Mathematics and Linear Systems Review

2.1: Matrix Algebra: Basics

- Vectors are quantities that contain magnitude and direction information
  - These quantities can be represented as a linear combination of basis vectors
    $$\boldsymbol{x} = x_1\boldsymbol{e}_1 + x_2\boldsymbol{e}_2 + \cdots x_n\boldsymbol{e}_n$$
    where $\{\boldsymbol{e}_1,\ \boldsymbol{e}_2,\ \ldots,\boldsymbol{e}_n\}$ define the fundamental directions in the given n-dimensional space.
  - This representation gives rise to a simple shorthand notation that will be used consistently throughout this course,
    $$\boldsymbol{x} = [x_1\ x_2\ x_3\ \cdots\ x_n]^T$$
- A matrix can be viewed as a "two-dimensional vector":
  $$A = \begin{bmatrix} a_{11} & \cdots & a_{1m} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nm} \end{bmatrix}$$
  - Notation $\Rightarrow A$ is n rows long by m columns wide, or more compactly, A is $(n \times m)$.
  - A common application of matrices is to summarize information about a set of simultaneous equations; although there are many other ways matrices may arise.

Concepts in Matrix Algebra

Transpose $\left(A^T\right) \Rightarrow$ interchange rows and columns

- $(AB)^T = B^T A^T$
- $\boldsymbol{x}^T \boldsymbol{y} = \boldsymbol{x} \cdot \boldsymbol{y}$ (a scalar quantity, inner product)
- $\boldsymbol{x}\,\boldsymbol{y}^T = A$ (a matrix in dyadic form, outer product)

Determinant $(|A|) \Rightarrow$ only defined for square matrices

$$|A| = a_{11}C_{11} + a_{12}C_{12} + \cdots + a_{1n}C_{1n}$$

where C_{1i} is the cofactor of a_{1i}. Note that the determinant can be expanded about any row or column of A

- $|AB| = |A|\,|B|$
- $|A| = \left|A^T\right|$

Inverse $\left(A^{-1}\right) \Rightarrow A^{-1}A = AA^{-1} = I$

- $A^{-1} = {}^1/_{|A|}\,\{\text{adj}\,A\}$ $\qquad \text{adj}\,A = \{\text{cofactor}(A)\}^T$
- $(AB)^{-1} = B^{-1}A^{-1}$
- $\left(A^T\right)^{-1} = \left(A^{-1}\right)^T$
- $\left|A^{-1}\right| = {}^1/_{|A|}$

Trace $(\text{trace}\,A) \Rightarrow \sum_{i=1}^{n} a_{ii}$ (defined for square matrices only)

- $\text{trace}\,A + B = \text{trace}\,A + \text{trace}\,B$
- $\text{trace}\,ABC = \text{trace}\,BCA = \text{trace}\,CAB$

Special Matrices • Symmetric $A^T = A$

- Skew Symmetric $A^T = -A$
- Orthogonal $A^{-1} = A^T$

Partitioned Matrices

- Many of the matrix operations presented above can be generalized to much larger matrices using simple notation through partitioning.
- What is a partitioned matrix?
  - A matrix that has been subdivided into smaller matrices

$$A = \left[\begin{array}{cc|cc} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ \hline a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{array}\right] = \left[\begin{array}{c|c} A_{11} & A_{12} \\ \hline A_{21} & A_{22} \end{array}\right]$$

$$\boldsymbol{x} = \left[\begin{array}{c} x_1 \\ x_2 \\ \hline x_3 \\ x_4 \end{array}\right] = \left[\begin{array}{c} X_1 \\ \hline X_2 \end{array}\right]$$

$$\Rightarrow A\boldsymbol{x} = \left[\begin{array}{c} A_{11}X_1 + A_{12}X_2 \\ \hline A_{21}X_1 + A_{22}X_2 \end{array}\right]$$

Some Properties of Partitioned Matrices

- Block Diagonal Matrices $\Rightarrow$

$$D = \left[\begin{array}{c|c|c} A_{11} & 0 & 0 \\ \hline 0 & A_{22} & 0 \\ \hline 0 & 0 & A_{33} \end{array}\right]$$

  - $|D| = |A_{11}|\,|A_{22}|\,|A_{33}|$
  - $D^{-1} = \text{diag}\left\{A_{11}^{-1},\ A_{22}^{-1},\ A_{33}^{-1}\right\}$

- Arbitrarily Partitioned Matrices $\Rightarrow$

$$M = \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]$$

where the blocks are not necessarily square.

– Definitions:

SCHUR COMPLEMENT of $A \Rightarrow D - CA^{-1}B$

SCHUR COMPLEMENT of $D \Rightarrow A - BD^{-1}C$

This leads to the following form of the block matrix inverse:

$$M^{-1} = \left[\begin{array}{c|c} (A - BD^{-1}C)^{-1} & -(A - BD^{-1}C)^{-1}BD^{-1} \\ \hline -D^{-1}C(A - BD^{-1}C)^{-1} & D^{-1} + D^{-1}C(A - BD^{-1}C)^{-1}BD^{-1} \end{array}\right]$$

If A is 2×2 ,

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

then we can write

$$A^{-1} = \frac{1}{ad - bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

which is the form of the familiar 2×2 inverse.

2.2: Matrix Algebra: Linear Independence and Rank

- Another important concept associated with n-dimensional vector spaces is linear independence.
- Definition: A set of $(n \times 1)$ vectors $(\boldsymbol{a}_1, \boldsymbol{a}_2, \ldots, \boldsymbol{a}_m)$ is *linearly dependent* if and only if

$$x_1\boldsymbol{a}_1 + x_2\boldsymbol{a}_2 + \cdots + x_m\boldsymbol{a}_m = 0 \;\Rightarrow\; x_1 = x_2 = \cdots = x_m = 0$$

  - In other words, for this condition to be true, there must exist a vector $\boldsymbol{a}_k$ that can be obtained as a combination of one or more of the other vectors.
  - If this condition cannot be met, then the vectors are *linearly independent*.

The concept of linear independence can also be related to the *rank* of a matrix,

$$x_1\boldsymbol{a}_1 + x_2\boldsymbol{a}_2 + \cdots + x_m\boldsymbol{a}_m = \left[\, \boldsymbol{a}_1 \,\middle|\, \boldsymbol{a}_2 \,\middle|\, \cdots \,\middle|\, \boldsymbol{a}_m \,\right] \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_m \end{bmatrix} = 0$$

The columns of a matrix can be thought of as vectors and the definition for the rank of a matrix now becomes apparent:

$\operatorname{rank} A =$ maximum number of linearly independent rows or columns of A

- if A is $(n \times m)$ with $n < m$, then $\operatorname{rank}\{A\} \le n$
- if A is $(n \times m)$ with $n > m$, then $\operatorname{rank} A \le m$

- if A is $(n \times n)$ and –
  - $\operatorname{rank} A < n$, then $|A| = 0,\ A^{-1}$ does not exist and A is said to be *singular*
  - $\operatorname{rank} A = n$, then the rows/columns of A form a basis set

Example

$$\boldsymbol{a}_1 = \begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix} \qquad \boldsymbol{a}_2 = \begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix} \qquad \boldsymbol{a}_3 = \begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix}$$

- Clearly,

$$\boldsymbol{a}_1 = \boldsymbol{a}_2 \ \Rightarrow\ x_1\boldsymbol{a}_1 + x_2\boldsymbol{a}_2 + 0 \cdot \boldsymbol{a}_3 = 0$$

  for any case where

$$x_1 = -x_2$$

- So $(\boldsymbol{a}_1,\ \boldsymbol{a}_2,\ \boldsymbol{a}_3)$ are not linearly independent, which implies

$$A = \begin{bmatrix} 3 & 3 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix} \Rightarrow \text{ singular}$$

2.3: Matrix Algebra: Eigenvalues / Eigenvectors

- One particularly useful property of a square matrix is that there exists a set of n scalars, λ_i, and a corresponding set of n vectors, $\boldsymbol{w}_i$, such that:

$$A\boldsymbol{w}_i = \lambda_i \boldsymbol{w}_i \qquad i = 1,\ \ldots,\ n$$

  - The n *eigenvalues* of A can be identified by finding the n roots of

$$|\lambda_i I - A| = 0$$

- The eigenvalues of A make the matrix $\lambda I - A$ singular so that there exists a non-zero vector $\boldsymbol{w}$ which satisfies $(\lambda I - A)\,\boldsymbol{w} = 0$ for each λ.
- These vectors are called the *eigenvectors* of A.

– Using partitioned matrices, the eigenvalue/eigenvector relationship can be written as:

$$A\left[\begin{array}{c|c|c|c} \boldsymbol{w}_1 & \boldsymbol{w}_2 & \cdots & \boldsymbol{w}_n \end{array}\right] = \left[\begin{array}{c|c|c|c} \boldsymbol{w}_1 & \boldsymbol{w}_2 & \cdots & \boldsymbol{w}_n \end{array}\right]\Lambda$$

where

$$\Lambda = \begin{bmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n \end{bmatrix}$$

$$\Rightarrow \quad AW = W\Lambda$$

– In the (common) case where the n eigenvectors of A are linearly independent, the inverse W^{-1} exists and we can write the eigenvalue-eigenvector decomposition of A as

$$A = W\Lambda W^{-1}$$

which is a special form of similarity transformation.

Example

$$A = \begin{bmatrix} 3 & 3 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

$$|\lambda I - A| = (\lambda - 2)\left(\lambda^2 - 4\lambda\right) = 0 \Rightarrow \lambda = 4,\ 2,\ 0$$

$$\lambda_1 = 4 \qquad \boldsymbol{w_1} = \begin{bmatrix} 3 & 1 & 0 \end{bmatrix}^T$$

$$\lambda_2 = 2 \qquad \boldsymbol{w_2} = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}^T$$

$$\lambda_3 = 0 \qquad \boldsymbol{w_3} = \begin{bmatrix} 1 & -1 & 0 \end{bmatrix}^T$$

so we can write the eigen-decomposition as

$$A = \begin{bmatrix} 3 & 3 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix} = \begin{bmatrix} 3 & 0 & 1 \\ 1 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 4 & & \\ & 2 & \\ & & 0 \end{bmatrix} \begin{bmatrix} 0.25 & 0.25 & 0 \\ 0 & 0 & 1 \\ 0.25 & -0.75 & 0 \end{bmatrix}$$

- Eigenvectors are unique up to a scaling factor
  - It is customary to scale the eigenvectors so that they each have unity magnitude (Matlab's *eig* routine generates the eigenvectors this way)
  - In this case the decomposition from the example is written

$$A = \begin{bmatrix} 0.9487 & 0 & -0.7071 \\ 0.3162 & 0 & 0.7071 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 4 & & \\ & 2 & \\ & & 0 \end{bmatrix} \begin{bmatrix} 0.7906 & 0.7906 & 0 \\ 0 & 0 & 1 \\ -0.3536 & 1.0607 & 0 \end{bmatrix}$$

- *Dual Eigenvectors* are defined from the inverse of the eigenvector matrix

$$W^{-1} = \begin{bmatrix} \boldsymbol{v}_1^T \\ \hline \vdots \\ \hline \boldsymbol{v}_n^T \end{bmatrix} = V$$

and satisfy the dual eigenvalue problem:

$$\boldsymbol{v}^T A = \lambda \boldsymbol{v}^T$$

2.4: Matrix Algebra: Vector Calculus

Matrix Calculus

- Differentiation and integration of a vector with respect to a scalar:

$$d\boldsymbol{x}/ds = \begin{bmatrix} dx_1/ds \\ dx_2/ds \\ \vdots \\ dx_b/ds \end{bmatrix} \qquad \int \boldsymbol{x}\,ds = \begin{bmatrix} \int x_1 ds \\ \int x_2 ds \\ \vdots \\ \int x_n ds \end{bmatrix}$$

- Differentiation of a scalar with respect to a vector (gradient):

$$\partial s(\boldsymbol{x})/\partial \boldsymbol{x} = \begin{bmatrix} \partial s/\partial x_1 & \partial s/\partial x_2 & \cdots & \partial s/\partial x_n \end{bmatrix}$$

- Differentiation of a vector with respect to a vector (Jacobian):

$$\partial \boldsymbol{a}(\boldsymbol{x})/\partial \boldsymbol{x} = \begin{bmatrix} \partial a_1/\partial x_1 & \partial a_1/\partial x_2 & \cdots & \partial a_1/\partial x_n \\ \partial a_2/\partial x_1 & \ddots & & \vdots \\ \vdots & & \ddots & \vdots \\ \partial a_m/\partial x_1 & \partial a_m/\partial x_2 & \cdots & \partial a_m/\partial x_n \end{bmatrix}$$

- Second derivative of a scalar with respect to a vector (Hessian):

$$\partial^2 s(\boldsymbol{x})/\partial \boldsymbol{x}^2 = \begin{bmatrix} \partial^2 s/\partial x_1^2 & \partial^2 s/\partial x_1 \partial x_2 & \cdots & \partial^2 s/\partial x_1 \partial x_n \\ \partial^2 s/\partial x_2 \partial x_1 & \ddots & & \vdots \\ \vdots & & \ddots & \vdots \\ \partial^2 s/\partial x_n \partial x_1 & \partial^2 s/\partial x_n \partial x_2 & \cdots & \partial^2 s/\partial x_n^2 \end{bmatrix}$$

- Some additional results you may find useful:

$$\begin{aligned} d/dt(A^{-1}) &= -A^{-1}\dot{A}A^{-1} \\ \partial/\partial A\,\{\mathrm{trace}\,(A)\} &= I \\ \partial/\partial A\,\{\mathrm{trace}\,(BAD)\} &= B^T D^T \\ \partial/\partial A\,\left\{\mathrm{trace}\,(ABA^T)\right\} &= 2AB \\ \partial/\partial A\,\{|BAD|\} &= |BAD|\,A^{-T} \end{aligned}$$

Taylor Series Expansions

- In this course, we will focus primarily on optimizing scalar functions of a set of variables
- The approach taken to accomplish this task relies heavily on the use of *Taylor Series expansions:*
  - Expansion of $f(x)$ around a point x_0 (where x is a scalar variable):

$$f(x) = f(x_0) + \left(\frac{df}{dx}\right)|_{x_0}\,(x - x_0) + \left(\frac{1}{2}\right)\left(\frac{d^2 f}{dx^2}\right)|_{x_0}\,(x - x_0)^2 + \cdots$$

  - But what if $\boldsymbol{x}$ is a vector variable?

$$f(\boldsymbol{x}) = f(\boldsymbol{x}_0) + \left(\frac{\partial f}{\partial \boldsymbol{x}}\right)|_{\boldsymbol{x}_0}(\boldsymbol{x} - \boldsymbol{x}_0) + \left(\frac{1}{2}\right)(\boldsymbol{x} - \boldsymbol{x}_0)^T\left(\frac{\partial^2 f}{\partial \boldsymbol{x}^2}\right)(\boldsymbol{x} - \boldsymbol{x}_0) + \cdots$$

- How is a Taylor series useful for finding the minima (and maxima) of a function $f(x)$?

– Consider scalar x with $f(x_0)$ a minimum

- $f(x) > f(x_0)$ for all x in a neighborhood of x_0

$$\Rightarrow \quad \frac{df}{dx}|_{x_0}(x-x_0) + \left(\frac{1}{2}\right)\left(\frac{d^2 f}{dx^2}\right)|_{x_0}(x-x_0)^2 + \cdots > 0$$

- For x very close to x_0, $\left(\frac{df}{dx}\right)(x-x_0)$ dominates the expression on the left side of the inequality
- Since x is arbitrary, $x - x_0$ can be both positive and negative $\Rightarrow$ $\frac{df}{dx}|_{x_o}$ must be zero!!
- If this is the case, $\left(\frac{1}{2}\right)\left(\frac{d^2 f}{dx^2}\right)(x-x_0)^2$ dominates the expression on the left hand side of the inequality; and since $(x-x_0)^2 > 0$,

$$\frac{d^2 f}{dx^2}|_{x_0} > 0$$

– So, the Taylor Series allows us to establish important conditions for use in identifying minima (and maxima) of a function

– Similar conditions can be developed for problems where x is a vector variable:

$$\frac{\partial f}{\partial \boldsymbol{x}} = 0$$

$$(\boldsymbol{x} - \boldsymbol{x}_0)^T \left(\frac{\partial^2 f}{\partial \boldsymbol{x}^2}\right)|_{x_0}(\boldsymbol{x} - \boldsymbol{x}_0) > 0$$

- But how can I tell whether this is true for all $\boldsymbol{x}$?

Quadratic Forms

- The expression above is a special type of scalar $\Rightarrow$ one that is written in a *quadratic form*

$$\text{General Quadratic Form} \quad \Rightarrow \quad \boldsymbol{x}^T A \boldsymbol{x}$$

- The matrix, A , associated with a quadratic form has special characteristics which describe the properties of $\boldsymbol{x}^T A \boldsymbol{x}$:
  - A can always be written as a symmetric matrix by decomposing into its symmetric and anti-symmetric parts:

$$A_s = 1/2\left(A + A^T\right)$$
$$A_a = 1/2\left(A - A^T\right)$$
$$\Rightarrow A = A_s + A_a$$
$$\boldsymbol{x}^T A \boldsymbol{x} \;=\; \boldsymbol{x}^T A_s \boldsymbol{x} + \boldsymbol{x}^T A_a \boldsymbol{x} \;=\; \boldsymbol{x}^T A_s \boldsymbol{x} + 1/2\left\{\boldsymbol{x}^T A \boldsymbol{x} - \boldsymbol{x}^T A^T \boldsymbol{x}\right\}$$
$$\Rightarrow \boldsymbol{x}^T A \boldsymbol{x} = \boldsymbol{x}^T A_s \boldsymbol{x}$$

  - A is:

positive definite	$(A > 0)$	if $\boldsymbol{x}^T A \boldsymbol{x} > 0 \quad \forall \boldsymbol{x} \neq 0$
positive semidefinite	$(A \geq 0)$	if $\boldsymbol{x}^T A \boldsymbol{x} \geq 0 \quad \forall \boldsymbol{x} \neq 0$
negative semidefinite	$(A \leq 0)$	if $\boldsymbol{x}^T A \boldsymbol{x} \leq 0 \quad \forall \boldsymbol{x} \neq 0$
negative definite	$(A < 0)$	if $\boldsymbol{x}^T A \boldsymbol{x} < 0 \quad \forall \boldsymbol{x} \neq 0$

  - If A is positive definite:
    - $|A| > 0$
    - $B^T A B > 0$ if B is real and nonsingular or if B has maximum column rank
    - $A^{-1} > 0$

- ∘ $A^n > 0$
- ∘ $\exists$ a nonsingular $B \ni A = B^T B$ (matrix square root)

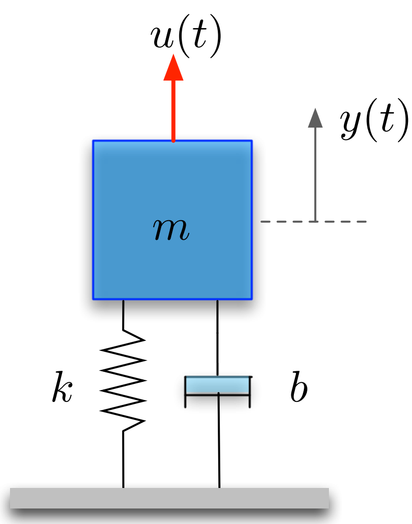
– Tests for definiteness:

$$\lambda_i > 0 \Rightarrow A > 0$$
$$\lambda_i \geq 0 \Rightarrow A \geq 0$$
$$\lambda_i \leq 0 \Rightarrow A \leq 0$$
$$\lambda_i < 0 \Rightarrow A < 0$$

– Test for positive definiteness: determinant of every principal subminor of A is positive!

2.5 Linear Systems: State-Space Representations

- What do we mean by a "state-space" representation
  - A representation of the dynamics of an n^{th} -order system as a system of first-order differential equations in an n-vector called the system *state vector*
- A classic example is given by second-order spring-mass-damper:



  - Writing the equations of motion according to Newton's 2^{nd} Law,

$$m\ddot{y}(t) = u(t) - b\dot{y}(t) - ky(t)$$
$$\ddot{y}(t) = \left(\frac{1}{m}\right)[u(t) - b\dot{y}(t) - ky(t)]$$

- Now defining the state vector

$$\boldsymbol{x}(t) = \begin{bmatrix} y(t) \\ \dot{y}(t) \end{bmatrix}$$

we have

$$\dot{\boldsymbol{x}}(t) = \begin{bmatrix} \dot{y}(t) \\ \ddot{y}(t) \end{bmatrix} = \begin{bmatrix} \dot{y}(t) \\ -\frac{k}{m}y(t) - \frac{b}{m}\dot{y}(t) + \frac{1}{m}u(t) \end{bmatrix}$$

- We can write this in the general form,

$$\dot{\boldsymbol{x}}(t) = Ax(t) + Bu(t)$$

  where A and B are constant matrices.

- Complete the picture by setting $y(t)$ as a function of $\boldsymbol{x}(t)$. The general form is the linear equation,

$$y(t) = C\boldsymbol{x}(t) + Du(t)$$

  where C and D are constant matrices.

- Thus we have the fundamental form for a linear state-space model:

$$\begin{aligned} \dot{\boldsymbol{x}}(t) &= A\boldsymbol{x}(t) + Bu(t) \\ y(t) &= C\boldsymbol{x}(t) + Du(t) \end{aligned}$$

  where $u(t)$ is the input, $x(t)$ is the "state", and A, B, C, D are constant matrices.

Definition: The *state* of a system at time t_0 is the minimum amount of information at t_0 that, together with the input $u(t)$, $t \geq t_0$, uniquely determines the behavior of the system for all $t \geq t_0$.

- A state-space description may also be defined for more general systems where parameters are time-varying, i.e., the matrices A, B, C, D are not constant.
  - For this case, we may express the general state equation as,

$$\dot{\boldsymbol{x}}(t) = f\,(\boldsymbol{x}(t), \boldsymbol{u}(t), t)$$

    or if the system is linear,

$$\dot{\boldsymbol{x}}(t) = A(t)\boldsymbol{x}(t) + B(t)\boldsymbol{u}(t)$$

Note that the vector-valued term $\boldsymbol{u}(t)$ allows for a multiple-input system.

- The time-domain solution of this set of *linear*, *time-varying* differential equations is given by:

$$\boldsymbol{x}(t) = \Phi(t, t_0)\boldsymbol{x}(t_0) + \int_{t_0}^{t} \Phi(t, \tau)B(\tau)\boldsymbol{u}(\tau)d\tau$$

where $\Phi(t, t_0)$ is referred to as the state transition matrix.

  – $\Phi(t, t_0)$ satisfies the matrix differential equation

$$\frac{d}{dt}\{\Phi(t, t_0)\} = A(t)\Phi(t, t_0)$$

$$\Phi(t_0, t_0) = I$$

  – Other properties of $\Phi(t, t_0)$:
    - $\Phi(t_2, t_1)\Phi(t_1, t_0) = \Phi(t_2, t_0)$
    - $\Phi^{-1}(t, t_0)$ exists for all t, t_0 and $\Phi^{-1}(t, t_0) = \Phi(t_0, t)$

- For linear, time-invariant systems, the state transition matrix takes a much simpler form:

$$\Phi(t, t_0) = e^{A(t-t_0)}$$

where

$$e^{At} = I + At + \frac{1}{2}A^2t^2 + \frac{1}{3!}A^3t^3 + \cdots$$

  – So, for linear, time-invariant systems,
    - $\Phi(t_2, t_1) = \Phi(t_2 - t_1) = \Phi(\triangle t)$
    - $\Phi^{-1}(t_2 - t_1) = \Phi(t_1 - t_2)$

  – Some standard methods to compute e^{At} :

- $e^{At} = \mathcal{L}^{-1}\left\{(sI - A)^{-1}\right\}$
- $A = W\Lambda W^{-1} \quad \Rightarrow \quad e^{At} = We^{\Lambda t}W^{-1}$ (where $e^{\Lambda t} = \mathrm{diag}\left\{e^{\lambda_1 t},\, \ldots,\, e^{\lambda_n t}\right\}$
- Numerical calculation of series expansion with truncation to get $e^{A\triangle t}$

Variational Equations

- Why are linear equations important in a non-linear world? Because they allow us to expand the solution of a non-linear problem to a wide range of similar problems

$$\dot{\boldsymbol{x}} = f(\boldsymbol{x},\, \boldsymbol{u},\, t)$$

$$\dot{\boldsymbol{x}} + \delta\dot{\boldsymbol{x}} = f(\boldsymbol{x} + \delta\boldsymbol{x},\, \boldsymbol{u} + \delta\boldsymbol{u},\, t)$$

$$\dot{\boldsymbol{x}} + \delta\dot{\boldsymbol{x}} \approx f(\boldsymbol{x},\, \boldsymbol{u},\, t)|_{\boldsymbol{x}_0,\boldsymbol{u}_0} + {}^{\partial f}/_{\partial \boldsymbol{x}}|_{\boldsymbol{x}_0,\boldsymbol{u}_0}\delta\boldsymbol{x} + {}^{\partial f}/_{\partial \boldsymbol{x}}|_{\boldsymbol{x}_0,\boldsymbol{u}_0}\delta\boldsymbol{u}$$

$$\Rightarrow\; \delta\dot{\boldsymbol{x}} = A\delta\boldsymbol{x} + B\delta\boldsymbol{u}$$

This gives a linear differential equation describing small purturbations about a nominal trajectory

State Transformations

- Another important characteristic of state variable representations is that they are not unique. Consider the linear state-space equation

$$\dot{\boldsymbol{x}} = A\boldsymbol{x} + B\boldsymbol{u}$$

  – Let T be a nonsingular, constant matrix such that

$$\acute{\boldsymbol{x}} = T\boldsymbol{x}$$

$$\dot{\boldsymbol{x}} = T^{-1}\dot{\acute{\boldsymbol{x}}} = AT^{-1}\acute{\boldsymbol{x}} + B\boldsymbol{u}$$

$$\Rightarrow \dot{\acute{x}} = TAT^{-1}\acute{x} + TBu$$

– Example state-space representations include: modal form, control canonical form, and observer canonical form - all model the *same* system

Discrete-Time State Space Representations

- If we assume that u can be approximated by a piecewise constant over every interval $kT \le t \le (k+1)T$, then the continuous-time state space representation can be extended to discrete-time systems:

$$x([k+1]T) = \Phi([k+1]T, kT)x(kT) + \int_{kT}^{(k+1)T} \Phi([k+1]T, \tau)Gu(\tau)d\tau$$

$$x([k+1]T) = \Phi x(kT) + \left\{\int \Phi G d\tau\right\} u(kT)$$

– So, for linear time-invariant systems,

$$x(k+1) = A_D x(k) + B_D u(k)$$

where

$$A_D = \Phi(T) \qquad B_D = \int_0^T \Phi(\tau)G d\tau$$

Note: $x(k+1)$ can also be written as:

$$x(k+1) = A_D^k x(0) + \sum_{i=0}^{k-1} A_D^{k-1-i} B_D u(i)$$

2.6: Linear Systems: Controllability and Observability

Linear, Time-Invariant State Space Representations

- Continuous time

$$\dot{\boldsymbol{x}} = A\boldsymbol{x} + B\boldsymbol{u}$$
$$\boldsymbol{y} = C\boldsymbol{x} + D\boldsymbol{u}$$

For strictly proper systems, $D = 0$, and we can write the solution as

$$\boldsymbol{x}(t) = e^{At}\boldsymbol{x}_0 + \int_0^t e^{A(t-\tau)}G\boldsymbol{u}(\tau)d\tau$$

- Discrete time

$$\boldsymbol{x}_{k+1} = A_D\boldsymbol{x}_k + B_D\boldsymbol{u}_k$$
$$\boldsymbol{y}_k = C\boldsymbol{x}_k + D\boldsymbol{u}_k$$

For strictly proper systems, $D = 0$, and we can write the solution as

$$\boldsymbol{x}_{k+1} = A_D^k\boldsymbol{x}_0 + \sum_{i=0}^{k-1} A_D^{k-1-i}B_D\boldsymbol{u}_i$$

Introduction to Controllability and Observability

- Two particularly important questions to address for linear systems are whether or not:

  1. we can use our inputs to drive the system to an arbitrary state
  2. we can use our outputs to reconstruct the states

- Linear Algebra Preliminaries:

  – Consider a set of linear algebraic equations defined by:

  $$R\boldsymbol{\alpha} = \boldsymbol{\beta}$$

  where R is a $(p \times q)$ matrix, $\boldsymbol{\alpha}$ is a $(q \times 1)$ vector and $\boldsymbol{\beta}$ is a $(p \times 1)$ vector

– When will solutions to this set of equations exist?

1. if $p = q$, a unique solution will exist provided $|R| \neq 0$
2. if $p > q$, (more equations than unknowns), a unique solution given by $\boldsymbol{\alpha} = (R^T R)^{-1} R^T \boldsymbol{\beta}$ will exist provided $\mathrm{rank}(R) = q$
3. if $p < q$, (fewer equations than unknowns),
   (a) an infinite number of solutions will exist for any $\boldsymbol{\beta}$ provided $\mathrm{rank}(R) = p$
   (b) but if $\mathrm{rank}(R) < p$, an infinite number of solutions will exist only if $\boldsymbol{\beta}$ lies in a certain subspace of the q- dimensional space (i.e., $\boldsymbol{\beta}$ cannot be arbitrary!)

- These concepts will help us to address our two questions above

Controllability

Definition $\Rightarrow$

A state is controllable at $t = t_0$ if there exists a finite $t_1 > t_0$ such that, for any $\boldsymbol{x}(t_0)$ and $\boldsymbol{x}(t_1)$, there exists an input $\boldsymbol{u}(t)$, $t \in [t_0,\ t_1]$, which transfers $\boldsymbol{x}(t_0)$ to $\boldsymbol{x}(t_1)$.

If all states are controllable for all t_0, then the system is controllable.

- Necessary & Sufficient Conditions for Controllability:

1. Continuous $\Rightarrow$ $\mathrm{rank}\left[\, B \,\middle|\, AB \,\middle|\, A^2B \,\middle|\, \cdots \,\middle|\, A^{n-1}B \,\right] = n$
2. Discrete $\Rightarrow$ $\mathrm{rank}\left[\, B_D \,\middle|\, A_D B_D \,\middle|\, A_D^2 B_D \,\middle|\, \cdots \,\middle|\, A_D^{n-1} B_D \,\right] = n$

- Where do these conditions come from? We'll consider the discrete-time case here.

– From above,

$$\boldsymbol{x}_{k+1} = A_D^k \boldsymbol{x}(0) + \sum_{i=0}^{k-1} A_D^{k-1-i} B_D \boldsymbol{u}_i$$

$$= A_D^k \boldsymbol{x}(0) + \left[\begin{array}{c|c|c|c|c} A_D^{k-1} B_D & A_D^{k-2} B_D & \cdots & A_D B_D & B_D \end{array}\right] \begin{bmatrix} \boldsymbol{u}_0 \\ \boldsymbol{u}_1 \\ \vdots \\ \boldsymbol{u}_{k-2} \\ \boldsymbol{u}_{k-1} \end{bmatrix}$$

$$\Rightarrow \quad \left[\begin{array}{c|c|c|c} A_D^{k-1} B_D & \cdots & A_D B_D & B_D \end{array}\right] \boldsymbol{v} = \boldsymbol{x}_{k+1} - A_D^k \boldsymbol{x}_0$$

– From our linear algebra preliminaries, there will be an infinite number of solutions, $\boldsymbol{v}$, for any arbitrary $\boldsymbol{x}_{k+1}$ and $\boldsymbol{x}_0$ only if

$$\text{rank}[\mathcal{C}] = \text{rank}\left[\begin{array}{c|c|c} A_D^{k-1} B_D & \cdots & B_D \end{array}\right] = n$$

Observability

Definition $\Rightarrow$

A state is observable at $t = t_0$ if, by observing the output $\boldsymbol{y}(t)$ during a finite time interval $[t_0,\ t_1]$, the state $\boldsymbol{x}(t_0)$ can be determined.
If all states are observable for every t_0, then the system is observable

- Necessary & Sufficient Conditions for Observability:
  1. Coninuous $\Rightarrow$ $\text{rank}\left[\begin{array}{c|c|c|c} C^T & A^T C^T & \cdots & (A^T)^{n-1} C^T \end{array}\right] = n$
  2. Discrete $\Rightarrow$ $\text{rank}\left[\begin{array}{c|c|c|c} C^T & A_D^T C^T & \cdots & (A_D^T)^{n-1} C^T \end{array}\right] = n$
- These conditions can be developed in a manner analogous to the development for controllability

Useful Matrix Identities

Basic Relationships

$$
\begin{aligned}
A(B+C) &= AB + AC \\
(A+B)^T &= A^T + B^T \\
(AB)^T &= B^T A^T \\
(AB)^{-1} &= B^{-1} A^{-1} \\
(A^{-1})^T &= (A^T)^{-1}
\end{aligned}
$$

Useful Derivative Identities

Gradients

$$
\begin{aligned}
\frac{\partial}{\partial \boldsymbol{x}}(\boldsymbol{y}^T \boldsymbol{x}) &= \frac{\partial}{\partial \boldsymbol{x}}(\boldsymbol{x}^T \boldsymbol{y}) = \boldsymbol{y}^T \\
\frac{\partial}{\partial \boldsymbol{x}}(\boldsymbol{y}^T A^T \boldsymbol{x}) &= \frac{\partial}{\partial x}(\boldsymbol{x}^T A \boldsymbol{y}) = \boldsymbol{y}^T A^T \\
\frac{\partial}{\partial \boldsymbol{x}}(\boldsymbol{x}^T A \boldsymbol{x}) &= \boldsymbol{x}^T (A + A^T) \\
\frac{\partial}{\partial \boldsymbol{x}}(\boldsymbol{x}^T Q \boldsymbol{x}) &= 2\boldsymbol{x}^T Q \;\; (Q \text{ symmetric}) \\
\frac{\partial}{\partial \boldsymbol{x}}([\boldsymbol{x}-\boldsymbol{y}]^T Q [\boldsymbol{x}-\boldsymbol{y}]) &= 2[\boldsymbol{x}-\boldsymbol{y}]^T Q \\
\frac{\partial}{\partial \boldsymbol{x}}(\boldsymbol{y}^T \boldsymbol{f}(\boldsymbol{x})) &= \frac{\partial}{\partial \boldsymbol{x}}\left(\boldsymbol{f}^T(\boldsymbol{x})\boldsymbol{y}\right) = \boldsymbol{y}^T \frac{\partial \boldsymbol{f}}{\partial \boldsymbol{x}} \\
\frac{\partial}{\partial \boldsymbol{x}}\left(\boldsymbol{y}^T(\boldsymbol{x})\boldsymbol{f}(\boldsymbol{x})\right) &= \frac{\partial}{\partial \boldsymbol{x}}\left(\boldsymbol{f}^T(\boldsymbol{x})\boldsymbol{y}(\boldsymbol{x})\right) = \boldsymbol{y}^T \frac{\partial \boldsymbol{f}}{\partial \boldsymbol{x}} + \boldsymbol{f}^T \frac{\partial \boldsymbol{y}}{\partial \boldsymbol{x}}
\end{aligned}
$$

Hessians

$$\frac{\partial^2}{\partial \boldsymbol{x}^2}(\boldsymbol{x}^T A \boldsymbol{x}) = A + A^T$$

$$\frac{\partial^2}{\partial \boldsymbol{x}^2}(\boldsymbol{x}^T Q \boldsymbol{x}) = 2Q$$

$$\frac{\partial^2}{\partial \boldsymbol{x}^2}([\boldsymbol{x} - \boldsymbol{y}]^T Q [\boldsymbol{x} - \boldsymbol{y}]) = 2Q$$

Jacobians

$$\frac{\partial}{\partial \boldsymbol{x}}(A\boldsymbol{x}) = A$$

$$\frac{\partial}{\partial \boldsymbol{x}}(s(\boldsymbol{x})\boldsymbol{f}(\boldsymbol{x})) = s\frac{\partial \boldsymbol{f}}{\partial \boldsymbol{x}} + \boldsymbol{f}\frac{\partial s}{\partial \boldsymbol{x}}$$

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