Multivariable Control in the Frequency Domain

The Control Problem

- Consider the input-output relationship
$$y = Gu + G_d d$$
where y is the output (or controlled) variable, u is the input (or manipulated) variable, and d is a disturbance.
- GOAL OF CONTROL:
$$\min\{r - y\}$$
where r is a system reference (or setpoint).
- We can classify two basic types of control problem:
  - Tracking problem - have y follow r as closely as possible
  - Regulator problem - counteract effects of d

Major Difficulties

- Plant Uncertainty
  - In the real world, our mathematical models G and G_d are not perfect, e.g.,
$$G_p = G + \triangle$$
where G_p is the "real" plant, and $\triangle$ is the "uncertainty".
  - We can classify the following cases:

- ◦ NOMINAL STABILITY - system is stable with no model uncertainty
- ◦ NOMINAL PERFORMANCE - system satisfies performance specifications with no model uncertainty
- ◦ ROBUST STABILITY - system is stable for all "uncertain" plants
- ◦ ROBUST PERFORMANCE - sytem satisfies performance specifications for all "uncertain" plants

Why the Frequency Domain?

- Most advanced control methods are developed and implemented in a time-domain framework
  - Time-domain is natural and easy to understand
  - Implementation - whether continuous or discrete - must take place in the time-domain
  - Powerful mathematical tools exist that make time-domain analysis particularly useful
- Nonetheless, the frequency domain offers insightful approaches to addressing performance, stability and robustness
  - Nyquist and Nyquist-like tests handle both absolute and relative stability
  - Descriptions of plant uncertainty are natural and easy to apply
  - New tools like H-infinity ($\mathscr{H}^\infty$) allow for design to worst-case uncertainty bounds
  - Matrix transfer functions (*transfer matrices*) give geometric insight into multi-input-multi-output behavior

Transfer Functions

- Recall that a continuous-time SISO transfer function relates input-to-output as a ratio of polynomials:

$$G(s) = \frac{b_m s^m + b_{m-1}s^{m-1} + \cdots + b_1 s + b_0}{s^n + a_{n-1}s^{n-1} + \cdots + a_1 s + a_0}$$

  where n is the system order, m is the order of the zero polynomial, and $n-m$ is the *relative order* of the system.

- We state the following definitions:
  - $G(s)$ is strictly proper if $\lim_{s\to\infty} G(s) = 0$
  - $G(s)$ is semi-proper (or bi-proper) if $\lim_{s\to\infty} G(s) = D$
  - $G(s)$ is proper if it is strictly proper or semi-proper
  - $G(s)$ is improper if $\lim_{s\to\infty} G(s) \to \infty$

- For multi-input-multi-output (MIMO) systems, $G(s)$ is a *matrix* of SISO transfer functions:

$$G(s) = \begin{bmatrix} g_{11}(s) & g_{12}(s) & \cdots & g_{1m}(s) \\ g_{21}(s) & g_{22}(s) & \cdots & g_{2m}(s) \\ \vdots & \vdots & \ddots & \vdots \\ g_{l1}(s) & g_{l2}(s) & \cdots & g_{lm}(s) \end{bmatrix}$$

  - Here the system has m inputs and l outputs

Scaling

- Scaling of variables is important in practical implementations as it makes system analysis and controller design simpler

SISO

- Unscaled variables may be written:

$$\hat{y} = \hat{G}\hat{u} + \hat{G}_d\hat{d}$$
$$\hat{e} = \hat{y} - \hat{r}$$

where here the 'hat' (ˆ) indicates the variables are unscaled.

- Scaled versions may be expressed by dividing each variable by its *maximum expected value*:

$$d = \hat{d}/\hat{d}_{max}$$
$$u = \hat{u}/\hat{u}_{max}$$

- Since a primary objective of control is to minimize control error $\hat{e}$, it is common to scale with respect to maximum control error, i.e.,

$$y = \hat{y}/\hat{e}_{max}$$
$$r = \hat{r}/\hat{e}_{max}$$
$$e = \hat{e}/\hat{e}_{max}$$

MIMO

- In the MIMO case, we express scaling factors as diagonal matrices:

$$\boldsymbol{d} = D_d^{-1}\hat{\boldsymbol{d}}$$
$$\boldsymbol{u} = D_u^{-1}\hat{\boldsymbol{u}}$$
$$\boldsymbol{y} = D_e^{-1}\hat{\boldsymbol{y}}$$
$$\boldsymbol{e} = D_e^{-1}\hat{\boldsymbol{e}}$$
$$\boldsymbol{r} = D_e^{-1}\hat{\boldsymbol{r}}$$

  - Substituting into the output equation above, we can write

$$D_e \boldsymbol{y} = \hat{G} D_u \boldsymbol{u} + \hat{G}_d D_d \boldsymbol{d}$$
$$D_e \boldsymbol{e} = D_e \boldsymbol{y} - D_e \boldsymbol{r}$$

- Introducing scaled transfer functions

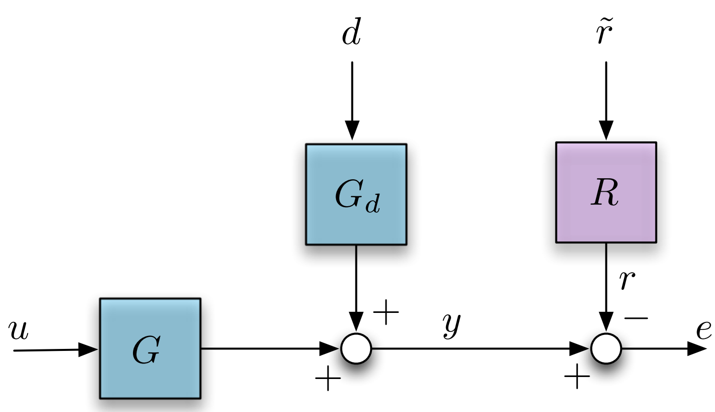
$$G = D_e^{-1} \hat{G} D_u$$
$$G_d = D_e^{-1} \hat{G}_d D_d$$

we can model in terms of scaled variables:

$$\boldsymbol{y} = G\boldsymbol{u} + G_d \boldsymbol{d}$$
$$\boldsymbol{e} = \boldsymbol{y} - \boldsymbol{r}$$

Example

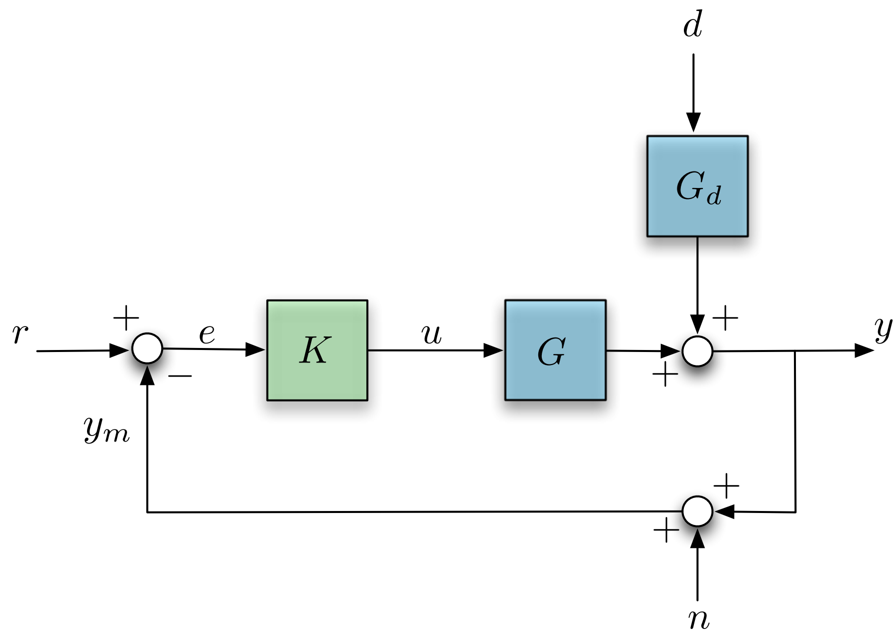
- Define a model in terms of scaled variables:



- Then the control objective may be described as follows:
  - For $|\boldsymbol{d}(t)| \le 1$ and $|\tilde{\boldsymbol{r}}(t)| \le 1$, manipulate $\boldsymbol{u}$ with $|\boldsymbol{u}(t)| \le 1$ such that $|\boldsymbol{e}(t)| = |\boldsymbol{y}(t) - \boldsymbol{r}(t)| \le 1$.

General Control Problem Formulation

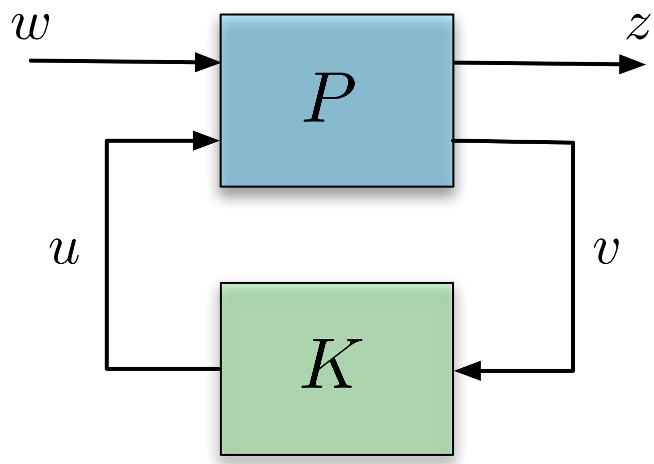
- The figure below shows a one-degree-of-freedom control configuration with negative feedback



  – Nomenclature is as follows

Symbol	Description
G	plant model
G_d	disturbance model
$\boldsymbol{r}$	reference inputs (commands, setpoints)
$\boldsymbol{d}$	disturbances (process noise, etc.)
$\boldsymbol{n}$	measurement noise
$\boldsymbol{y}$	plant outputs (controlled variables)
$\boldsymbol{y}_m$	measured plant outputs
$\boldsymbol{u}$	plant inputs (manipulated variables)

- A much more general framework for multivariable control system descriptions is given by the following *general control configuration*:

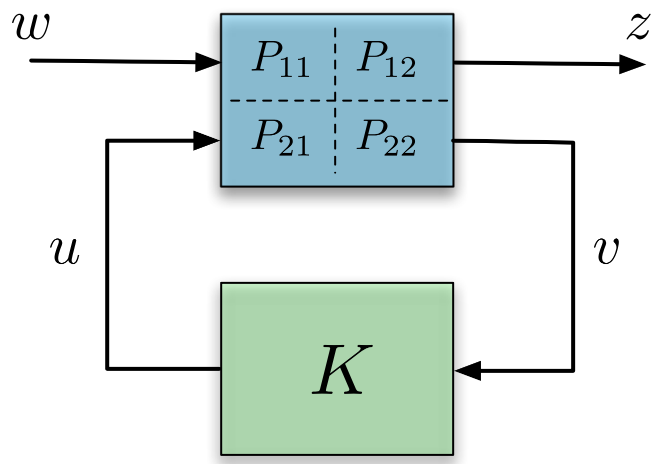


– Nomenclature is summarized below

Symbol	Description
P	generalized plant model
$\boldsymbol{w}$	exogenous inputs: commands, disturbances and noise
$\boldsymbol{z}$	exogenous outputs: 'error' signals to be minimized
$\boldsymbol{v}$	controller inputs (e.g., 'error' signal for one-DOF controller)
$\boldsymbol{u}$	control signals
K	controller model

– Generalized plant model will include G and G_d and the interconnection structure with the controller; it may also contain weighting functions

- The general control configuration is often portrayed in the following partitioned form



- Writing the corresponding input-output equation, we have

$$\begin{aligned} \boldsymbol{z} &= P_{11}\boldsymbol{w} + P_{12}\boldsymbol{u} \\ \boldsymbol{v} &= P_{21}\boldsymbol{w} + P_{22}\boldsymbol{u} \\ \boldsymbol{u} &= K\boldsymbol{v} \end{aligned}$$

  - It is easy to see that with appropriate definitions for the partitions of P, we can model a great variety of feedback control structures
  - For example, if we wish to put the single degree-of-freedom control system into the general control configuration, we make the following assignments,

exogenous inputs	$\boldsymbol{r}$, $\boldsymbol{d}$ and $\boldsymbol{n}$
exogenous outputs	$\boldsymbol{e} = \boldsymbol{r} - \boldsymbol{y}$

  - Therefore, we define the vector of exogenous inputs

$$\boldsymbol{w} = \begin{bmatrix} \boldsymbol{r} \\ \hline \boldsymbol{d} \\ \hline \boldsymbol{n} \end{bmatrix}$$

  - ...and the corresponding vector of exogenous outputs

$$z = \left[\begin{array}{c|c|c} P_{111} & P_{112} & P_{113} \end{array}\right] w$$

where we have subpartitioned the P_{11} block of the generalized plant.

- Equating the input-output equations, we obtain

$$\begin{aligned} z &= r - y \\ &= r - (Gu + G_d d) \\ &= r - Gu - G_d d \end{aligned}$$

giving,

$$\begin{aligned} P_{11} &= \left[\begin{array}{c|c|c} I & -G_d & 0 \end{array}\right] \\ P_{21} &= \left[\begin{array}{c|c|c} I & -G_d & -I \end{array}\right] \\ P_{12} &= P_{22} = -G \end{aligned}$$

- So, the entire P matrix has the form

$$P = \left[\begin{array}{ccc|c} I & -G_d & 0 & -G \\ \hline I & -G_d & -I & -G \end{array}\right]$$

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