-1} & (I + GK)^{-1} \end{bmatrix} \begin{bmatrix} \boldsymbol{d}_u \\ \boldsymbol{d}_y \end{bmatrix}$$

Theorem 4.4 Assume that the transfer functions $G(s)$ and $K(s)$ contain no unstable hidden modes. Then the feedback system above is *internally stable* if and only if all four closed-loop transfer function matrices are stable.

Theorem 4.5 Assume there are no RHP pole-zero cancellations between $G(s)$ and $K(s)$. Then the feedback system above is internally stable if and only if one of the four closed-loop transfer function matrices is stable.

Implications of the internal stability requirement

1. If $G(s)$ has a RHP zero at z, then $L = GK$, $T = GK\,(I + GK)^{-1}$, $SG = \left(I + GK^{-1}\right)G$, $L_I = KG$ and $T_I = KG\,(I + KG)^{-1}$ will each have a RHP zero at z .
2. If $G(s)$ has a RHP pole at p, then $L = GK$ and $L_I = KG$ also have a RHP pole at p, while $S + (I + GK)^{-1}$, $KS = K\,(I + GK)^{-1}$ and $S_I = (I + GK)^{-1}$ have a RHP zero at p.

REMARK: "Perfect control" implies $S \approx 0$ and $T \approx 1$:

$\text{RHP zero} \Rightarrow$ perfect control impossible

$\text{RHP pole} \Rightarrow$ perfect control possible

Stabilizing Controllers

- It was shown by Youla et al (1976) that is is possible to generate a convenient parameterization of all stabilizing controllers for a given system.
- We will examine the case of stable plants below.

Stable plants

Lemma 4.1 For a stable plant $G(s)$ the negative feedback system of the previous section is internally stable if and ony if the term $Q = K\,(I + GK)^{-1}$ is stable.

PROOF: The four transfer functions are

$$K(I+GK)^{-1} = Q$$
$$(I+GK)^{-1} = I - GQ$$
$$(I+KG)^{-1} = I - QG$$
$$G(I+KG)^{-1} = G(I-QG)$$

which are clearly all stable if and only if G and Q are stable.

- By solving the first of these expressions with respect to the controller K, we obtain a parameterization of *all stabilizing negative feedback controllers* for the stable plant $G(s)$:

$$K_{\text{stab}} = (I - QG)^{-1} Q = Q(I - GQ)^{-1}$$

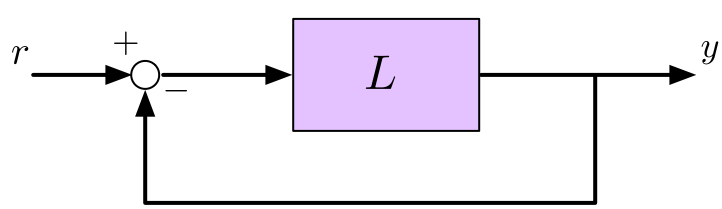
where the parameter Q is *any stable transfer function matrix*.

Stability Analysis in the Frequency Domain

- Stability of a linear system is equivalent to the system having no poles in the closed RHP
- In this section, we will examine the use of frequency domain techniques to determine closed-loop stability from the open-loop transfer matrix frequency response $L(j\omega)$

Open- and closed-loop characteristic polynomials

- Consider the unity feedback system shown below



- If $L(s)$ has a state-space realization

$$L = \begin{bmatrix} A_{ol} & B_{ol} \\ C_{ol} & D_{ol} \end{bmatrix}$$

then

$$L(s) = C_{ol}\left(sI - A_{ol}\right)^{-1} B_{ol} + D_{ol}$$

- The poles of $L(s)$ are the roots of the *open-loop* characteristic polynomial

$$\phi_{ol}(s) = \det\left(sI - A_{ol}\right)$$

- Assume there are no RHP pole-zero cancellations between G and K
  - From Theorem 4.5 internal stability of the *closed-loop* system is equivalent to stability of $S(s) = (I + L(s))^{-1}$
- The realization of $S(s)$ can be derived as follows:

$$\begin{aligned} \dot{\boldsymbol{x}} &= A_{ol}\boldsymbol{x} + B_{ol}\left(\boldsymbol{r} - \boldsymbol{y}\right) \\ -\boldsymbol{e} &= \boldsymbol{r} - \boldsymbol{y} = \boldsymbol{r} - C_{ol}\boldsymbol{x} - D_{ol}\left(\boldsymbol{r} - \boldsymbol{y}\right) \\ \boldsymbol{r} - \boldsymbol{y} &= \left(I + D_{ol}\right)^{-1}\left(\boldsymbol{r} - C_{ol}\boldsymbol{x}\right) \end{aligned}$$

$$\dot{\boldsymbol{x}} = \left(A_{ol} - B_{ol}\left(I + D_{ol}\right)^{-1} C_{ol}\right)\boldsymbol{x} + B_{ol}\left(I + D_{ol}\right)^{-1}\boldsymbol{r}$$

  - Therefore the state matrix of $S(s)$ is:

$$A_{cl} = A_{ol} - B_{ol}\left(I + D_{ol}\right)^{-1} C_{ol}$$

  - And the closed-loop characteristic polynomial is:

$$\phi_{cl}(s) = \det\left(sI - A_{ol} + B_{ol}\left(I + D_{ol}\right)^{-1} C_{ol}\right)$$

Relationship between characteristic polynomials

- From the expression

$$L(s) = C_{ol}\,(sI - A_{ol})^{-1}\,B_{ol} + D_{ol}$$

we can write

$$\det\,(I + L(s)) = \det\left(I + C_{ol}\,(sI - A_{ol})^{-1}\,B_{ol} + D_{ol}\right)$$

- Recalling Schur's formula,

$$\det\begin{bmatrix} A & B \\ C & D \end{bmatrix} = \det\,(A)\cdot\det\left(D - CA^{-1}B\right)$$

we can write (with $A = I + D_{ol}$, $B = -C_{ol}$, $D = sI - A_{ol}$ and $C = B_{ol}$),

$$\det\,(I + L(s)) = \frac{\phi_{cl}(s)}{\phi_{ol}(s)}\cdot c$$

where $c = \det\,(I + D_{ol})$ is a constant.

- NOTE:

$$\det\begin{bmatrix} I + D_{ol} & -C_{ol} \\ B_{ol} & sI - A_{ol} \end{bmatrix}$$

$$= \det\,[I + D_{ol}]\,\det\left[sI - A_{ol} + B_{ol}\,(I + D_{ol})^{-1}\,C_{ol}\right]$$

$$= \det\,[sI - A_{ol}]\,\det\left[I + D_{ol} + C_{ol}\,(sI - A_{ol})^{-1}\,B_{ol}\right]$$

Small Gain Theorem

- Consider the spectral radius of the loop gain frequency response, which is defined at each frequency as

$$\rho\,(L(j\omega) \triangleq \max_i |\lambda_i\,(L(j\omega))|$$

Theorem 4.6 SPECTRAL RADIUS STABILITY CONDITION. Consider a system with a stable loop transfer function $L(s)$. Then the closed-loop system is stable if

$$\rho\left(L\left(j\omega\right)\right) < 1 \quad \forall\omega$$

PROOF: Assume the system is unstable. Therefore $\det\left(I + L(s)\right)$ encircles the origin, and there is an eigenvalue, $\lambda_i\left(L\left(j\omega\right)\right)$ which is larger than 1 at some frequency. If $\det\left(I + L\left(s\right)\right)$ does encircle the origin, then there must exist a gain $\epsilon \in (0,\, 1]$ and a frequency ω' such that

$$\det\left(I + \epsilon L\left(j\omega'\right)\right) = 0$$

or

$$\prod_i \lambda_i\left(I + \epsilon L\left(j\omega'\right)\right) = 0$$

$$\begin{aligned}
&\Leftrightarrow 1 + \epsilon\lambda_i\left(L\left(j\omega'\right)\right) = 0 \quad \text{for some } i \\
&\Leftrightarrow \lambda_i\left(L\left(j\omega'\right)\right) = -\frac{1}{\epsilon} \quad \text{for some } i \\
&\Rightarrow \left|\lambda_i\left(j\omega'\right)\right| \ge 1 \quad \text{for some } i \\
&\Leftrightarrow \rho\left(L\left(j\omega'\right)\right) \ge 1
\end{aligned}$$

- INTERPRETATION:
  - If the system gain is less than 1 in all directions (all eigenvalues) and for all frequencies, then all signal deviations will eventually die out $\rightarrow$ the system is stable
  - Spectral radius theorem is *conservative* because no phase information is considered

Small Gain Theorem. Consider a system with a stable loop transfer function matrix $L(s)$. Then the closed-loop system is stable if

$$\|L\,(j\omega)\| < 1 \quad \forall\omega$$

where $\|L\|$ denotes any matrix norm satisfying $\|AB\| \leq \|A\| \cdot \|B\|$, e.g., the singular value $\sigma\,(L)$.

- NOTE: The small gain theorem is generally more conservative than the spectral radius condition