Optimization Methods for Systems & Control

1.1: What is Optimization?

- Optimization is ever-present in the world around us –
  - People optimize to obtain maximum efficiencies, to maximize rates-of-return, or to achieve optimized performance of engineering designs, for example
  - Nature optimizes to achieve states of minimum energy
- Optimization can be described as the science of determining the "best" solution to a given problem
  - In particular, we shall examine problems that can be described in mathematical terms
- Problems described mathematically are called *models* – which have applications in almost every branch of science and technology
- Numerical (or computational) optimization was virtually unknown before 1940, so it's still a relatively 'young' mathematical topic. The advent of the digital computer obviously accelerated this development.
- There is no universal optimization algorithm. There are instead many techniques available for solving optimization problems - the choice is largely driven by the nature of the problem and any *a priori* assumptions.

Optimization_Basics

- Most real-world optimization problems cannot be solved!
  - We must make approximations and simplifications to get to a meaningful result
  - One key exception: convex optimization
- Mathematically, optimization is the minimization or maximization of a function subject to constraints
- The general form of a mathematical optimization problem is given below:

$$\begin{array}{ll} \min\limits_{\boldsymbol{u}\in\mathbb{R}^n} & L(\boldsymbol{u}) \\ \text{subject to} & c_i(\boldsymbol{u}) \le b_i \quad i = 1, \dots, m \end{array}$$

  - $\boldsymbol{u} = [u_1, u_2, \dots, u_n]$ vector of optimization variables (unknowns)
  - $L : \mathbb{R}^n \to \mathbb{R}$: objective function
  - $c_i : \mathbb{R}^n \to \mathbb{R}$: constraint functions
- The optimal solution $\boldsymbol{u}^*$ has the smallest value of L among all vectors $\boldsymbol{u}$ that satisfy the constraints c_i

Example: Parameter Optimization

Consider the nonlinear objective function,

$$L(\boldsymbol{u}) = \frac{u_2}{(1 + u_1^2 + u_2^2)}$$

where the two-dimensional parameter vector is given by:

$$\boldsymbol{u} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

- How do we go about finding a solution to the parameter optimization problem given above?
  - Early methods were *ad hoc*, without much theoretical background
    - One such method involved repeated bisection in each of the variables in an attempt to find the minimum
    - Another generates points $\boldsymbol{u}^{(k)}$ at random within a region, selecting the one which gives best function value
    - Generally, these methods are impractical as computational effort goes up rapidly with number of variables
  - One successful such method, however, is known as the *simplex method,* where a simplex is defined as a set of $n+1$ equidistant points in $\mathbb{R}^n$, e.g., an equilateral triangle $(n=2)$ or tetrahedron $(n=3)$. See Figure 1.1

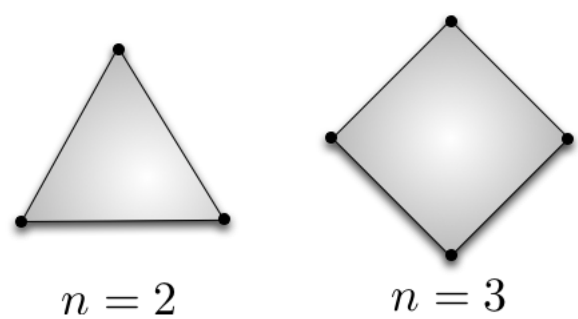

Figure 1.1 Regular simplexes

  - The method follows:
    - Step 1: determine vertex at which function value is greatest

- ◦ Step 2: reflect vertex in centroid of other n vertices to form new simplex
- ◦ Step 3: determine vertex (other than the new one) with largest function value
- ◦ Step 4: repeat Steps 2 - 3 until vertex has been in simplex for M iterations, then contract simplex.

– Figure 1.2 shows the typical progress of a simplex iteration

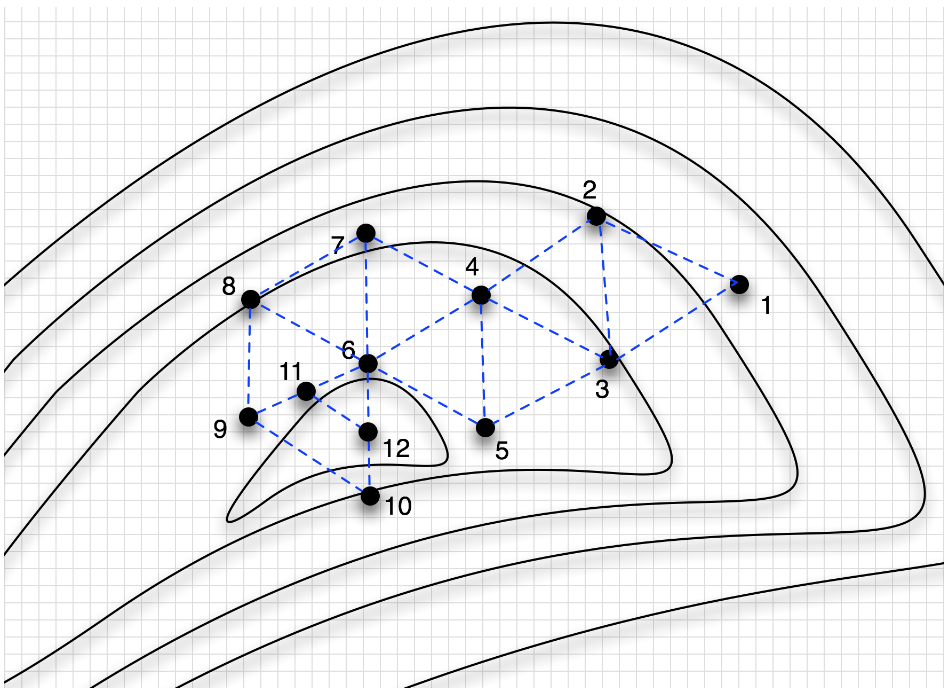

Figure 1.2 Simplex method in two variables

- Analytically, we know from calculus that the extremum of a function will be found at a *stationary point,* that is, where the first order derivatives vanish

  – Using this approach we calculate the first derivative of L with respect to $\boldsymbol{u}$ and find those values of $\boldsymbol{u}$ where the derivative is equal to zero:

$$\frac{\partial L}{\partial \boldsymbol{u}} = 0$$

$\Rightarrow$

$$\frac{\partial L}{\partial u_1} = \frac{-2u_1u_2}{(1+u_1^2+u_2^2)^2} = 0$$

$$\Rightarrow \quad u_1u_2 = 0$$

$$\frac{\partial L}{\partial u_2} = \frac{-2u_2^2}{(1+u_1^2+u_2^2)^2} + \frac{1}{(1+u_1^2+u_2^2)} = 0$$

$$= -2u_2^2 + 1 + u_1^2 + u_2^2 = 0$$

$$\Rightarrow \quad 1 + u_1^2 - u_2^2 = 0$$

This gives the solution:

$$u_1 = 0$$
$$u_2 = \pm 1$$

- Since there are two solutions, we need to determine which corresponds to the minimum. For this simple problem, we may simply substitute back into the objective function to see which is smaller.
- In this case the solution $\boldsymbol{u}^* = [0,\ -1]$ gives $L(\boldsymbol{u}^*) = -0.5$, which is indeed the function's *global minimum*.
- We can easily visualize a function of two variables, so plotting the function surface in this case is also helpful. See Figure 1.3 .
- For most practical problems, it won't always be possible to calculate derivatives explicitly
- Furthermore, there is generally no guarantee that a minimum found is in fact a global minimum

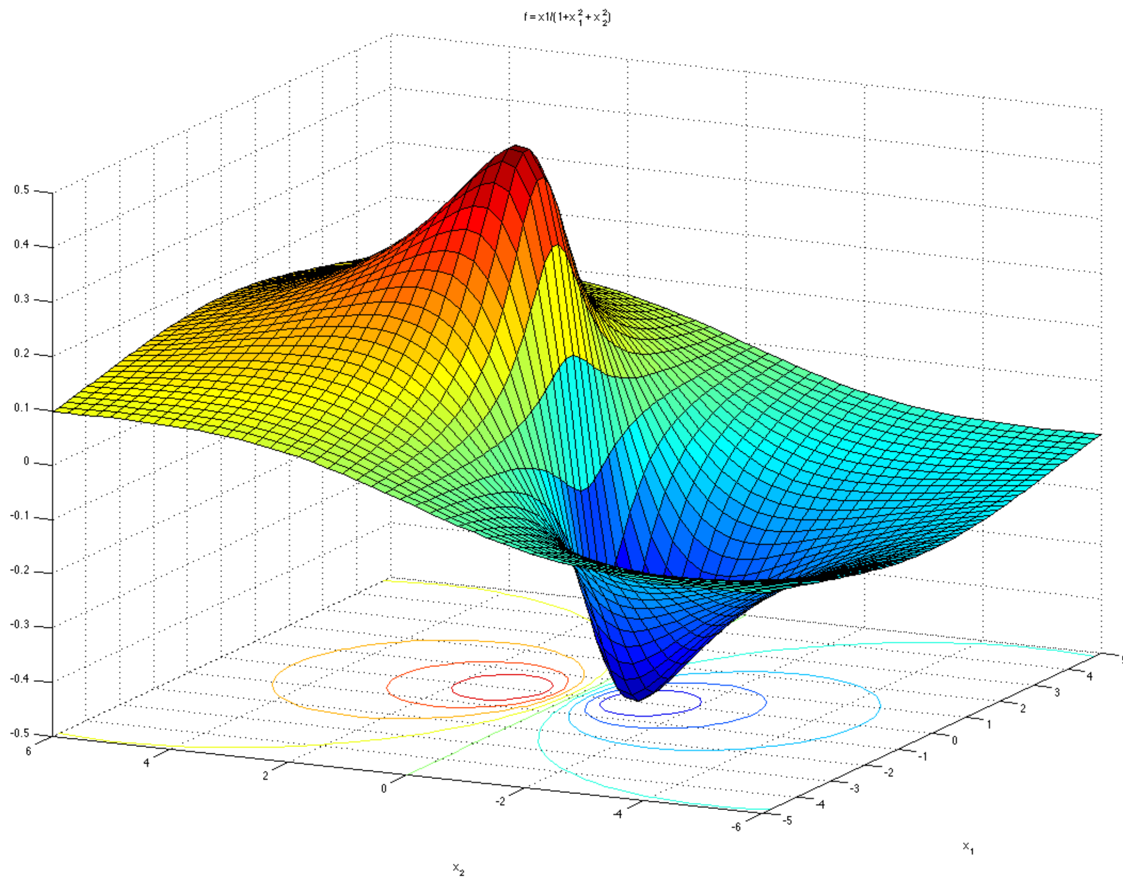

Figure 1.3 3D function surface: $L(\boldsymbol{u}) = \dfrac{u_2}{(1+u_1^2+u_2^2)}$

- As a result, many important optimization methods approximate complicated, nonlinear functions, at least locally, with much simpler well-behaved functions (e.g., quadratic functions)
- Assuring that a miminum is global is a much harder problem and is normally addressed with sophisticated search strategies

1.2: Course Outline

- The main objective of this course is to introduce fundamental optimization methods important to solving engineering problems
- In particular, we'll address the following topics:
  - Mathematics and Linear Systems Review
    - Matrix Algebra
    - Eigenvalues / Eigenvectors
    - Linear Systems: State-Space Representation
  - Parameter Optimization: Unconstrained
    - Necessary and Sufficient Conditions for a Local Minimum
    - Descent Algorithms
    - Line Search Algorithms
    - Newton-like Methods
  - Parameter Optimization: Constrained
    - Equality Constraints
    - Lagrange Multipliers
    - Numerical Algorithms
    - Linear Programming
    - Quadratic Programming
  - Dynamic Systems Optimization
    - Discrete-time Systems: Single and Multi-Stage
    - Two-Point Boundary Value Problems
    - Minimum-Time Problems
  - Linear Quadratic Optimal Control

- ◦ Dynamic Programming
- ◦ Calculus of Variations
- ◦ Minimum Principle
- ◦ Dynamic Root Locus

– Applications

- ◦ Model Predictive Control
- ◦ Linear Quadratic Optimal Control