Dynamic Systems Optimization

Disrete-Time Optimization: Single-Stage Systems

- To begin our invesitigation of optimization of dynamic systems, we'll focus on the most simple dynamic problem ⇒ single-stage discrete-time systems:

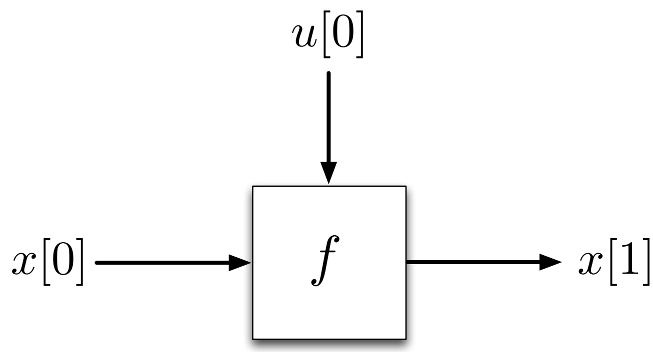

Figure 5.1 Single-stage system

$$\boldsymbol{x}[1] = \boldsymbol{f}\,\{\boldsymbol{x}[0],\, \boldsymbol{u}[0]\}$$

where

$$J = \varphi\,\{\boldsymbol{x}[1]\} + L\,\{\boldsymbol{x}[0],\, \boldsymbol{u}[0]\}$$

- what are the unknowns in this problem?
  - $\boldsymbol{x}[1]$, $\boldsymbol{u}[0]$ ($\boldsymbol{x}[0]$ is a known initial condition)
  - what control do we have?
    - $\boldsymbol{u}[0]$ ($\boldsymbol{x}[1]$ will be fixed by the dynamic constraint once $\boldsymbol{u}[0]$ is known.)

Result: This dynamic optimization problem is no more than a parameter optimization problem with equality constraints.

- So, we'll use the Lagrange multiplier technique that we developed previously in order to attack this problem

$$\bar{J} = \varphi\{\boldsymbol{x}[1]\} + L\{\boldsymbol{x}[0],\, \boldsymbol{u}[0]\} + \boldsymbol{\lambda}^T[1]\{\boldsymbol{f}\,(\boldsymbol{x}[0],\, \boldsymbol{u}[0]) - \boldsymbol{x}[1]\}$$

  – Define: $H = L + \boldsymbol{\lambda}^T \boldsymbol{f}$

$$\bar{J} = \phi\{\boldsymbol{x}[1]\} + H\{\boldsymbol{x}[0],\, \boldsymbol{u}[0],\, \boldsymbol{\lambda}[1]\} - \boldsymbol{\lambda}^T[1]\boldsymbol{x}[1]$$

$$\delta\bar{J} = \left\{\frac{\partial\varphi}{\partial\boldsymbol{x}[1]} - \boldsymbol{\lambda}^T[1]\right\} d\boldsymbol{x}[1] + \frac{\partial H}{\partial\boldsymbol{u}[0]} d\boldsymbol{u}[0] + \frac{\partial H}{\partial\boldsymbol{x}[0]} d\boldsymbol{x}[0]$$

  – to find the minimum value of $\bar{J}$ (and hence J), we set $\delta\bar{J} = 0$

    ◦ by choosing $\boldsymbol{\lambda}^T[1] = \dfrac{\partial\varphi}{\partial\boldsymbol{x}[1]}$, we avoid determining $d\boldsymbol{x}[1]$ in terms of $d\boldsymbol{u}[0]$

    ◦ $\boldsymbol{x}[0]$ given $\Rightarrow\ d\boldsymbol{x}[0] = 0$

    ◦ therefore, a stationary point of $\bar{J}$ will be obtained if

$$\frac{\partial H}{\partial\boldsymbol{u}[0]} = 0$$

    ◦ so, the conditions for a stationary point in this problem are:

1. $\dfrac{\partial \varphi}{\partial \boldsymbol{x}[1]} - \boldsymbol{\lambda}^T[1] = 0$

2. $\dfrac{\partial H}{\partial \boldsymbol{u}[0]} = 0$

3. $\boldsymbol{x}[1] = f\,\{\boldsymbol{x}[0],\, \boldsymbol{u}[0]\}$

- these conditions provide $2n + m$ equations in $2n + m$ unknowns $\Rightarrow$ enough information to solve the problem

Introduction to Multi-Stage Systems

- An obvious extension of the results above is to consider a system which changes dynamically over a series of stages:

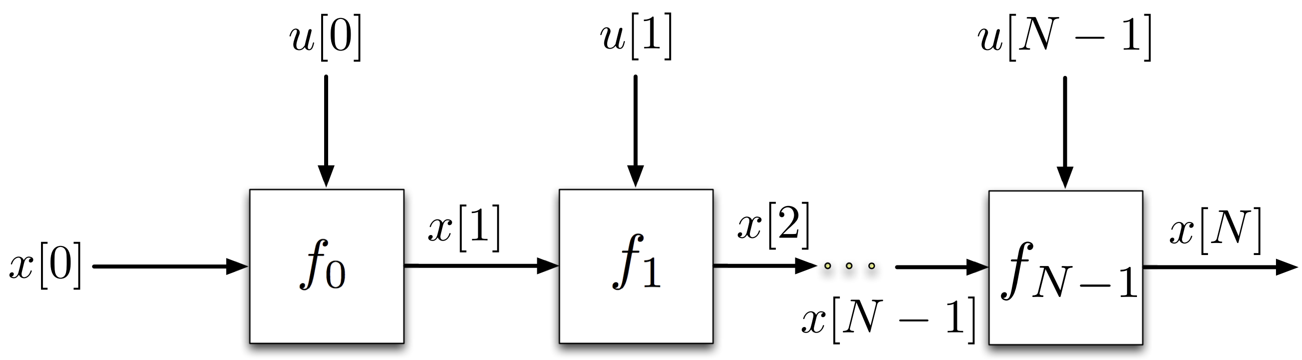

Figure 5.2 Multi-stage system

$$J = \varphi\,\{\boldsymbol{x}[N]\} + \sum_{k=0}^{N-1} L^{(k)}\,\{\boldsymbol{x}[k],\, \boldsymbol{u}[k]\}$$

where φ is defined as the *terminal cost* and L as the *running cost*.

[NOTE: both $\boldsymbol{f}$ and L could change at each stage; for simplicity however, we will assume that this does not happen.]

- Our goal: select the parameters $\{u[k];\ k = 0,\ 1, \ldots,\ N-1\}$ and identify the corresponding parameters $\{x[k];\ k = 1,\ 2, \ldots,\ N\}$ to minimize the cost
- How should we attack this problem? LAGRANGE MULTIPLIERS
  - remember that we now have a set of N vector constraints that must be adjoined to the cost

$$\bar{J} = J + \sum_{k=0}^{N-1} \boldsymbol{\lambda}^T[k+1]\left\{\boldsymbol{f}\ (\boldsymbol{x}[k],\ \boldsymbol{u}[k]) - \boldsymbol{x}[k+1]\right\}$$

  [Note that when constraints are met, $\bar{J} = J$]
  - we'll now extend the definition of H:

$$H[k] = H\{\boldsymbol{x}[k],\ \boldsymbol{u}[k],\ \boldsymbol{\lambda}[k]\} = L\{\boldsymbol{x}[k],\ \boldsymbol{u}[k]\}$$

$$+\boldsymbol{\lambda}^T[k+1]f\ \{\boldsymbol{x}[x],\ \boldsymbol{u}[k]\}$$

$$\Rightarrow \bar{J} = \varphi\{\boldsymbol{x}[N]\} + H[0] + \sum_{k=1}^{N-1}\left\{H[k] - \boldsymbol{\lambda}^T[k]\boldsymbol{x}[k]\right\} - \boldsymbol{\lambda}^T[N]\boldsymbol{\lambda}[N]$$

  - and now, since we've introduced Lagrange multipliers, we can take the first variation of $\bar{J}$ treating $\boldsymbol{u}[k]$ and $\boldsymbol{x}[k]$ as if they were independent:

$$\delta\bar{J} = \frac{\partial\varphi}{\partial\boldsymbol{x}[N]}d\boldsymbol{x}[N] + \sum_{k=1}^{N-1}\left\{\frac{\partial H[k]}{\partial\boldsymbol{x}[k]} - \boldsymbol{\lambda}^T[k]\right\}d\boldsymbol{x}[k]$$

$$+\sum_{k=1}^{N-1}\frac{\partial H[k]}{\partial\boldsymbol{u}[k]}d\boldsymbol{u}[k] + \frac{\partial H[0]}{\partial\boldsymbol{u}[0]}d\boldsymbol{u}[0]$$

$$+\frac{\partial H[0]}{\partial\boldsymbol{x}[0]}d\boldsymbol{x}[0] - \boldsymbol{\lambda}^T[N]d\boldsymbol{x}[N]$$

◦ let's examine this expression and determine the conditions for a stationary point ...

- To develope necessary conditions for a stationary point, let's first simplify the expression
  - remember, $\boldsymbol{\lambda}^T[k]$ is arbitrary, so we an choose it as we wish

$$\boldsymbol{\lambda}^T[N] = \frac{\partial \varphi}{\partial \boldsymbol{x}[N]} \qquad \boldsymbol{\lambda}^T[k] = \frac{\partial H[k]}{\partial \boldsymbol{x}[k]},\ k = 1, 2, \ldots, N-1$$

$$\Rightarrow \delta \bar{J} = \sum_{k=0}^{N-1} \frac{\partial H[k]}{\partial \boldsymbol{u}[k]} d\boldsymbol{u}[k] + \frac{\partial H[0]}{\partial \boldsymbol{x}[0]} d\boldsymbol{x}[0]$$

  - if initial conditions are given, $d\boldsymbol{x}[0] = 0$ and $\delta \bar{J}$ simplifies to

$$\delta \bar{J} = \sum_{k=0}^{N-1} \frac{\partial H[k]}{\partial \boldsymbol{u}[k]} d\boldsymbol{u}[k]$$

  - thus, a necessary condition for a stationary point is:

$$\frac{\partial H[k]}{\partial \boldsymbol{u}[k]} = 0,\ k = 0, 1, \ldots, N-1$$

- Now that we've developed this stationarity condition, how do we solve the problem?
  - what are the unknowns involved?

$$\begin{aligned} &\boldsymbol{u}[k];\ k = 0, 1, \ldots, N-1 \quad (m * N) \\ &\boldsymbol{x}[k];\ k = 1, 2, \ldots, N \quad (n * N) \\ &\boldsymbol{\lambda}[k];\ k = 0, , 1, \ldots, N \quad (n * N) + n \end{aligned}$$

  - and what are the equations available to solve for these unknowns?

$$
\begin{aligned}
(m * N) \qquad 0 &= \frac{\partial H[k]}{\partial \boldsymbol{u}[k]} = \frac{\partial L}{\partial \boldsymbol{u}[k]} + \boldsymbol{\lambda}^T[k+1]\frac{\partial \boldsymbol{f}}{\partial \boldsymbol{u}[k]}, \; k = 0, \ldots, N-1 \\
(n * N) \quad \boldsymbol{\lambda}^T[k] &= \frac{\partial H[k]}{\partial \boldsymbol{x}[k]} \quad k = 0, \ldots, N-1 \\
n \quad \boldsymbol{\lambda}^T[N] &= \frac{\partial \varphi}{\partial \boldsymbol{x}[N]}
\end{aligned}
$$

$$
(n * N) \;\; \boldsymbol{x}[k+1] = \boldsymbol{f}\,\{\boldsymbol{x}[k], \boldsymbol{u}[k]\} = \left\{\frac{\partial H[k]}{\partial \boldsymbol{\lambda}^T[k+1]}\right\}, \; k = 0, \ldots, N-1
$$

– so, we have $N(2n+m)+n$ equations in $N(2n+m)+n$ unknowns $\Rightarrow$ solution exists if the equations are independent

Example: Discrete-time Brachistochrone Problem

DESCRIPTION: a bead slides on a wire in a constant gravity field. The inclination angle, θ, may be changed at time intervals $\triangle t$.

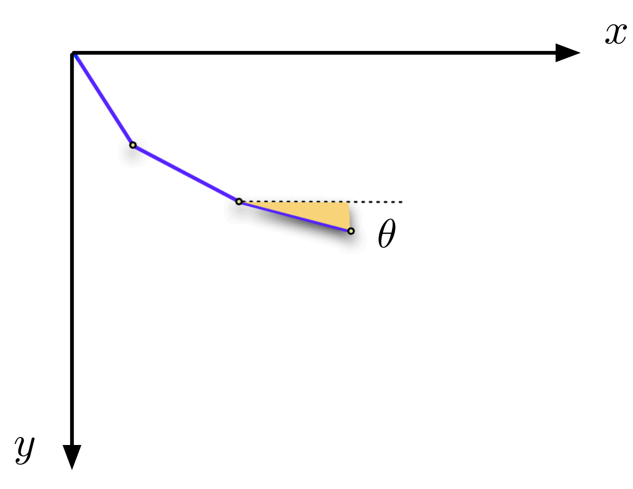

Figure 5.3 Brachistochrone Problem

Find $\theta[k]$ for $k = 0, 1, \ldots, N-1$ to maximize horizontal distance x at time t_f with $\triangle t = t_f/N$.

SOLUTION $\Rightarrow$

1. The velocity and position along the wire at each corner point can be identified from:

$$v[k+1] = v[k] + g\sin\theta[k]\triangle t$$

$$\ell[k+1] = \ell[k] + v[k]\triangle t + \frac{1}{2}g\sin\theta[k]\triangle t^2$$

(a) N can be introduced into the problem by normalizing the variables:

$$\tilde{v}[k] = \frac{v[k]}{gt_f} \qquad \tilde{\ell}[k] = \frac{\ell[k]}{gt_f^2}$$

$$\Rightarrow \tilde{v}[k+1] = \tilde{v}[k] + \frac{1}{N}\sin\theta[k]$$

$$\triangle\tilde{\ell}[k] = \tilde{\ell}[k+1] - \tilde{\ell}[k] = \frac{1}{N}\tilde{v}[k] + \frac{1}{2N^2}\sin\theta[k]$$

2. Using $\triangle\tilde{\ell}$, the x and y positions of the corner points can be identified:

$$\tilde{x}[k+1] = \tilde{x}[k] + \triangle\tilde{\ell}[k]\cos\theta[k]$$

$$\tilde{y}[k+1] = \tilde{y}[k] + \triangle\tilde{\ell}[k]\sin\theta[k]$$

3. The results above define the dynamics of the problem:

$$\text{constrained variables}: v,\ x,\ y \qquad \{\boldsymbol{x}\text{ vector}\}$$

$$\text{control variables}: \quad \theta \qquad \{\boldsymbol{u}\text{ vector}\}$$

4. What about the cost?

$$J = -\tilde{x}[N]$$

(a) why minus? we want to maximize the final horizontal position, $\tilde{x}$

(b) using our previous definitions, $\phi = -\tilde{x}[N]$ and $L = 0$

$$\Rightarrow J = -x[N] + \sum_{k=1}^{N-1}\left\{\lambda_v[k+1]\left\{\tilde{v}[k] + \frac{1}{N}\sin\theta[k]\right\}\right.$$

$$\left. +\lambda_x[k+1]\left\{\tilde{x}[k] + \triangle\tilde{\ell}[k]\cos\theta[k]\right\}\right.$$

$$+\lambda_y[k+1]\left\{\tilde{y}[k] + \triangle\tilde{\ell}[k]\sin\theta[k]\right\}\Big\}$$

$$-\sum_{k=1}^{N}\left\{\lambda_v[k]\tilde{v}[k] + \lambda_x[k]\tilde{x}[k] + \lambda_y[k]\tilde{y}[k]\right\}$$

i. why three λ's? because we have three "states"

5. Equations required to solve this problem:

(a) Dynamic constraints $(3 * N)$:

$$\tilde{v}[k+1] = \tilde{v}[k] + \frac{1}{N}\sin\theta[k]; \quad \tilde{v}[0] = 0$$

$$\tilde{x}[k+1] = \tilde{x}[k] + \triangle\tilde{\ell}[k]\cos\theta[k]; \quad \tilde{x}[0] = 0$$

$$\tilde{y}[k+1] = \tilde{y}[k] + \triangle\tilde{\ell}[k]\sin\theta[k]; \quad \tilde{y}[0] = 0$$

(b) Geometric constraint (N):

$$\triangle\ell[k] = \frac{1}{N}v[k] + \frac{1}{2N^2}\sin\theta[k]$$

(c) Costate equations:

$$\boldsymbol{\lambda}^T[k] = \frac{\partial H[k]}{\partial \boldsymbol{x}[k]}$$

$$\lambda_v[k] = \lambda_v[k+1] + \lambda_x[k+1]\left(\frac{1}{N}\right)\cos\theta[k] + \lambda_y[k+1]\left(\frac{1}{N}\right)\sin\theta[k]$$

$$\lambda_x[k] = \lambda_x[k+1]$$

$$\lambda_y[k] = \lambda_y[k+1]$$

$$\lambda_v[N] = 0 \qquad \lambda_x[N] = -1 \qquad \lambda_y[N] = 0$$

(d) Optimality condition:

$$\frac{\partial H[k]}{\partial \boldsymbol{u}[k]} = 0$$

$$\left(\frac{1}{N}\right)\lambda_v[k+1]\cos\theta[k]-\lambda_x[k+1]\triangle\tilde{\ell}[k]\sin\theta[k]$$

$$+\lambda_x[k+1]\left(\frac{1}{2N^2}\right)\cos^2\theta[k]+\lambda_y[k+1]\triangle\tilde{\ell}[k]\cos\theta[k]$$

$$+\lambda_y[k+1]\left(\frac{1}{2N^2}\right)\sin\theta[k]\cos\theta[k]=0$$

$$\left(\frac{1}{N}\right)\lambda_v[k+1]\cos\theta[k]-\triangle\tilde{\ell}[k]\sin\theta[k]+\left(\frac{1}{2N^2}\right)\cos^2\theta[k]=0$$

i. Note that $\theta[k]$ is a function of $\lambda_v[k+1]$ and $\tilde{v}[k]$

- Even in this relatively simple example, the number of equations to be solved suggests that the solution process will be extremely complicated. The existence of non-linear relationships makes the process even worse! So what can we do? Develop numerical methods.

- What do we have available for use?

  – a set of difference equations that develop forward in time; i.e., the *state equations*:

$$\boldsymbol{x}[k+1]=\boldsymbol{f}\left\{\boldsymbol{x}[k],\,\boldsymbol{u}[k]\right\}\quad \boldsymbol{x}[0]=\boldsymbol{x}_0$$

  – a set of difference equations that develop backward in time; i.e., the *co-state equations*:

$$\boldsymbol{\lambda}[k]=\left\{\frac{\partial \boldsymbol{f}}{\partial \boldsymbol{x}[k]}\right\}^T\boldsymbol{\lambda}[k+1]+\left\{\frac{\partial L}{\partial \boldsymbol{x}[k]}\right\}^T$$

$$\boldsymbol{\lambda}[N]=\left\{\frac{\partial \varphi}{\partial \boldsymbol{x}[N]}\right\}^T$$

- These two difference equations are coupled and define a TWO-POINT BOUNDARY VALUE PROBLEM
  - the boundary conditions are split between the end points
  - once $\boldsymbol{\lambda}[k+1]$ and $\boldsymbol{x}[k]$ are known, $\boldsymbol{u}[k]$ can be computed using the algebraic equations defined by the optimality condition

$$\frac{\partial H[k]}{\partial \boldsymbol{u}[k]} = 0$$

- So, solving a two-point boundary value problem provides a means of identifying the solution of our dynamic optimization problem
  - curiosity: even though we don't care about $\boldsymbol{\lambda}[k]$, we must calculate it to identify $\boldsymbol{u}[k]$

Solution methods for two-point boundary value problems

[A] Shooting Method

1. Guess $\boldsymbol{\lambda}[0]$
2. Compute $\boldsymbol{\lambda}[1]$ using the co-state difference equations
3. Compute $\boldsymbol{u}[0]$ using $\boldsymbol{x}[0]$, $\boldsymbol{\lambda}[1]$, and the stationarity conditions
4. Compute $\boldsymbol{x}[1]$ using the state difference equations
5. Continue steps (2) through (4) up to time N
6. If $\boldsymbol{\lambda}^T[N] = \dfrac{\partial \varphi}{\partial \boldsymbol{x}[N]}$, the solution is correct; but if $\boldsymbol{\lambda}^T[N] \neq \dfrac{\partial \varphi}{\partial \boldsymbol{x}[N]}$, a new $\boldsymbol{\lambda}[0]$ must be chosen and steps (2) through (5) repeated
   (a) how do you choose $\boldsymbol{\lambda}[0]$? It's an art.
   (b) you might try varying each element of $\boldsymbol{\lambda}[0]$ individually to observe the sensitivity of the results to these changes, and then use this information to select the new $\boldsymbol{\lambda}[0]$

- Problem $\Rightarrow$ the process is very sensitive to the initial guess; the solution may not converge unless the first guess is pretty accurate

[B] Gradient Method

1. Guess all of the control variables $\{\boldsymbol{u}[k];\ k = 0,\ 1,\ \ldots,\ N-1\}$
2. Compute $\boldsymbol{x}[k]$ using the state difference equations
3. Compute $\boldsymbol{\lambda}[k]$ backwards using the co-state difference equation
4. Stop when all ${}^{\partial H[k]}/_{\partial \boldsymbol{u}[k]}$ are sufficiently close to zero
   (a) Why? We want to set $\delta \bar{J} = \sum_{k=0}^{N-1} {}^{\partial H[k]}/_{\partial \boldsymbol{u}[k]} d\boldsymbol{u}[k] = 0$ which can only happen when ${}^{\partial H[k]}/_{\partial \boldsymbol{u}[k]} = 0$
   (b) A useful criterion is the following RMS-type measurement:
   $$\left[\left(\frac{1}{N}\right)\sum_{k=0}^{N-1}\left\{\frac{\partial H[k]}{\partial \boldsymbol{u}[k]}\right\}\left\{\frac{\partial H[k]}{\partial \boldsymbol{u}[k]}\right\}^T\right]^{\frac{1}{2}} < \epsilon$$
5. If the stopping criterion is not satisfied, another guess at the control variables must be made
   (a) This can be done during step (3) by setting
   $$\boldsymbol{u}_{NEW}[k] = \boldsymbol{u}[k] - K\frac{\partial H[k]}{\partial \boldsymbol{u}[k]} \quad \text{for some } K > 0$$
   (b) Why does this work?
      i. if ${}^{\partial H}/_{\partial \boldsymbol{u}} > 0$, then $d\boldsymbol{u} < 0$ will produce $\delta \bar{J} < 0$ and hence $\bar{J}$ will decrease
      ii. if ${}^{\partial H}/_{\partial \boldsymbol{u}} < 0$, then $d\boldsymbol{u} > 0$ will produce $\delta J \bar{<} 0$ and hence $\bar{J}$ will decrease

- Problem $\Rightarrow$ just like the parameter optimization problem, the selection of K here is tricky

Continuous-Time Optimization: Fixed-time, No Terminal Constraints

- Dynamic optimization problems for continuous-time systems are problems of the *calculus of variations*
  - can be considered as limiting cases of discrete-time systems where the time interval between steps becomes infinitesimally small
- Consider the system described by the non-linear differential equations:

$$\dot{\boldsymbol{x}} = \boldsymbol{f}\,(\boldsymbol{x},\, \boldsymbol{u},\, t)\,; \quad x(t_0) = x_0; \quad t_0 \le t \le t_f$$

and performance (cost) function:

$$J = \varphi\left\{\boldsymbol{x}(t_f),\, t_f\right\} + \int_{t_0}^{t_f} L\left\{\boldsymbol{x}(t),\, \boldsymbol{u}(t),\, t\right\} dt$$

- GOAL: minimize J by selecting $\boldsymbol{u}(t)$ (and $\boldsymbol{x}(t)$)
- We first define the augmented cost function $\bar{J}$ by adjoining the system differential equations to J with Lagrange multipliers:

$$\bar{J} = \varphi\left\{\boldsymbol{x}(t_f),\, t_f\right\} + \int_{t_0}^{t_f} \left\{L\left[\boldsymbol{x},\, \boldsymbol{u},\, t\right] + \boldsymbol{\lambda}^T\left(f\left[\boldsymbol{x},\, \boldsymbol{u},\, t\right] - \dot{\boldsymbol{x}}\right)\right\} dt$$

- Define the *Hamiltonian* function,

$$H = L + \boldsymbol{\lambda}^T \boldsymbol{f}$$

  - NOTE:

$$\int_{t_0}^{t_f} -\boldsymbol{\lambda}^T \dot{\boldsymbol{x}}\, dt = \lambda^T \boldsymbol{x}\Big|_{t_0}^{t_f} + \int_{t_0}^{t_f} \dot{\boldsymbol{\lambda}}^T \boldsymbol{x}\, dt$$

where we have used inegration by parts →

$$dv = \dot{\boldsymbol{x}}\, dt$$
$$v = \boldsymbol{x}$$

$$du = -\dot{\boldsymbol{\lambda}}^T dt$$
$$u = -\boldsymbol{\lambda}^T$$

– this gives ⇒

$$\bar{J} = \varphi\left\{\boldsymbol{x}(t_f),\, t_f\right\} + \boldsymbol{\lambda}^T(t_0)\boldsymbol{x}(t_0) - \boldsymbol{\lambda}^T(t_f)\boldsymbol{x}(t_f) + \int_{t_0}^{t_f}\left(H + \dot{\boldsymbol{\lambda}}^T\boldsymbol{x}\right)dt$$

- Now, vary parameters,

$$\delta\bar{J} = \left.\frac{\partial\varphi}{\partial\boldsymbol{x}(t_f)}\right|\delta\boldsymbol{x}(t_f) + \boldsymbol{\lambda}^T(t_0)\delta\boldsymbol{x}(t_0) - \boldsymbol{\lambda}^T(t_f)\delta\boldsymbol{x}(t_f)$$
$$+ \int_{t_0}^{t_f}\left\{\left(\frac{\partial H}{\partial\boldsymbol{x}} + \dot{\boldsymbol{\lambda}}^T\right)\delta\boldsymbol{x} + \frac{\partial H}{\partial\boldsymbol{u}}\delta\boldsymbol{u}\right\}dt$$

– we choose the Lagrange multipliers such that the coefficients of $\delta\boldsymbol{x}$ vanish:

$$\frac{\partial H}{\partial\boldsymbol{x}} + \dot{\boldsymbol{\lambda}}^T = 0$$

– by the boundary conditions at t_f we have :

$$\frac{\partial\varphi}{\partial\boldsymbol{x}(t_f)} - \boldsymbol{\lambda}^T(t_f) = 0$$

– finally, for an extremum δJ must be zero for arbitrary $\delta\boldsymbol{u}$, giving:

$$\frac{\partial H}{\partial\boldsymbol{u}} = 0$$

– the three equations above are the EULER-LAGRANGE EQUATIONS of the calculus of variations

- So, to solve for the optimal control $\boldsymbol{u}(t)$ that minimizes the performance function J, we need to solve the following differential equations:

$$\dot{\boldsymbol{x}} = f(\boldsymbol{x},\, \boldsymbol{u},\, t)$$

$$\dot{\boldsymbol{\lambda}} = -\left(\frac{\partial \boldsymbol{f}}{\partial \boldsymbol{x}}\right)^T \boldsymbol{\lambda} - \left(\frac{\partial L}{\partial \boldsymbol{x}}\right)^T$$

– where $\boldsymbol{u}(t)$ is determined by

$$\frac{\partial H}{\partial \boldsymbol{u}} = 0 \quad \text{or} \quad \left(\frac{\partial \boldsymbol{f}}{\partial \boldsymbol{u}}\right)^T \boldsymbol{\lambda} + \left(\frac{\partial L}{\partial \boldsymbol{u}}\right)^T = 0$$

- The boundary conditions are again split - some given for $t = t_0$ and some given for $t = t_f$:

$$\boldsymbol{x}(t_0) = \boldsymbol{x}_0$$

$$\boldsymbol{\lambda}(t_f) = \left(\frac{\partial \varphi}{\partial \boldsymbol{x}}\right)^T$$

- So, again, as in the discrete-time multi-stage problem, we're faced with a two-point boundary value problem

Example: Minimum Energy Room Temperature Control

- PROBLEM STATEMENT: We desire to heat a room using the least possible energy. Let T be the temperature in the room, T_a the ambient temperature outside (assumed constant) and $h(t)$ the rate of heat supplied to the room.
- Simplified dynamic equations may be written,

$$\dot{T} = -a\left(T - T_a\right) + bh$$

for some constants a and b which depend on room design and construction.

– we define the state as

$$x(t) \equiv T(t) - T_a(t)$$

and the control input as

$$u(t) \equiv h(t)$$

– thus, we may express the scalar state equation as:

$$\dot{x} = -ax + bu$$

- In order to control the room room temperature on the time interval $[t_0,\, t_f]$ with least energy, we define the cost function,

$$J = \frac{1}{2}k\left(x(t_f) - x_d\right)^2 + \frac{1}{2}\int_{t_0}^{t_f} u(t)^2 dt$$

for some weighting, k

- SOLUTION:

– Find augmented cost function, $\bar{J}$

$$\bar{J} = J + \int_{t_0}^{t_f} \boldsymbol{\lambda}^T \left(\boldsymbol{f} - \dot{\boldsymbol{x}}\right) dt$$

– Satisfy Euler-Lagrange equations:

$$\frac{\partial \varphi}{\partial t_f} - \lambda(t_f) = 0$$

$$\frac{\partial H}{\partial \boldsymbol{x}} + \dot{\boldsymbol{\lambda}} = 0$$

$$\frac{\partial H}{\partial u} = 0$$

and state equation:

$$\dot{x} = -ax + bu$$

∘ recall,

$$\begin{aligned} H &= L + \lambda f \\ &= \frac{1}{2}u^2 + \lambda(-ax + bu) \end{aligned}$$

◦ so we get →

$$\dot{\lambda} = a\lambda \qquad \lambda(t_f) = k\left[x(t_f) - x_d\right]$$
$$\dot{x} = -ax + bu \qquad x(t_0) = 0$$

$$u + \lambda b = 0 \qquad \Rightarrow \quad u = -b\lambda$$

– Substituting for u,

$$\dot{x} = -ax - b^2\lambda$$
$$\dot{\lambda} = a\lambda$$

or, writing in matrix form,

$$\begin{bmatrix} \dot{x} \\ \dot{\lambda} \end{bmatrix} = \begin{bmatrix} -a & -b^2 \\ 0 & a \end{bmatrix} \begin{bmatrix} x \\ \lambda \end{bmatrix}$$
$$\dot{z} = Az$$

– Solving,

$$z = e^{At} z(0)$$

$$\begin{aligned} x(t) &= e^{-at}x(0) - \frac{b^2}{a} \cdot \frac{1}{2}\left(e^{at} - e^{-at}\right)\lambda(0) \\ &= e^{-at}x(0) - \frac{b^2}{a}\sinh(at)\lambda(0) \\ \lambda(t) &= e^{at}\lambda(0) \end{aligned}$$

- But, we don't know $\lambda(0)$

– however,

$$\begin{aligned} \lambda(0) &= \lambda(t_f)e^{-at_f} \\ &= k\left[x(t_f) - x_d\right]e^{-at_f} \end{aligned}$$

- but where to we get e^{-at_f}?

$$x(t_f) = -\frac{b^2}{2a}\left(e^{at_f} - e^{-at_f}\right)e^{-at_f}\lambda(t_f)$$
$$= -\frac{b^2}{2a}\left(1 - e^{-2at_f}\right)\lambda(t_f)$$

  and

$$\lambda(t_f) = k\left[x(t_f) - x_d\right]$$

- Solving,

$$x(t_f) = \frac{x_d}{1 + \frac{ae^{at_f}}{b^2 k \sinh(at_f)}}$$

$$\lambda(t_f) = \frac{-2x_d ak}{2a - b^2 k\left(1 - e^{-2at_f}\right)}$$

$\Rightarrow$

$$u(t) = \frac{x_d abke^{at}}{ae^{at_f} + b^2 k \sinh(at_f)}$$

$$x(t) = \frac{x_d b^2 k \sinh(at)}{ae^{at_f} + b^2 k \sinh(at_f)}$$

- Putting some numbers to the example, let

$$a = 0.4$$
$$b = 0.8$$

  - assume $t_0 = 0$ and $t_f = 20$
  - further assume $T_a = 5$, $T_0 = T_a$, and define a target temperature of $T_d = 20\,(x_d = 15)$
  - the following plots show resulting temperature and control effort plots for weightings $k = 1,\ 2,\ 4,\ 16$

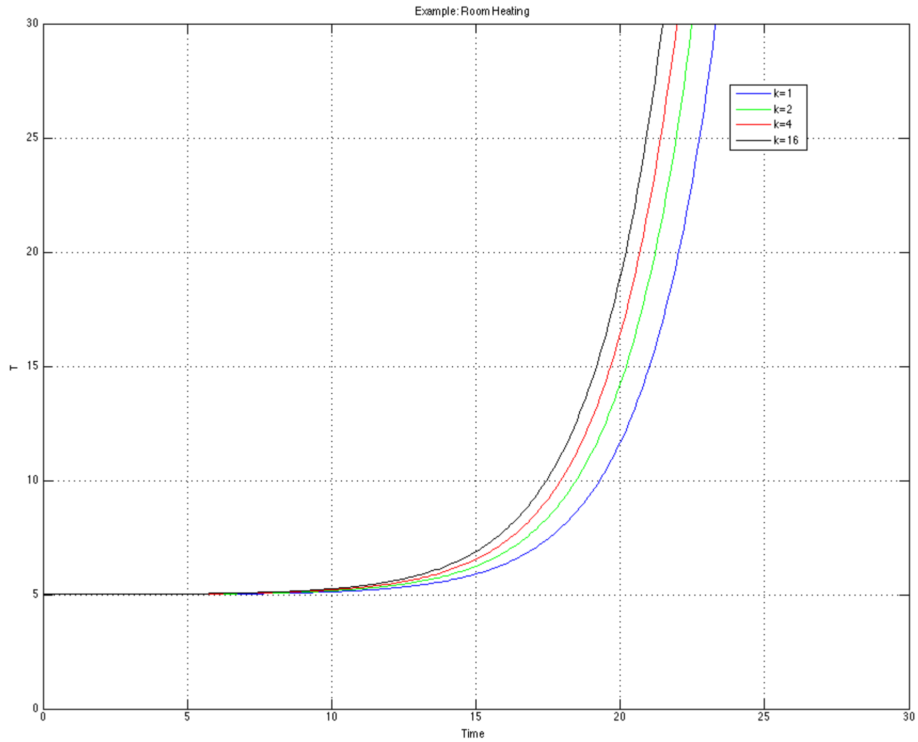

Figure 5.4 Room Heating Example: Temperature T

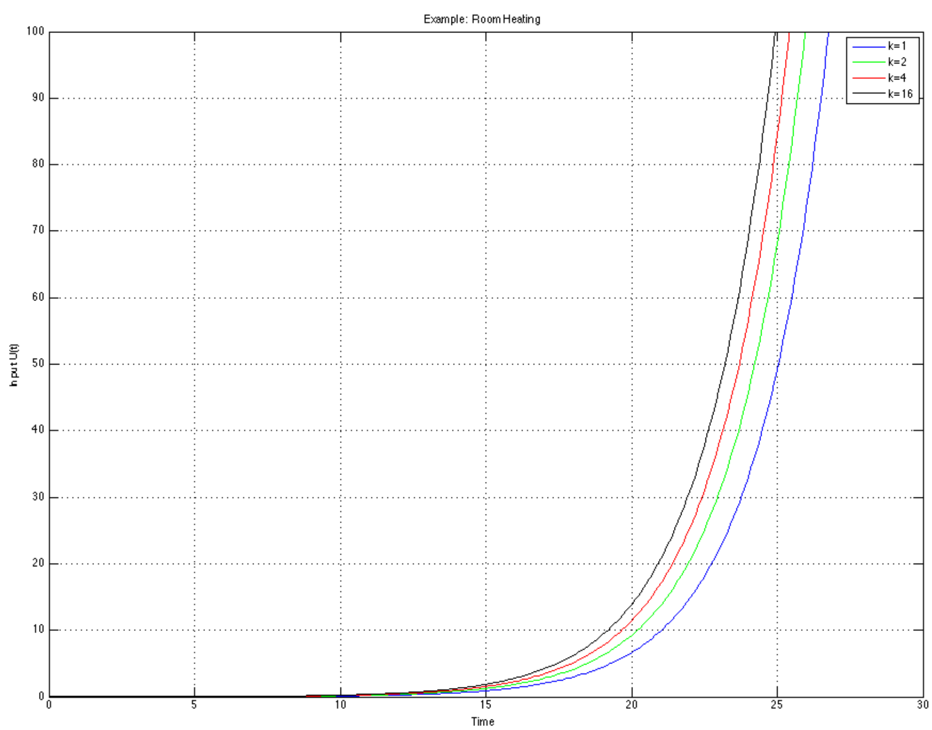

Figure 5.5 Room Heating Example: Control Input U

Hamilton's Principle

- In mechanics, the motion of a conservative system from time t_0 to t_f is such that the integral

$$J = \int_{t_0}^{t_f} (T - V)\, dt$$

has a stationary value.

- Here we define

$$\begin{aligned} T &= \text{kinetic energy} \\ V &= \text{potential energy} \\ \boldsymbol{x} &= [q_1,\, q_2, \ldots, q_n] \Rightarrow \text{vector of generalized coordinates} \\ \boldsymbol{u} &= [\dot{q}_1,\, \dot{q}_2, \ldots,\, \dot{q}_n] \Rightarrow \text{vector of generalized velocities} \end{aligned}$$

  - A dynamic constraint is then given by:

$$\dot{\boldsymbol{x}} = \boldsymbol{u}$$

- The Hamiltonian is then

$$\begin{aligned} H &= (T - V) + \boldsymbol{\lambda}^T \boldsymbol{u} \\ &= L + \boldsymbol{\lambda}^T \boldsymbol{u} \end{aligned}$$

- Once again the necessary conditions for stationarity may be written,

$$\begin{aligned} \frac{\partial H}{\partial \boldsymbol{u}} &= 0 \\ \dot{\boldsymbol{\lambda}}^T &= -\frac{\partial H}{\partial \boldsymbol{x}}, \quad \boldsymbol{\lambda}(t_f) = \boldsymbol{0} \\ \dot{\boldsymbol{x}} &= \boldsymbol{u}, \quad \boldsymbol{x}(0) = \boldsymbol{x}_0 \end{aligned}$$

- Now,

$$\frac{d}{dt}\left\{\frac{\partial H}{\partial \boldsymbol{u}}\right\} = 0$$

$$\frac{\partial H}{\partial \boldsymbol{u}} = \frac{\partial L}{\partial \boldsymbol{u}} + \boldsymbol{\lambda}^T$$

- So we can write,

$$\begin{aligned} \frac{d}{dt}\left\{\frac{\partial H}{\partial \boldsymbol{u}}\right\} &= \frac{d}{dt}\left\{\frac{\partial L}{\partial \boldsymbol{u}}\right\} + \dot{\boldsymbol{\lambda}}^T \\ &= \frac{d}{dt}\left\{\frac{\partial L}{\partial \boldsymbol{u}}\right\} - \frac{\partial L}{\partial \boldsymbol{x}} = 0 \end{aligned}$$

$\Rightarrow$ Equations of Motion

- Consider,

$$\frac{dH}{dt} = \frac{\partial L}{\partial t} + \frac{\partial H}{\partial \boldsymbol{x}}\frac{\partial \boldsymbol{x}}{\partial t} + \frac{\partial H}{\partial \boldsymbol{u}}\frac{\partial \boldsymbol{u}}{\partial t} + \dot{\boldsymbol{\lambda}}^T \boldsymbol{f}$$

  - at the optimal solution,

$$\frac{\partial H}{\partial \boldsymbol{u}} = 0$$

$$\frac{\partial H}{\partial \boldsymbol{x}} = -\dot{\boldsymbol{\lambda}}^T$$

  - if J is not an explicit function of t, then

$${}^{\partial L}\!/_{\partial t} = 0 \quad \Rightarrow \quad {}^{dH}\!/_{dt} = 0 \quad \Rightarrow \quad H(t) = \text{constant}$$

Continuous-Time Optimization: Fixed-Time, Terminal Constraints

- We'll now look at a Calculus of Variations approach to solving a slightly more difficult optimization problem: one with constraints at the terminal time

PROBLEM:

Dynamic System	$\dot{\boldsymbol{x}} = f(\boldsymbol{x}, \boldsymbol{u}, t)$	$\boldsymbol{x}(t_0) = \boldsymbol{x}_0$
Cost	$J = \varphi\left\{\boldsymbol{x}\left(t_f\right)\right\} + \int_{t_0}^{t_f} L\{\boldsymbol{x}, \boldsymbol{u}, t\}\, dt$	
Terminal Constraints	$\boldsymbol{\psi}\left\{\boldsymbol{x}\left(t_f\right)\right\} = \boldsymbol{c}$	$\boldsymbol{\psi}$ is $q \times 1,\ q \le n$
Goal	select $\boldsymbol{u}(t)$ to minimize J subject to terminal constraints	

SOLUTION:

- As before, we'll adjoin the constraints to the cost function using Lagrange multipliers
  - the difference is that we now have two types of constraint:
    1. intermediate system constraints $\Rightarrow\ \dot{\boldsymbol{x}} = f(\boldsymbol{x}, \boldsymbol{u}, t)$
    2. terminal constraints $\Rightarrow\ \boldsymbol{\psi}\left\{\boldsymbol{x}(t_f)\right\} = \boldsymbol{c}$

$$\Rightarrow \quad \bar{J} = \varphi\left\{\boldsymbol{x}(t_f)\right\} + \boldsymbol{\nu}^T\left\{\boldsymbol{\psi}\left[\boldsymbol{x}(t_f)\right] - \boldsymbol{c}\right\} + \int_{t_0}^{t_f} \left\{L(\boldsymbol{x}, \boldsymbol{u}, t) + \boldsymbol{\lambda}^T(\boldsymbol{f} - \dot{\boldsymbol{x}})\right\} dt$$

- Note here that φ is an aggregate function of the final states that we want to make "small" in some sense; $\boldsymbol{\psi}$ is a vector function of the final states that we want "fixed" at $\boldsymbol{c}$.

- Furthermore, if we define $\Phi\left\{\boldsymbol{x}(t_f)\right\} = \varphi + \boldsymbol{\nu}^T\{\boldsymbol{\psi} - \boldsymbol{c}\}$, the problem looks identical to the one examined previously and can be solved in the same way

– there are some mathematical distinctions that must be made, however

1. the system must be controllable so that it is possible to reach the specified terminal constraints
2. $\delta \boldsymbol{u}$ is no longer arbitrary since the only admissible values for $\delta \boldsymbol{u}$ are ones which will ensure that the terminal constraints remain satisfied

- Let's now step through the solution process:

$$\bar{J} = \Phi\left\{\boldsymbol{x}(t_f)\right\} + \int_{t_0}^{t_f} \left\{H(\boldsymbol{x},\, \boldsymbol{u},\, t) - \boldsymbol{\lambda}^T \dot{\boldsymbol{x}}\right\} dt$$

– integrating by parts yields:

$$\bar{J} = \Phi\left\{\boldsymbol{x}(t_f)\right\} + \boldsymbol{\lambda}_0^T \boldsymbol{x}_0 - \boldsymbol{\lambda}_f^T \boldsymbol{x}_f + \int_{t_0}^{t_f} \left\{H(\boldsymbol{x},\, \boldsymbol{u},\, t) + \dot{\boldsymbol{\lambda}}^T \boldsymbol{x}\right\} dt$$

– taking the first variation of $\bar{J}$ yields:

$$\delta \bar{J} = \left\{\frac{\partial \Phi}{\partial \boldsymbol{x}(t_f)} - \boldsymbol{\lambda}_f^T\right\} \delta \boldsymbol{x}(t_f) + \boldsymbol{\lambda}_0^T \delta \boldsymbol{x}_0 + \frac{\partial \Phi}{\partial \boldsymbol{c}} \delta \boldsymbol{c}$$

$$+ \int_{t_0}^{t_f} \left\{\left[\frac{\partial H}{\partial \boldsymbol{x}} + \dot{\boldsymbol{\lambda}}^T\right] \delta \boldsymbol{x} + \frac{\partial H}{\partial \boldsymbol{u}} \delta \boldsymbol{u}\right\} dt$$

– eliminate the $\delta \boldsymbol{x}$ terms by selecting appropriate Lagrange multipliers,

$$\dot{\boldsymbol{\lambda}} = -\frac{\partial H}{\partial \boldsymbol{x}}^T = -\frac{\partial f}{\partial \boldsymbol{x}}^T \boldsymbol{\lambda} - \frac{\partial L}{\partial \boldsymbol{x}}^T \quad \text{[costate equations]}$$

$$\boldsymbol{\lambda}^T(t_f) = \frac{\partial \Phi}{\partial \boldsymbol{x}(t_f)} = \frac{\partial \varphi}{\partial \boldsymbol{x}(t_f)} + \boldsymbol{\nu}^T \frac{\partial \boldsymbol{\psi}}{\partial \boldsymbol{x}(t_f)} \quad \text{[boundary conditions]}$$

– can we ignore $\delta \boldsymbol{x}_0$ and $\delta \boldsymbol{c}$? Normally yes because $\boldsymbol{x}_0$ and $\boldsymbol{c}$ are fixed $\Rightarrow \; \delta \boldsymbol{x}_0 = \delta \boldsymbol{c} = 0$

- ◦ I've included these terms in $\delta \bar{J}$ to make a point about $\boldsymbol{\lambda}_0$ and $\boldsymbol{\nu} = -{}^{\partial \Phi}/_{\partial \boldsymbol{c}}{}^T$:
  - · $\boldsymbol{\lambda}_0$ and $\boldsymbol{\nu}$ represent the sensitivity of $\bar{J}$ to changes in initial conditions and terminal constraints respectively
  - · so, if I know $\boldsymbol{\lambda}_0$ and $\boldsymbol{\nu}$, I can estimate (to first order) the changes in J that would be caused by changing $\boldsymbol{x}_0$ and $\boldsymbol{c}$

– using these results, we can simplify $\delta \bar{J}$ to ⇒

$$\delta \bar{J} = \int_{t_0}^{t_f} \frac{\partial H}{\partial \boldsymbol{u}} \delta \boldsymbol{u} \, dt$$

– and so δJ can only be non-negative for admissible δu if

$$\frac{\partial H}{\partial \boldsymbol{u}}^T = \frac{\partial \boldsymbol{f}}{\partial \boldsymbol{u}}^T \boldsymbol{\lambda} + \frac{\partial L}{\partial \boldsymbol{u}}^T = 0 \quad \text{[stationarity conditions]}$$

- ◦ the stationarity conditions are a set of algebraic equations that will be used to define $\boldsymbol{u}$

– finally, we must remember that all of the constraints must be satisfied

$$\dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x}, \boldsymbol{u}, t), \quad \boldsymbol{x}(t_0) = \boldsymbol{x}_0$$

$$\boldsymbol{\psi}\left[\boldsymbol{x}(t_f)\right] = \boldsymbol{c}$$

Unknowns	Equations
$\boldsymbol{x}(t) \rightarrow n \times 1$	$\dot{\boldsymbol{x}} = \boldsymbol{f} \rightarrow n$
$\boldsymbol{u}(t) \rightarrow m \times 1$	${}^{\partial H}/_{\partial \boldsymbol{u}} = 0 \rightarrow m$
$\boldsymbol{\lambda}(t) \rightarrow n \times 1$	$\dot{\boldsymbol{\lambda}} = -{}^{\partial H}/_{\partial \boldsymbol{x}}{}^T \rightarrow n$
$\boldsymbol{\nu} \rightarrow q \times 1$	$\boldsymbol{\psi} = \boldsymbol{c} \rightarrow q$

- This is still a difficult two-point boundary value problem (now with extra parameters $\boldsymbol{\nu}$ to be identified)

Example [Kirk, pg. 313, Problem 5-12]

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = -x_2 + u$$
$$x_1(0) = x_2(0) = 0$$

- Cost function:

$$J = \frac{1}{2}\{x_1(2) - 5\}^2 + \frac{1}{2}\{x_2(2) - 2\}^2 + \frac{1}{2}\int_0^2 u^2 dt$$

- Terminal constraint:

$$x_1(2) + 5x_2(2) = 15$$

- Augmented cost function:

$$\bar{J} = \frac{1}{2}(x_1 - 5)^2 + \frac{1}{2}(x_2 - 2)^2 + \nu\{x_1 + 5x_2 - 15\}$$

$$+ \int_0^2 \left\{\frac{1}{2}u^2 + \lambda_1 x_2 + \lambda_2(-x_2 + u) - \lambda_1\dot{x}_1 - \lambda_2\dot{x}_2\right\} dt$$

  – after integration by parts,

$$\bar{J} = \frac{1}{2}(x_1 - 5)^2 + (x_2 - 2)^2 + \nu(x_1 + 5x_2 - 15)$$

$$+\lambda_1(0)x_1(0) + \lambda_2(0)x_2(0) - \lambda_1(2)x_1(2) - \lambda_2(2)x_2(2)$$

$$+ \int_0^2 \left\{H - \dot{\lambda}_1 x_1 - \dot{\lambda}_2 x_2\right\} dt$$

$$\left[\text{where } H = {}^1\!/\!_2 u^2 + \lambda_1 x_2 + \lambda_2(-x_2 + u)\right]$$

- Costate equations:

$$\dot{\boldsymbol{\lambda}} = \frac{-\partial H}{\partial \boldsymbol{x}} \Rightarrow \begin{matrix} \dot{\lambda}_1 = & 0 \\ \dot{\lambda}_2 = & -\lambda_1 + \lambda_2 \end{matrix} \Rightarrow \begin{matrix} \lambda_1 = & k_1 \\ \lambda_2 = & k_1\left(1 - e^t\right) + k_2 e^t \end{matrix}$$

– boundary conditions ⇒

$$\frac{\partial \Phi}{\partial x_1(2)} = x_1 - 5 + \nu - \lambda_1(2) = 0$$

$$\frac{\partial \Phi}{\partial x_2(2)} = x_2 - 2 + 5\nu - \lambda_2(2) = 0$$

– ⇒

$$x_1(2) - 5 + \nu = k_1$$

$$x_2(2) - 2 + 5\nu = k_1\left(1 - e^2\right) + k_2 e^2$$

- Stationarity condition:

$$\frac{\partial H}{\partial u} = u + \lambda_2 = 0 \;\Rightarrow\; u = -\lambda_2$$

- State equations:

1. $\dot{x}_2 = -x_2 - \lambda_2 \quad \Rightarrow$

$$(s+1)\,X_2(s) = -\frac{k_1}{s} - \frac{(k_2 - k_1)}{(s-1)}$$

$$X_2(s) = -\frac{k_1}{s\,(s+1)} - \frac{(k_2 - k_1)}{(s-1)\,(s+1)}$$

⇒

$$x_2(t) = k_1\left(-1 + 1/2e^{-t} + 1/2e^{t}\right) + k_2\left(1/2e^{-t} - 1/2e^{t}\right)$$

2. $\dot{x}_1 = x_2 \quad \Rightarrow$

$$x_1(t) = \int_0^t x_2(\tau)\,d\tau$$

⇒

$$x_1(t) = k_1\left(-t - 1/2e^{-t} + 1/2e^{t}\right) + k_2\left(1 - 1/2e^{-t} - 1/2e^{t}\right)$$

- To finish the problem, we must identify k_1, k_2, and ν

– collect boundary conditions ⇒

$$x_1(2) - 5 + \nu = k_1$$

$$x_2(2) - 2 + 5\nu = k_1\left(1 - e^2\right) + k_2 e^2$$

$$x_1(2) = k_1\left(-2 - {}^1\!/_2 e^{-2} + {}^1\!/_2 e^2\right) + k_2\left(1 - {}^1\!/_2 e^{-2} - {}^1\!/_2 e^2\right)$$

$$x_2(2) = k_1\left(-1 + {}^1\!/_2 e^{-2} + {}^1\!/_2 e^2\right) + k_2\left({}^1\!/_2 e^{-2} - {}^1\!/_2 e^2\right)$$

$$x_1(2) + 5x_2(2) = 15$$

– expressing in matrix form we have:

$$\begin{bmatrix} 1 & 0 & 1 & -1 & 0 \\ 0 & 1 & 5 & 6.3891 & -7.3891 \\ 1 & 0 & 0 & -1.6269 & 2.7622 \\ 0 & 1 & 0 & -2.7622 & 3.6269 \\ 1 & 5 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1(2) \\ x_2(2) \\ \nu \\ k_1 \\ k_2 \end{bmatrix} = \begin{bmatrix} 5 \\ 2 \\ 0 \\ 0 \\ 15 \end{bmatrix}$$

– solving,

$$\begin{bmatrix} x_1(2) \\ x_2(2) \\ \nu \\ k_1 \\ k_2 \end{bmatrix} = \begin{bmatrix} 3.0576 \\ 2.3885 \\ -0.6553 \\ -2.5976 \\ -2.6369 \end{bmatrix}$$

– since $u = -\lambda_2 = -k_1\left(1 - e^t\right) - k_2 e^t$,

$$u^*(t) = 2.5976 + 0.03927e^t$$

– the optimal cost is computed as:

$$J^* = 1.8864 + 0.07546 + \frac{1}{2}\int_0^2 \left(2.5976 + 0.03927e^t\right)^2 dt = 9.3818$$

- Question: What would the cost be if I changed the terminal constraint to $x_1(2) + 5x_2(2) = 15.3$?

  – First order approximation:

  $$\delta J = -\nu\delta c = -(-.6553)(.3) = 0.1966 \;\Rightarrow\; J = 9.5784$$

– Actual: $J = 9.5829$

- In many instances, the terminal constraints placed on the problem examined above are not functions of the final states, but rather constraints on the final states themselves

$$\psi_i\left\{\boldsymbol{x}(t_f)\right\} = x_i(t_f) = c_i, \quad i = 1, 2, \ldots, q; \; q \le n$$

  – does this change the general solution process? NO!
  – what does change?

$$\boldsymbol{\lambda}(t_f) = \frac{\partial \Phi}{\partial \boldsymbol{x}(t_f)} \quad \Rightarrow \quad \lambda_i(t_f) = \begin{cases} \nu_i & i = 1, 2, \ldots, q \\ \dfrac{\partial \varphi}{\partial x_i(t_f)} & i = q+1, \ldots, n \end{cases}$$

    ◦ NOTE: φ will not be a function of $(x_i;\; i = 1, \ldots, q)$ because these states are already constrained

$$\boldsymbol{\psi}\left\{\boldsymbol{x}(t_f)\right\} = \boldsymbol{c} \quad \Rightarrow \quad x_i(t_f) = c_i$$

Example: [continuation of previous problem]

- Let

$$J = \frac{1}{2}\int_0^2 u^2 dt$$
$$x_1(2) = 5$$
$$x_2(2) = 2$$

  – H is the same $\Rightarrow$ $\lambda_1(t),\ \lambda_2(t),\ x_1(t),\ x_2(t)$ same form as before
  – Two constraints $\Rightarrow$ 4 unknowns $(\nu_1,\ \nu_2,\ k_1,\ k_2)$ to identify

$$\lambda_1(2) = \nu_1 = k_1$$

$$\lambda_2(2) = \nu_2 = k_1\left(1 - e^2\right) + k_2 e^2$$
$$x_1(2) = 5$$
$$x_2(2) = 2$$

- Result:

$$k_1 = -7.292$$
$$k_2 = -6.105$$

$$u^*(t) = 7.292 - 1.187e^t$$

$$J^* = 16.75$$

Changing Hard Terminal Constraints to Soft Constraints

- In many situations, the fixed-time terminal constraint problem may be too difficult to solve analytically so we must resort to numerical solutions
  - can the software developed for the no terminal constraint problem be used here without modification? YES!
  - how? change the hard constraints to soft constraints with weighting and apply one of the algorithms presented previously

$$\varphi(t_f) \to \varphi(t_f) + \sum_i w_i \left\{\psi_i - c_i\right\}^2$$

  - what does this approach imply? that we'll be satisfied with small deviations from the terminal constraints

Continuous-Time Optimization: Free Time Problems

- In the previous section, we examined the continuous-time optimization problem assuming the final time was fixed; but in many cases, the final time may be free and will act as another parameter to be selected in the optimization process
  - but any changes in $x(t_f)$ are not independent of changes in t_f !

$$dx(t_f) = \delta x(t_f) + \dot{x}(t_f)dt_f$$

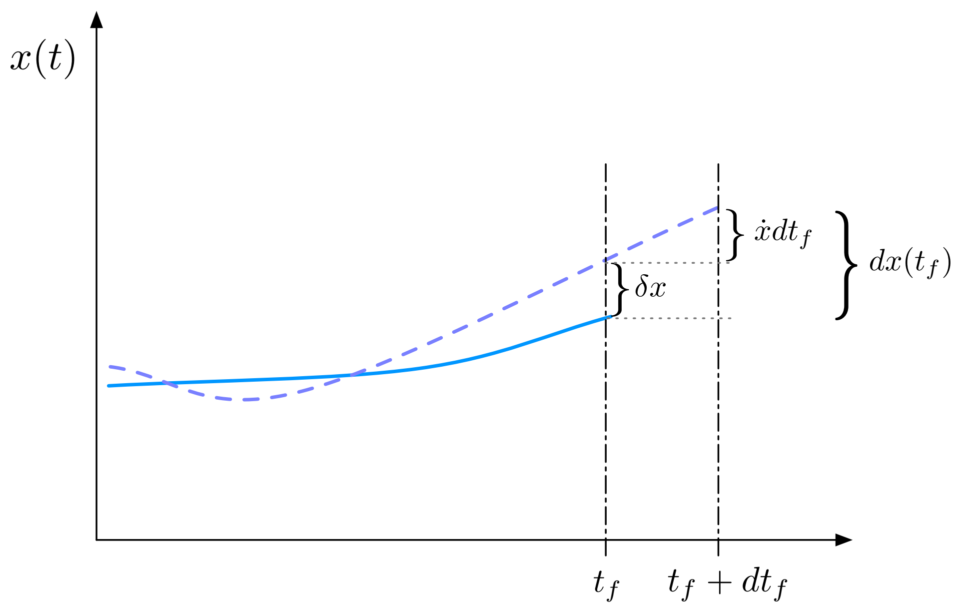

Figure 5.6 Free-Time Problem

  - so in this problem, we have to worry about both types of change in $x(t_f)$ $\Rightarrow$ our problem will be slightly more complicated

- How do we solve it? CALCULUS OF VARIATIONS

PROBLEM:

Dynamic System	$\dot{\boldsymbol{x}} = \boldsymbol{f}\,(\boldsymbol{x},\,\boldsymbol{u},\,t)$	$\boldsymbol{x}(t_0) = \boldsymbol{x}_0$
Cost	$J = \varphi\left\{\boldsymbol{x}\left(t_f\right)\right\} + \int_{t_0}^{t_f} L\left\{\boldsymbol{x},\,\boldsymbol{u},\,t\right\} dt$	
Terminal Constraints	$\boldsymbol{\psi}\left\{\boldsymbol{x}\left(t_f\right),\,t_f\right\} = \boldsymbol{c}$	$\boldsymbol{\psi}$ is $q \times 1,\ q \le n$
Goal	select $\boldsymbol{u}(t)$ and t_f to minimize J subject to terminal constraints	

SOLUTION:

⇒

$$\begin{aligned}\bar{J} &= \varphi + \boldsymbol{\nu}^T\left\{\boldsymbol{\psi} - \boldsymbol{c}\right\} + \int_{t_0}^{t_f}\left\{L + \boldsymbol{\lambda}^T\left(\boldsymbol{f} - \dot{\boldsymbol{x}}\right)\right\} dt \\ &= \Phi + \int_{t_0}^{t_f}\left\{L + \boldsymbol{\lambda}^T\left(\boldsymbol{f} - \dot{\boldsymbol{x}}\right)\right\} dt\end{aligned}$$

- Now, we can take the differential of $\bar{J}$:

$$\begin{aligned}d\bar{J} =& \frac{\partial\Phi}{\partial\boldsymbol{x}(t_f)} d\boldsymbol{x}(t_f) + \frac{\partial\Phi}{\partial t_f} dt_f \\ &+ \left\{H - \boldsymbol{\lambda}^T\dot{\boldsymbol{x}}\right\}\Big|_{t_f} dt_f - \left\{H - \boldsymbol{\lambda}^T\dot{\boldsymbol{x}}\right\}\Big|_{t_0} dt_0 \\ &+ \int_{t_0}^{t_f}\left\{\frac{\partial H}{\partial\boldsymbol{x}}\delta\boldsymbol{x} + \frac{\partial H}{\partial\boldsymbol{u}}\delta\boldsymbol{u} - \boldsymbol{\lambda}^T\delta\dot{\boldsymbol{x}}\right\} dt - \boldsymbol{\nu}^T\delta\boldsymbol{c}\end{aligned}$$

⇒

$$d\bar{J} = \frac{\partial\Phi}{\partial\boldsymbol{x}(t_f)} d\boldsymbol{x}(t_f) + \frac{\partial\Phi}{\partial t_f} dt_f$$

$$+L(t_f)dt_f - L(t_0)dt_0$$

$$+\int_{t_0}^{t_f}\left\{\left[\frac{\partial H}{\partial \boldsymbol{x}} + \dot{\boldsymbol{\lambda}}^T\right]\delta \boldsymbol{x} + \frac{\partial H}{\partial \boldsymbol{u}}\delta \boldsymbol{u}\right\}dt$$

$$+\boldsymbol{\lambda}^T(t_0)\delta \boldsymbol{x}(t_0) - \boldsymbol{\lambda}^T(t_f)\delta \boldsymbol{x}(t_f)$$

$$-\boldsymbol{\nu}^T\delta \boldsymbol{c}$$

- What is new here? Since t_f is free, we must account for the fact that the final value of L may vary $\left(L(t_f)dt_f\right)$ as well as the fact that $\boldsymbol{x}(t_f)$ may vary in two ways $(d\boldsymbol{x} = \delta \boldsymbol{x} + \dot{\boldsymbol{x}}dt)$

$\Rightarrow$

$$d\bar{J} = \left\{\frac{\partial \Phi}{\partial \boldsymbol{x}(t_f)} - \boldsymbol{\lambda}^T(t_f)\right\}\delta \boldsymbol{x}(t_f) + \left\{\frac{\partial \Phi}{\partial t_f} + \frac{\partial \Phi}{\partial \boldsymbol{x}(t_f)}\dot{\boldsymbol{x}}(t_f) + L(t_f)\right\}dt_f$$

$$+\int_{t_0}^{t_f}\left\{\left[\frac{\partial H}{\partial \boldsymbol{x}} + \dot{\boldsymbol{\lambda}}^T\right]\delta \boldsymbol{x} + \frac{\partial H}{\partial \boldsymbol{u}}\delta \boldsymbol{u}\right\}dt$$

$$+\boldsymbol{\lambda}^T(t_0)\delta \boldsymbol{x}(t_0) - \boldsymbol{\nu}^T\delta \boldsymbol{c} - L(t_0)dt_0$$

- The rest of the process is the same as before:
  - Costate Equations: (choose $\boldsymbol{\lambda}$ to eliminate $\delta \boldsymbol{x}$)

$$\dot{\boldsymbol{\lambda}}^T = -\frac{\partial H}{\partial x} \qquad \boldsymbol{\lambda}^T(t_f) = \frac{\partial \Phi}{\partial x(t_f)}$$

  - Stationarity Condition:

$$\frac{\partial H}{\partial \boldsymbol{u}} = 0$$

  - Constraints:
    1. dynamic $\Rightarrow$ $\dot{\boldsymbol{x}} = \boldsymbol{f}\{\boldsymbol{x}, \boldsymbol{u}, t\}$
    2. terminal $\Rightarrow$ $\boldsymbol{\psi}\left\{\boldsymbol{x}(t_f),\, t_f\right\} = \boldsymbol{c}$

– Transversality Condition (introduced because t_f is free):

$$\frac{\partial \Phi}{\partial t_f} + \frac{\partial \Phi}{\partial \boldsymbol{x}(t_f)}\dot{\boldsymbol{x}}(t_f) + L(t_f) = 0$$

∘ but,

$$\boldsymbol{\lambda}^T(t_f) = \frac{\partial \Phi}{\partial \boldsymbol{x}(t_f)} \quad \text{and} \quad \dot{\boldsymbol{x}}(t_f) = \boldsymbol{f}\left\{\boldsymbol{x}(t_f), \boldsymbol{u}(t_f), t_f\right\}$$

$$\frac{\partial \Phi}{\partial t_f} + \left(L + \boldsymbol{\lambda}^T \boldsymbol{f}\right)_{t_f} = 0$$

or

$$H(t_f) = -\frac{\partial \Phi}{\partial t_f}$$

- Are these results surprising? NO!

1. we can think of the unspecified terminal time problem as a family of fixed terminal time problems from which we must select the one which minimizes the cost

   ⇒ so all conditions derived previously for the fixed time problem must still apply

2. but there must be another condition (the transversality condition) available to determine the optimal value of t_f

Example

$$\dot{x}_1 = x_2 \qquad x_1(0) = x_2(0) = 0$$
$$\dot{x}_2 = -x_2 + u$$

$$J = \frac{1}{2}\left\{x_1(t_f) - 5\right\}^2 + \frac{1}{2}\left\{x_2(t_f) - 2\right\}^2 + \frac{1}{2}\int_0^{t_f} u^2 dt$$

$$x_1(t_f) + 5x_2(t_f) = 15$$

- What equations do we use to solve this problem? same as before...

$$\lambda_1 = k_1$$
$$\lambda_2 = k_1\left(1 - e^t\right) + k_2 e^t$$

$$x_1(t) = k_1\left(-t - 1/2e^{-t} + 1/2e^t\right) + k_2\left(1 - 1/2e^{-t} - 1/2e^t\right)$$
$$x_2(t) = k_1\left(-1 + 1/2e^{-t} + 1/2e^t\right) + k_2\left(1/2e^{-t} - 1/2e^t\right)$$

$$u = -\lambda_2$$

- and...

$$x_1(t_f) - 5 + \nu = k_1$$
$$x_2(t_f) - 2 + 5\nu = k_1\left(1 - e^{t_f}\right) + k_2 e^{t_f}$$

$$x_1(t_f) = k_1\left(-t_f - 1/2e^{-t_f} + 1/2e^{t_f}\right) + k_2\left(1 - 1/2e^{-t_f} - 1/2e^{t_f}\right)$$
$$x_2(t_f) = k_1\left(-1 + 1/2e^{-t_f} + 1/2e^{t_f}\right) + k_2\left(1/2e^{-t_f} - 1/2e^{t_f}\right)$$

$$x_1(t_f) + 5x_2(t_f) = 15$$

  – giving 5 equations but 6 unknowns $\left(x_1,\ x_2,\ k_1,\ k_2,\ \nu,\ t_f\right)$

$$H(t_f) = -\frac{\partial \Phi}{\partial t_f} = 0 \ \Rightarrow\ \frac{1}{2}u^2 + (\lambda_1 - \lambda_2)\, x_2 + \lambda_2 u = 0$$

$$\left\{\lambda_1(t_f) - \lambda_2(t_f)\right\} x_2(t_f) - \frac{1}{2}\lambda_2^2(t_f) = 0$$

- Are these problems easy to solve? NO!
  – I'll give you a chance to try one in the homework!

Continuous Time Optimization: Minimum Time Problems

- A special case of the free terminal time problem is the minimum time problem
  - the goal of this problem is to minimize the elapsed time required to transfer the system from a specified initial state to some specified final condition
    - for this problem to make sense, at least one state must be specified at $t = t_0$ and at least one constraint must be specified at $t = t_f$ (we have to do *something* in minimum time, or we won't do anything!)
  - this is a constrained, free terminal time problem; so all of the techniques developed previously apply
    - but what is the cost?

$$J = t_f - t_0 = \int_{t_0}^{t_f} 1dt$$

so, $\varphi = 0$ and $L = 1$

Example: Brachistochrone Problem ("shortest time")

- A mass m moves in a constant force field of magnitude g starting from rest at the origin. Find the minimum time path to reach a specified final point.

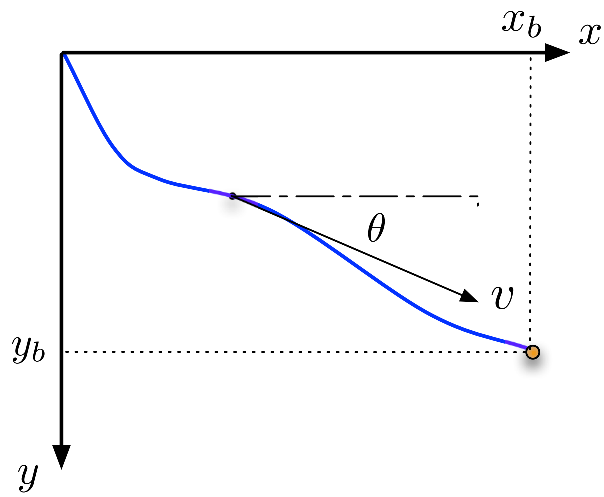

Figure 5.7 Brachistochrone Problem: Continous-Time

$$\dot{x} = v\cos\theta$$
$$\dot{y} = v\sin\theta$$

- But energy must be conserved,

$$\frac{1}{2}mv^2 + mgy = \text{constant}$$

$$\Rightarrow \quad v = \sqrt{2gy}$$

$$J = \int_{t_0}^{t_f} 1 dt$$

- Solving,

$$\bar{J} = \nu_1\left\{x(t_f) - x_b\right\} + \nu_2\left\{y(t_f) - y_b\right\}$$
$$+ \int_{t_0}^{t_f} \{1 + \lambda_1(v\cos\theta - \dot{x}) + \lambda_2(v\sin\theta - \dot{y})\}\, dt$$

$$H = 1 + \lambda_1(v\cos\theta) + \lambda_2(v\sin\theta)$$

$$\dot{\lambda}_1 = -\frac{\partial H}{\partial x} = 0$$

$$\dot{\lambda}_2 = -\frac{\partial H}{\partial y} = \{\lambda_1 \cos\theta + \lambda_2 \sin\theta\}\frac{\partial v}{\partial y} = \frac{g}{v}\{\lambda_1 \cos\theta + \lambda_2 \sin\theta\}$$

$$\frac{\partial H}{\partial u} = \frac{\partial H}{\partial \theta} = -\lambda_1 v \sin\theta + \lambda_2 v \cos\theta = 0 \;\Rightarrow\; \frac{\lambda_2}{\lambda_1} = \tan\theta$$

$$H(t_f) = -\frac{\partial \Phi}{\partial t_f} = 0 \quad \text{and} \quad \frac{\partial H}{\partial t} = 0$$

$\Rightarrow$ $H(t)$ is constant along the optimal path

- Therefore,

$$H(t) = 1 + \lambda_1 v \cos\theta + \lambda_2 v \sin\theta = 0$$

$$\begin{aligned} \dot{x} &= v\cos\theta & x(0) &= y(0) = 0 \\ \dot{y} &= v\sin\theta & & \\ x(t_f) &= x_b & y(t_f) &= y_b \\ \lambda_1(t_f) &= \nu_1 & \lambda_2(t_f) &= \nu_2 \end{aligned}$$

$$\left.\begin{aligned} \frac{\lambda_2}{\lambda_1} &= \tan\theta \\ \lambda_1 v\cos\theta + \lambda_2 v \sin\theta &= -1 \end{aligned}\right\} \Rightarrow \quad v\cos\theta + v\sin\theta\tan\theta = \frac{-1}{\lambda_1}$$

$$\lambda_1 = \frac{-\cos\theta}{v}$$

$$\lambda_2 = \frac{-\sin\theta}{v}$$

$$\dot{\lambda}_1 = \frac{v\,(\sin\theta)\,\dot{\theta} + {}^{g}\!/\!_{v}\,(\cos\theta)\,\dot{y}}{v^2} = 0 \quad (\lambda_1 \text{ is constant})$$

$$\Rightarrow \quad \dot{\theta} = -\frac{g}{v^2}\cdot\frac{\cos\theta}{\sin\theta}\cdot v\sin\theta = -\frac{g}{v}\cos\theta$$

- So, we have equations for $\dot{x},\ \dot{y},\ \dot{\theta}$, and 4 boundary conditions to solve for $x(t),\ y(t),\ \theta(t)$, and t_f ! But it's messy.
- Instead, let's treat θ as our independent variable.
  - $\lambda_1 =$ constant $\Rightarrow$

$$\begin{aligned}\frac{\cos\theta(t)}{v(t)} &= \frac{\cos\theta_f}{v_f}\\ \cos\theta &= \frac{v}{v_f}\cdot\cos\theta_f = \sqrt{\frac{y}{y_b}}\cdot\cos\theta_f\\ &\Rightarrow y(t) = y_b\cdot\frac{\cos^2\theta}{\cos^2\theta_f}\end{aligned}$$

  - $\dot{x} = v\cos\theta \Rightarrow$

$$\begin{aligned}\frac{dx}{d\theta}\cdot\dot{\theta} &= v\cos\theta\\ \frac{dx}{d\theta} &= -\frac{v^2}{g} = -2y\\ \Rightarrow\ \frac{dx}{d\theta} &= -\frac{2y_b}{\cos^2\theta_f}\cdot\cos^2\theta\\ x(t) &= \int_{\theta(t)}^{\theta_f}\frac{dx}{d\theta}d\theta\end{aligned}$$

$$\Rightarrow\ x(t) = x_b + \frac{y_b}{2\cos^2\theta_f}\left\{2\left(\theta_f-\theta\right)+\sin 2\theta_f - \sin 2\theta\right\}$$

$$y(0) = 0\ \Rightarrow\ \theta(0) = 90^\circ$$

$$x(0) = 0\ \Rightarrow\ \text{solve for } \theta_f$$

  - if we know $x(t)$ and $y(t)$, we can find $\theta(t)$. A feedback law!

- Time-to-go can now be calculated by integrating $\dot{\theta} = g\lambda_1 = \text{constant}$

Example: Zermelo's Problem

- Consider a ship travelling through a region of strong currents subject to the following conditions:

  1. the velocity of the current in the x-direction is a linear function of y
  2. the velocity of the current in the y-direction is zero
  3. the ship has constant speed (v) , but can change its heading (θ)

- Find the minimum time path from a given initial position to a specified final position.

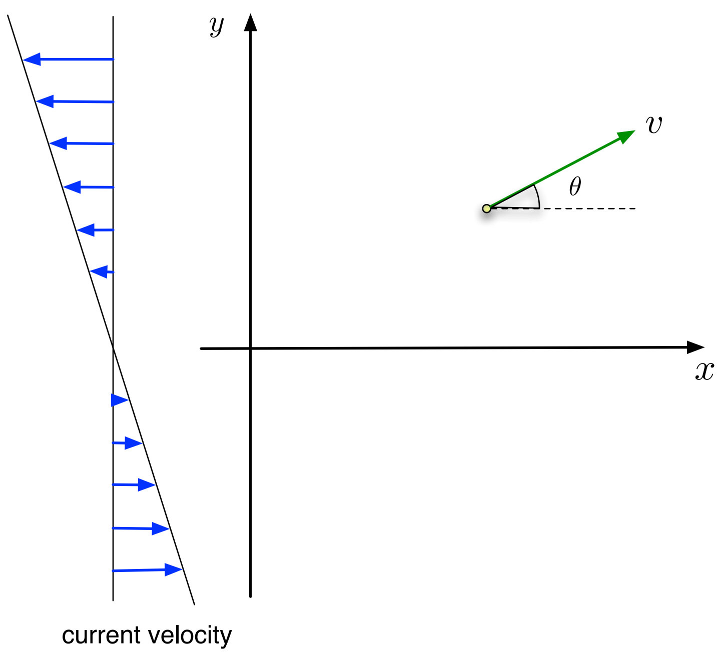

Figure 5.8 Zermelo's Problem

$$\dot{x} = v\cos\theta + u$$
$$\dot{y} = v\sin\theta$$
$$u = -\frac{v}{h}\cdot y$$

where h is a normalizing constant.

$$J = \int_{t_0}^{t_f} 1 dt$$

$$x(t_f) = y(t_f) = 0$$

$$\Phi = \nu_x x(t_f) + \nu_y y(t_f)$$

$$H = 1 + \lambda_x \left\{ v\cos\theta - \frac{v}{h}\cdot y \right\} + \lambda_y v \sin\theta$$

- Costate Equations:

$$\dot{\lambda}_x = -\frac{\partial H}{\partial x} = 0, \qquad \lambda_x(t_f) = \nu_x$$

$$\dot{\lambda}_y = -\frac{\partial H}{\partial y} = \lambda_x \cdot \frac{v}{h}, \qquad \lambda_y(t_f) = \nu_y$$

$\Rightarrow$

$$\lambda_x = k_1$$

$$\lambda_y = \frac{k_1 v}{h}\cdot t + k_2$$

- State Equations:

$$\dot{x} = v\cos\theta - \frac{v}{h}\cdot y \quad x(0) = x_0$$

$$\dot{y} = v\sin\theta \quad y(0) = y_0$$

- Stationarity Condition:

$$\frac{\partial H}{\partial \theta} = -\lambda_x v \sin\theta + \lambda_y v\cos\theta = 0 \;\Rightarrow\; \tan\theta = \frac{\lambda_y}{\lambda_x}$$

- Transversality Condition:

$$H(t_f) = -\frac{\partial \Phi}{\partial t_f} = 0 \;\Rightarrow\; \lambda_x \left\{ v\cos\theta - \frac{v}{h}\cdot y \right\} + \lambda_y v\sin\theta = -1$$

- Terminal Constraints:

$$x(t_f) = y(t_f) = 0$$

- It's a messy process to try to identify everything in terms of t; so let's eliminate t and make θ the independent variable
  - since H is not an explicit function of time, $H(t) =$ constant

$$H(t) = H(t_f) = 0$$

$\Rightarrow$

$$\begin{bmatrix} v\cos\theta - \frac{v}{h}\cdot y & v\sin\theta \\ -v\sin\theta & v\cos\theta \end{bmatrix}\begin{bmatrix} \lambda_x \\ \lambda_y \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \end{bmatrix}$$

$$\lambda_x = \frac{-\cos\theta}{v\left(1 - \frac{y}{h}\cos\theta\right)} \qquad \lambda_y = \frac{-\sin\theta}{v\left(1 - \frac{y}{h}\cos\theta\right)}$$

$$\dot{\lambda}_x = -\frac{\partial H}{\partial x} = 0 = \frac{\partial \lambda_x}{\partial \theta}\cdot\dot{\theta} + \frac{\partial \lambda_x}{\partial y}\cdot\dot{y} = 0$$

$$\dot{\theta} = \frac{-\frac{\partial \lambda_x}{\partial y}}{\frac{\partial \lambda_x}{\partial \theta}}\cdot\dot{y} \qquad \dot{y} = v\sin\theta$$

$\Rightarrow$

$$\dot{\theta} = \frac{v}{h}\cdot\cos^2\theta$$

$$\dot{x} = v\cos\theta - \frac{v}{h}\cdot y = \frac{dx}{d\theta}\cdot\dot{\theta} \;\Rightarrow\; \frac{dx}{d\theta} = \frac{(h\cos\theta - y)}{\cos^2\theta}$$

$$\dot{y} = v\sin\theta = \frac{dy}{d\theta}\cdot\dot{\theta} \;\Rightarrow\; \frac{dy}{d\theta} = \frac{h\sin\theta}{\cos^2\theta}$$

$$\int_y^{y_f} dy = \int_\theta^{\theta_f} \frac{h\sin\theta}{\cos^2\theta} d\theta \quad \Rightarrow \quad y(t) = h\left\{\sec\theta(t) - \sec\theta(t_f)\right\}$$

$$\int_x^{x_f} dx = \int_\theta^{\theta_f} \left\{h\sec\theta - h\sec^3\theta + h\sec\theta_f\sec^2\theta\right\} d\theta$$

$$\int_\theta^{\theta_f} \sec\theta_f\sec^2\theta d\theta = \sec\theta_f\left\{\tan\theta_f - \tan\theta\right\}$$

$$\int_\theta^{\theta_f} \sec\theta d\theta = \ln\left\{\frac{\sec\theta_f + \tan\theta_f}{\sec\theta + \tan\theta}\right\}$$

$$\int_\theta^{\theta_f} \sec^3\theta d\theta = \frac{1}{2}\frac{\sin\theta}{\cos^2\theta}\bigg|_\theta^{\theta_f} + \frac{1}{2}\int_\theta^{\theta_f}\sec\theta d\theta$$

$\Rightarrow$

$$x(t) = \frac{h}{2}\left\{\sec\theta(t_f)\left\{\tan\theta(t) - \tan\theta(t_f)\right\} + \tan\theta(t)\left\{\sec\theta(t_f) - \sec\theta(t)\right\}\right.$$
$$\left. + \ln\left[\frac{\tan\theta(t) + \sec\theta(t)}{\tan\theta(t_f) + \sec\theta(t_f)}\right]\right\}$$

- So if $x(t)$ and $y(t)$ are known, $\theta(t)$ and $\theta(t_f)$ can be calculated
- Feedback Control Law:

$$\lambda_x = \text{constant} \Rightarrow \frac{\cos\theta(t)}{v\left\{1 - \dfrac{y(t)}{h}\cos\theta(t)\right\}} = \frac{\cos\theta(t_f)}{v}$$

$\Rightarrow$

$$\cos\theta(t) = \frac{\cos\theta(t_f)}{1 + \dfrac{y(t)}{h}\cos\theta(t_f)}$$

[Note: $\theta(t_f)$ can be identified using initial conditions.]

- Time-to-go:

$$\dot{\theta} = \frac{v}{h}\cos^2\theta \quad \Rightarrow \quad \frac{v}{h}\left(t_f - t\right) = \tan\theta_f - \tan\theta$$

Discrete-TIme Optimization: Minimum Time Problems

- As we've seen in the previous sections, time free and minimum time problems are difficult to solve analytically; so in many of these situations, it would be nice to resort to numerical techniques to obtain a solution
- One way to do this is to examine a discrete-time problem that is equivalent in many ways to the continuous-time problems and examine this problem in light of our goal to obtain a computer algorithm to solve it.

Problem:

$$\boldsymbol{x}[k+1] = \boldsymbol{f}\,\{\boldsymbol{x}[k],\ \boldsymbol{u}[k],\ \Delta\} \quad \boldsymbol{x}[0] \text{ given}$$

$$J = N\Delta \quad \text{(with } N \text{ fixed)}$$

$$\psi\,\{\boldsymbol{x}[N]\} = \boldsymbol{c}$$

$$\text{Find } \{u[0],\, u[1],\, \ldots,\, u[N-1]\} \text{ and } \Delta \text{ to minimize } J$$

NOTE:

1. At least one terminal condition must be specified to define this problem (i.e., minimum time to do what?)
2. $\boldsymbol{x}[k+1]$ will generally be an explicit function of Δ

$$\dot{\boldsymbol{x}} = \boldsymbol{f} \;\Rightarrow\; \boldsymbol{x}[k+1] = \boldsymbol{x}[k] + \boldsymbol{f}\,\Delta$$

- The solution process examined here varies from the one we've used up to now because we want to develop a computer algorithm to solve the problem

– first, let's examine the effects of changing $u[k]$ and Δ on each of the end constraints

$$J_i = \psi_i - c_i$$

$$\bar{J}_i = \psi_i - c_i + \sum_{k=0}^{N-1} \lambda_i^T[k+1]\{\boldsymbol{f} - \boldsymbol{x}[k+1]\}$$

- now, we can take the first variation of $\bar{J}_i$

$$\begin{aligned}\delta \bar{J}_i &= \left\{\frac{\partial \psi_i}{\partial \boldsymbol{x}[N]} - \boldsymbol{\lambda}_i^T[N]\right\}\delta \boldsymbol{x}[N] \\ &+ \sum_{k=1}^{N-1}\left\{\boldsymbol{\lambda}_i^T[k+1]\frac{\partial \boldsymbol{f}}{\partial \boldsymbol{x}[k]} - \boldsymbol{\lambda}_i[k]\right\}\delta \boldsymbol{x}[k] \\ &+ \sum_{k=0} \boldsymbol{\lambda}_i^T[k+1]\frac{\partial f}{\partial \boldsymbol{u}[k]}\delta \boldsymbol{u}[k] + \sum_{k=0}^{N-1}\boldsymbol{\lambda}_i^T[k+1]\frac{\partial \boldsymbol{f}}{\partial \Delta}\delta\Delta\end{aligned}$$

$$\delta J_i = \left\{\frac{\partial \psi_i}{\partial x[N]} - \lambda_i^T[N]\right\}\delta x[N]$$

1. $\boldsymbol{\lambda}_i^T[k+1]\dfrac{\partial \boldsymbol{f}}{\partial \boldsymbol{u}[k]}$ indicates the effect of changing $\boldsymbol{u}[k]$ on the i^{th} constraint
2. $\boldsymbol{\lambda}_i^T[k+1]\dfrac{\partial \boldsymbol{f}}{\partial \Delta}$ indicates the effect of changing Δ on the i^{th} constraint
3. the following difference equation defines λ_i:

$$\boldsymbol{\lambda}_i[k] = \left\{\frac{\partial \boldsymbol{f}}{\partial \boldsymbol{x}[k]}\right\}^T \boldsymbol{\lambda}_i[k+1] \qquad \boldsymbol{\lambda}_i[N] = \frac{\partial \psi_i}{\partial x[N]}^T$$

– having developed expressions for $\delta \bar{J}_i$, we can adjoin these to the first variation of the original cost and attempt to set the result equatl to zero

$$\delta J + \sum_{i=1}^{q} \nu_i \delta \bar{J}_i = N\delta\Delta + \left[\sum_{i=1}^{q} \nu_i \left\{\sum_{k=0}^{N-1} \boldsymbol{\lambda}_i^T[k+1]\frac{\partial \boldsymbol{f}}{\partial \Delta}\right\}\right]\delta\Delta$$

$$+ \sum_{k=0}^{N-1}\left\{\sum_{i=1}^{q} \nu_i \boldsymbol{\lambda}_i^T[k+1]\frac{\partial \boldsymbol{f}}{\partial \boldsymbol{u}[k]}\right\}\delta \boldsymbol{u}[k]$$

– or in matrix notation,

$$\delta J = \boldsymbol{\nu}^T \delta \bar{J} = \left(N + \boldsymbol{\nu}^T H_\Delta\right)\delta\Delta + \sum_{k=0}^{N-1} \boldsymbol{\nu}^T H_u[k]\delta \boldsymbol{u}[k]$$

where H_Δ is a column vector whose i^{th} element is

$$\sum_{k=0}^{N-1} \boldsymbol{\lambda}_i^T[k+1]\frac{\partial \boldsymbol{f}}{\partial \Delta}$$

and $H_u[k]$ is a matrix whose i^{th}row is $\boldsymbol{\lambda}_i^T[k+1]\dfrac{\partial \boldsymbol{f}}{\partial \boldsymbol{u}[k]}$

– using the results above, we now have enough information to establish and implement a gradient algorithm to solve the problem:

STEP 1: Guess Δ and $\{u[k];\ k = 0, 1, \ldots, N-1\}$

STEP 2: Compute $\{x[k];\ k = 1, 2, \ldots, N\}$ using the dynamic constraints defined by the difference equations:

$$x[k+1] = f\,\{x[k],\ u[k],\ \Delta\} \qquad x[0] \text{ given}$$

STEP 3: Compute $\{\lambda_i[k];\ k = 0, 1, \ldots, N-1, \quad i = 1, 2, \ldots, q\}$ using the costate difference equations:

$$\boldsymbol{\lambda}_i[k] = \left\{\frac{\partial \boldsymbol{f}}{\partial \boldsymbol{x}[k]}\right\}^T \boldsymbol{\lambda}_i[k+1] \qquad \boldsymbol{\lambda}_i[N] = \frac{\partial x_i}{\partial \boldsymbol{x}[N]}^T$$

STEP 4: Compute H_Δ and $\{H_u[k];\ k = 0, 1, \ldots, N-1\}$ using expressions developed previously

STEP 5: Based on the available information, update Δ and $\{\boldsymbol{u}[k];\ k = 0,\ 1,\ \ldots,\ N-1\}$ and return to Step 2

- How?

$$\delta\Delta = -K_\Delta\left\{N + \boldsymbol{\nu}^T H_\Delta\right\} \qquad \delta\boldsymbol{u}[k] = -K_u H_u^T[k]\boldsymbol{\nu}$$

  - $K_\Delta > 0$ and $K_u > 0$ should be selected based on your physical understanding of the problem so that $\delta\Delta$ and $\delta\boldsymbol{u}[k]$ do not violate our first-order assumptions

- Why? If $\delta\Delta$ and $\delta\boldsymbol{u}[k]$ are chosen in this manner, then

$$\delta J + \boldsymbol{\nu}^T\delta J = -K_\Delta\left(N + \boldsymbol{\nu}^T H_\Delta\right)^2 - K_u\left\{\sum_{k=0}^{N-1}\boldsymbol{\nu}^T H_u[k]H_u^T[k]\boldsymbol{\nu}\right\}$$

  - so the cost will be reduced by choosing these values for δ_Δ and $\delta\boldsymbol{u}[k]$

- But to compute δ_Δ and $\delta\boldsymbol{u}[k]$, we need $\boldsymbol{\nu}$!

$$\begin{aligned}\delta\bar{J} &= H_\Delta\delta\Delta + \sum_{k=0}^{N-1}H_u[k]\delta\boldsymbol{u}[k]\\ &= -K_\Delta N H_\Delta - K_\Delta H_\Delta H_\Delta^T\boldsymbol{\nu} - \sum_{k=0}^{N-1}K_u H_u[k]H_u^T[k]\boldsymbol{\nu}\\ &= -K_\Delta\boldsymbol{q} - K_\Delta Q\boldsymbol{\nu}\end{aligned}$$

where

$$q = NH_\Delta \quad Q = H_\Delta H_\Delta^T + \frac{K_u}{K_\Delta}\sum_{k=0}^{N-1}H_u[k]H_u^T[k]$$

- So,

$$\boldsymbol{\nu} = -Q^{-1}\left\{\boldsymbol{q} + \frac{\partial\bar{J}}{K_\Delta}\right\}$$

- What is $\delta \bar{J}$?

$$\bar{J} = \boldsymbol{\psi} - \boldsymbol{c}$$

$$\bar{J}_{opt} = \bar{J} + \delta \bar{J} \;\Rightarrow\; \delta \bar{J} = -\bar{J}$$

To determine $\boldsymbol{\nu}$, calculate $\boldsymbol{\psi} - \boldsymbol{c}$ for given $\boldsymbol{x}[N]$

QUESTION: When do we stop?

- Looking at the first variation of the cost, we find that

$$N + \boldsymbol{\nu}^T H_{\Delta} = 0 \qquad \boldsymbol{\nu}^T H_u[k] = 0$$

- So, stop when

$$\left| N + \boldsymbol{\nu}^T H_{\Delta} \right| < \epsilon_1 \qquad \text{and} \qquad \frac{1}{N}\left\{ \sum_{k=0}^{N-1} \boldsymbol{\nu}^T H_u[k] H_u^T[k] \boldsymbol{\nu} \right\} < \epsilon_2$$

Continuous-Time Optimization: Equality Path Constraints

- Up to now, the only constraints that we've included in the optimization process (apart from the dynamic constraints imposed by the system) hae been end point constraints.
- Our goal in this section is to investigate the solution process in which constraints exist along the entire trajectory or at intermediate points (so-called "path constraints")

Integral Equality Contraints

- The first of these problems is the INTEGRAL CONSTRAINT problem:

$$\dot{\boldsymbol{x}} = \boldsymbol{f}\{\boldsymbol{x},\, \boldsymbol{u},\, t\} \qquad \boldsymbol{x}(0) = \boldsymbol{x}_0$$

$$J = \varphi\left\{\boldsymbol{x}_f,\, t_f\right\} + \int_{t_0}^{t_f} L\{\boldsymbol{x},\, \boldsymbol{u},\, t\}\, dt$$

$$\boldsymbol{\psi}\left\{\boldsymbol{x}_f,\, t_f\right\} = \boldsymbol{c}$$

$$\int_{t_0}^{t_f} N\left\{\boldsymbol{x},\, \boldsymbol{u},\, t\right\} dt = k \quad \text{(NEW constraint)}$$

- To solve this problem, we define a new additional state, $x_{n+1}(t)$, which satisfies the following equations:

$$\dot{x}_{n+1} = N\left\{\boldsymbol{x},\, \boldsymbol{u},\, t\right\} \qquad x_{n+1}(0) = 0 \qquad x_{n+1}(t_f) = k$$

  – by augmenting the state vector with this additional state, we can transform the new integral constraint into a form that we already know how to handle

- The solution process is now identical to the one we've already developed

  1. Adjoin the constraints to the cost:

$$\bar{J} = \varphi + \boldsymbol{\nu}^T\left\{\boldsymbol{\psi} - \boldsymbol{c}\right\} + \nu_{q+1}\left\{x_{n+1}(t_f) - k\right\} + \int_{t_0}^{t_f}\left\{L + \boldsymbol{\lambda}^T\left(\boldsymbol{f} - \dot{\boldsymbol{x}}\right) + \gamma\left(N - \dot{x}_{n+1}\right)\right\} dt$$

  2. Take the first variation of $\bar{J}$ (remembering to integrate by parts):

$$\delta\bar{J} = \left\{\frac{\partial\varphi}{\partial\boldsymbol{x}(t_f)} + \boldsymbol{\nu}^T\frac{\partial\boldsymbol{\psi}}{\partial\boldsymbol{x}(t_f)} - \boldsymbol{\lambda}^T(t_f)\right\}\delta\boldsymbol{x}(t_f) + \left\{\nu_{q+1} - \gamma(t_f)\right\}\delta x_{n+1}(t_f) + \int_{t_0}^{t_f}\left\{\left(\frac{\partial H}{\partial\boldsymbol{x}} + \dot{\boldsymbol{\lambda}}\right)\delta\boldsymbol{x} + \left(\frac{\partial H}{\partial x_{n+1}} + \dot{\gamma}\right)\delta x_{n+1} + \frac{\partial H}{\partial\boldsymbol{u}}\delta\boldsymbol{u}\right\} dt$$

  where $H = L + \boldsymbol{\lambda}^T\boldsymbol{f} + \gamma N$

NOTE: $\int_{t_0}^{t_f} \boldsymbol{\lambda}\,\delta\dot{\boldsymbol{x}}\,dt = \boldsymbol{\lambda}^T\delta\boldsymbol{x}\Big|_{t_0}^{t_f} - \int_{t_0}^{t_f} \dot{\boldsymbol{\lambda}}^T\delta\boldsymbol{x}\,dt$

3. Set each coefficient in $\delta\bar{J}$ to zero to identify the equations required to solve this problem:

$$\dot{\boldsymbol{\lambda}}^T = -\frac{\partial H}{\partial \boldsymbol{x}}, \qquad \lambda^T(t_f) = \frac{\partial \varphi}{\partial \boldsymbol{x}(t_f)} + \boldsymbol{\nu}^T\frac{\partial \boldsymbol{\psi}}{\partial \boldsymbol{x}(t_f)}$$

$$\dot{\gamma} = -\frac{\partial H}{\partial x_{n+1}} = 0, \qquad \gamma(t_f) = \nu_{q+1}$$

$$\frac{\partial H}{\partial \boldsymbol{u}} = 0$$

$$\dot{\boldsymbol{x}} = \boldsymbol{f}\,(\boldsymbol{x},\,\boldsymbol{u},\,t)\,, \qquad \boldsymbol{x}(0) = \boldsymbol{x}_0$$

$$\boldsymbol{\psi}\left(\boldsymbol{x}_f,\,t_f\right) = \boldsymbol{c}$$

$$x_{n+1}(t_f) = k$$

NOTE: $2n + m + q + 1$ equations in $2n + m + q + 1$ unknowns

Example: Maximum Area with Given Perimiter ("Dido's Problem")

PROBLEM SET-UP:

- Independent (time) variable: x
- Dependent variable: y
- Control variable: θ
- Dynamic constraint: $\dfrac{dy}{dx} = \tan\theta$ [NOTE: $-\pi/2 < \theta \le \pi/2$]
- Integral constraint: $L = \displaystyle\int_{-a}^{a} \frac{1}{\cos\theta}dx$ [$dx = d\ell\cos\theta$]

- Terminal constraint: $y(x_f) = y(a) = 0$
- Cost function: $J = -\int_{-a}^{a} y dx \qquad \Rightarrow$ minimize $-A$

SOLUTION:

1. Augment the states:

$$\frac{dz}{dx} = \sec\theta \qquad z(a) = L$$

2. Adjoin the cost:

$$\begin{aligned}\bar{J} &= \nu_y y(a) + \nu_z \{z(a) - L\} \\ &+ \int_{-a}^{a} \left\{ -y + \lambda_y \left(\tan\theta - \frac{dy}{dx} \right) + \gamma \left(\sec\theta - \frac{dz}{dx} \right) \right\} dx\end{aligned}$$

3. Take the first variation of $\bar{J}$:

$$\begin{aligned}\delta J &= \left(\nu_y - \lambda_y\right) \delta y(a) + (\nu_z - \gamma)\, \delta z(a) \\ &+ \int_{-a}^{a} \left\{ \left(\frac{\partial H}{\partial y} + \dot{\lambda}_y \right) \delta y + \left(\frac{\partial H}{\partial z} + \dot{\gamma} \right) \delta z + \frac{\partial H}{\partial \theta} \delta\theta \right\} dx\end{aligned}$$

where $H = -y + \lambda_y \tan\theta + \gamma \sec\theta$

4. Identify the equations to be solved:

$$\begin{aligned}\dot{\lambda}_y &= 1; \quad \lambda_y(a) = \nu_y \Rightarrow \lambda_y = x + k_1 \\ \dot{\gamma} &= 0; \quad \gamma(a) = \nu_z \Rightarrow \gamma = \nu_z\end{aligned}$$

$$\frac{\partial H}{\partial \theta} = \lambda_y \sec^2\theta + \gamma \tan\theta \sec\theta = 0 \quad \Rightarrow \quad \sin\theta = -\frac{\lambda_y}{\gamma}$$

$$\begin{aligned} dy &= \tan\theta dx & \lambda_y &= x + k_1 = -\gamma \sin\theta \\ dy &= -\gamma \sin\theta d\theta & x &= -\gamma \sin\theta - k_1 \\ y &= \gamma\cos\theta + k_2 & dx &= -\gamma\cos\theta d\theta \end{aligned}$$

$$\int_{-a}^{a}\left(\frac{1}{\cos\theta}\right)dx = \int_{\theta_i}^{\theta_f} -\gamma d\theta = -\gamma\left(\theta_f - \theta_i\right) = L$$

- Unknowns: $k_1,\ k_2,\ \gamma,\ \theta_f,\ \theta_i$
- Equations:

$$\begin{aligned} x(\theta_i) &= -a \qquad x(\theta_f) = a \\ y(\theta_i) &= 0 \qquad y(\theta_f) = 0 \\ L &= -\gamma\left(\theta_f - \theta_i\right) \end{aligned}$$

$$\begin{aligned} &(1) \quad \gamma\sin\theta_i + k_1 = a \\ &(2) \quad -\gamma\sin\theta_f - k_1 = a \\ &(3) \quad \gamma\cos\theta_i + k_2 = 0 \\ &(4) \quad \gamma\cos\theta_f + k_2 = 0 \\ &(5) \quad -\gamma\left(\theta_f - \theta_i\right) = L \end{aligned}$$

(3) & (4) $\rightarrow\ \theta_f = \pm\theta_i$, but $(5)\ \Rightarrow\ \theta_f = -\theta_i; \quad \gamma = \dfrac{L}{2\theta_i}$

(1) & (2) $\rightarrow\ k_1 = 0; \quad a = \gamma\sin\theta_i$

$$\Rightarrow \quad \frac{\sin\theta_i}{\theta_i} = \frac{2a}{L}$$

Since $-\dfrac{\pi}{2} < \theta < \dfrac{\pi}{2}$, this equation $\Rightarrow\ P < \pi a$

$$\begin{aligned} k_1 &= 0 \\ \gamma &= \frac{L}{2\theta_i} \\ \theta_f &= -\theta_i \end{aligned}$$

$$k_2 = -\frac{L\cos\theta_i}{2\theta_i}$$

- What does all of this mean?

$$x = -\frac{L}{2\theta_i}\sin\theta \qquad y = \frac{L}{2\theta_i}\{\cos\theta - \cos\theta_i\}$$

$$\Rightarrow \quad x^2 + \left\{y + \frac{L}{2\theta_i}\cos\theta_i\right\}^2 = \left(\frac{L}{2\theta_i}\right)^2$$

- Therefore, the rope forms a circular arc of radius $L/2\theta_i$ centered at:

$$x = 0$$
$$y = -\frac{L}{2\theta_i}\cos\theta_i$$

- So far, we've examined path constraints by looking at "integral contraints"
- We'll continue the process by looking at equality constraints that must be satisfied along the entire optimal path

Control-Only Equality Constraints

- The standard (fixed-time, terminal constraint) problem is the same, but now we add an additional constraint on the controls:

$$\boldsymbol{G}\{u,\, t\} = \boldsymbol{k} \qquad t_o \le t \le t_f$$

[NOTE: $m \ge 2$ or else completely specified by $\boldsymbol{G}$]

$$\bar{J} = \varphi + \boldsymbol{\nu}^T(\boldsymbol{\psi} - \boldsymbol{c}) + \int_{t_0}^{t_f}\left\{L + \boldsymbol{\lambda}^T(\boldsymbol{f} - \dot{\boldsymbol{x}}) + \boldsymbol{\mu}^T(\boldsymbol{G} - \boldsymbol{k})\right\}dt$$

$$H = L + \boldsymbol{\lambda}^T\boldsymbol{f} + \boldsymbol{\mu}^T\boldsymbol{G}$$

- Since $\boldsymbol{G}$ is not a function of $\boldsymbol{x}$, all of the equations developed previously are valid except

$$\frac{\partial H}{\partial \boldsymbol{u}} = \frac{\partial L}{\partial \boldsymbol{u}} + \boldsymbol{\lambda}^T \frac{\partial \boldsymbol{f}}{\partial \boldsymbol{u}} + \boldsymbol{\mu}^T \frac{\partial \boldsymbol{G}}{\partial \boldsymbol{u}} = 0$$

$$\boldsymbol{G}(\boldsymbol{u},\, t) = \boldsymbol{k}$$

  – these two sets of equations provide enough information to identify $\boldsymbol{u}$ and $\boldsymbol{\mu}$

Control and State Equality Constraints

- Again, the standard problem is the same, but now our path constraints take the form:

$$\boldsymbol{G}\left\{\boldsymbol{x},\, \boldsymbol{u},\, t\right\} = \boldsymbol{k}$$

- $\bar{J}$ and H are identical to those shown above for the control-only equality constraints, so the equations required to solve this problem are:

$$\dot{\boldsymbol{\lambda}} = -\frac{\partial H}{\partial \boldsymbol{x}}^T = -\left\{\frac{\partial L}{\partial \boldsymbol{x}} + \boldsymbol{\lambda}^T \frac{\partial \boldsymbol{f}}{\partial \boldsymbol{x}} + \boldsymbol{\mu}^T \frac{\partial \boldsymbol{G}}{\partial \boldsymbol{x}}\right\}^T$$

$$\boldsymbol{\lambda}(t_f) = \frac{\partial \Phi}{\partial \boldsymbol{x}(t_f)} \qquad \text{where } \Phi = \varphi + \boldsymbol{\nu}^T\left[\boldsymbol{\psi} - \boldsymbol{c}\right]$$

$$\dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x},\, \boldsymbol{u},\, t), \qquad \boldsymbol{x}(0) = \boldsymbol{x}_0$$

$$\frac{\partial H}{\partial \boldsymbol{u}} = \frac{\partial L}{\partial \boldsymbol{u}} + \boldsymbol{\lambda}^T \frac{\partial \boldsymbol{f}}{\partial \boldsymbol{u}} + \boldsymbol{\mu}^T \frac{\partial \boldsymbol{G}}{\partial \boldsymbol{u}} = 0$$

$$\boldsymbol{G}(\boldsymbol{x},\, \boldsymbol{u},\, t) = \boldsymbol{k}$$

$$\psi\left\{\boldsymbol{x}(t_f),\, \boldsymbol{u},\, t\right\} = \boldsymbol{c}$$

  – Unknowns: $\boldsymbol{x},\ \boldsymbol{\lambda},\ \boldsymbol{\nu},\ \boldsymbol{\mu},\ \boldsymbol{u}$

State-Only Equality Constraints

- Once again, the standard problem is the same; but now our path constraints are a function of $\boldsymbol{x}$ and t only

$$\boldsymbol{G}(\boldsymbol{x},\ t) = \boldsymbol{k}$$

  - this is a somewhat more complicated problem because some of the elements of $\boldsymbol{x}(t)$ depend on other elements of $\boldsymbol{x}(t)$ as well as the previous $\boldsymbol{x}$ and $\boldsymbol{u}$

- There are a number of different ways to solve this problem:
  - Method 1:
    - solve for one subset of the states as a function of the remaining states and time
    - reduce the dimension of the state vector using the solution derived above
    - problem with this approach $\rightarrow$ the choice of state subsets is not unique, so some choices may produce more difficulties than others
  - Method 2:
    - convert the state-only constraints into control and state constraints
      - since $\boldsymbol{G}(\boldsymbol{x},\ t) = \boldsymbol{k}$ along the optimal path, $\partial \boldsymbol{G}/\partial t$ must be zero along the optimal path

$$\frac{\partial \boldsymbol{G}}{\partial t} = \frac{\partial \boldsymbol{G}}{\partial t} + \frac{\partial \boldsymbol{G}}{\partial \boldsymbol{x}}\frac{d\boldsymbol{x}}{dt} = \frac{\partial \boldsymbol{G}}{\partial t} + \frac{\partial \boldsymbol{G}}{\partial \boldsymbol{x}}\boldsymbol{f} = 0$$

      - in general, $\boldsymbol{f}$ is a function of $\boldsymbol{u}$, so $d\boldsymbol{G}/dt = 0$ is a control and state constraint

- ∘ but ${}^{d\boldsymbol{G}}/_{dt} = 0$ even when $\boldsymbol{G} \neq \boldsymbol{k}$, so we must add additional terminal constraints,

$$\boldsymbol{G}\left\{\boldsymbol{x}(t_f),\; t_f\right\} = \boldsymbol{k}$$

to ensure tha the proper constraint is applied

- ∘ if ${}^{dG}/_{dt}$ is not a function of u, additional derivatives can be taken and additional termianl constraints added

– Method 3: a computational alternative using "soft" constraints

- ∘ define a new cost function:

$$\bar{J} = J + K \int_{t_0}^{t_f} \{\boldsymbol{G} - \boldsymbol{k}\}^T \{\boldsymbol{G} - \boldsymbol{k}\}\, dt$$

- ∘ select K to establish the proper trade-off between the various elements of the cost

Continuous-Time Optimization: Inequality Constraints

- In many problems, we may not need to force the satisfaction of an equality constraint but instead may be forced to live with inequality constraints driven by physical attributes of the problem (e.g., limited fuel)

  – in most instances, these inequality constraints apply only to the available control variables

  – so, we'll focus our attention on control-only inequality constrained problems:

$$J = \varphi + \int_{t_0}^{t_f} L dt$$

$$\dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x},\, u,\, t)$$

$$\boldsymbol{\psi}\left\{\boldsymbol{x}(t_f),\; t_f\right\} = \boldsymbol{k}_1$$

$$\boldsymbol{c}\{\boldsymbol{u}(t),\ t\} \le k_2$$

STANDARD CALCULUS OF VARIATIONS APPROACH

$$J = \varphi + \boldsymbol{\nu}^T\{\boldsymbol{\psi} - \boldsymbol{k}_1\} + \int_{t_0}^{t_f}\left\{L + \boldsymbol{\lambda}^T(\boldsymbol{f} - \dot{\boldsymbol{x}}) + \boldsymbol{\mu}^T(\boldsymbol{c} - \boldsymbol{k}_2)\right\}dt$$

$$H = L + \boldsymbol{\lambda}^T\boldsymbol{f} + \boldsymbol{\mu}^T\boldsymbol{c}$$

- All of the equations developed previously still apply:

$$\dot{\boldsymbol{\lambda}}^T = -\frac{\partial H}{\partial \boldsymbol{x}} \qquad \boldsymbol{\lambda}^T(t_f) = \frac{\partial \Phi}{\partial \boldsymbol{x}(t_f)}$$

$$\dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x},\ \boldsymbol{u},\ t) \qquad \boldsymbol{x}(t_0) = \boldsymbol{x}_0$$

$$\frac{\partial H}{\partial \boldsymbol{u}} = 0$$

$$\boldsymbol{\psi} = \boldsymbol{k}_1$$

$$\boldsymbol{c} \le \boldsymbol{k}_2$$

NOTE:

$\boldsymbol{\mu} = 0$ if constraints are inactive

$\boldsymbol{\mu} > 0$ if constraints are active

ALTERNATIVE APPROACH: PONTRYAGIN'S MINIMUM PRINCIPLE

- Russian mathematician Pontryagin demonstrated that the optimal control must minimize the function

$$H = \boldsymbol{\lambda}^T\boldsymbol{f}$$

  for all admissable controls

- So what? We already knew that we had to solve ${}^{\partial H}/_{\partial \boldsymbol{u}} = 0$
  – in many inequality constrained problems, finding the $\boldsymbol{u}$ which minimizes H is obvious

– and Pontryagin proved that this $\boldsymbol{u}$ is the optimal one, so we don't need to add the complexities associated with additional Lagrange multipliers

Example:

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = -x_2 + u$$

$$J = \int_{t_0}^{t_f} \frac{1}{2}\left(x_1^2 + u^2\right) dt$$

t_f specified $\quad\quad$ $\boldsymbol{x}(t_f)$ free

1. No Control Constraints:

$$H = \frac{1}{2}\left(x_1^2 + u^2\right) + \lambda_1 x_2 + \lambda_2 (u - x_2)$$

$$\dot{\lambda}_1 = -x_1, \qquad \lambda_1(t_f) = 0$$
$$\dot{\lambda}_2 = \lambda_2 - \lambda_1, \qquad \lambda_2(t_f) = 0$$
$$u = -\lambda_2$$

- so, we augment the states to solve for $\boldsymbol{x}$ and $\boldsymbol{\lambda}$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{\lambda}_1 \\ \dot{\lambda}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & -1 & 0 & -1 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & -1 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \lambda_1 \\ \lambda_2 \end{bmatrix}$$

– recall that $u = -\lambda_2$

$$\Rightarrow \quad \boldsymbol{z}(t) = \mathcal{L}^{-1}\left\{(sI - A)^{-1}\right\} \boldsymbol{z}(0)$$

– and since $x_1(t_0),\ x_2(t_0),\ \lambda_1(t_f)$, and $\lambda_2(t_f)$ are known, $\lambda_1(0)$ and $\lambda_2(0)$ can be identified and used to describe $x(t)$, $\lambda(t)$, and $u(t)$

2. Constrained Control: What if $|u| \le 1$?

$$H = \frac{1}{2}\left(x_1^2 + u^2\right) + \lambda_1 x_2 + \lambda_2 u - \lambda_2 x_2$$

- using Pontryagin's Principle, the control that minimizes H is

$$u = -\lambda_2, \qquad \text{provided } |u| \le 1$$

  – what if $\lambda_2 > 1$? Pick $u = -1$ to minimize H

  – what if $\lambda_2 < -1$? Pick $u = +1$ to minimize H

  – what if $-1 < \lambda_2 < 1$? Pick $u = -\lambda_2$ to minimize H

- the solution to this problem is not the same as that for the unconstrained problem because u may not be continuous

  – so, we may need to solve the problem in parts by piecing together constrained and unconstrained arcs

Introduction to Linear Constraints

- A special case of the control inequality constraint problem in which Pontryagin's Principle plays a very important role occurs when the dynamic system constraints and the control variable inequality constraints are all linear:

$$\dot{\boldsymbol{x}} = A\boldsymbol{x} + Bu \qquad -1 \le u(t) \le 1$$

we'll let u be a scalar in the follwing development to keep things simple, but the ideas are easily extended

Example: Single-Axis Satellite Attitude Control Using Reaction Jets

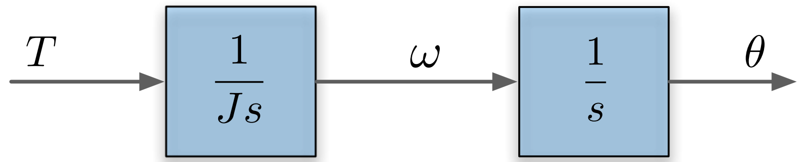

Figure 5.9 Single-Axis Satellite Attitude Control

- Define:

$$x_1 = \theta$$
$$x_2 = \omega$$

- Equations of motion:

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = \frac{1}{2}T$$

where T is the commanded input

- So, we can write,

$$\dot{\boldsymbol{x}} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \boldsymbol{x} + \begin{bmatrix} 0 \\ \dfrac{1}{J} \end{bmatrix} u$$

$$u = T \quad \text{and} \quad -1 \le \frac{T}{T_{max}} \le 1$$

- How do we solve this problem? It depends on what we want to do

1. minimum time $\Rightarrow$
   – since the system is linear and the performance index is linear, we should expect that the optimal solution requires a control that lies on the boundary of the feasible region

  – in addition, one or more changes in control may be required during operation $\rightarrow$ the control may suddently change from one point on the boundary to another [BANG-BANG CONTROL]

2. minimum fuel $\Rightarrow$
   – same linear system, but different linear cost $\rightarrow$ now saving fuel is more important than saving time
   – at certain times, it may be beneficial to turn the reaction jets off [BANG-OFF-BANG CONTROL]

3. minimum energy $\Rightarrow$
   – quadratic cost $\rightarrow$ variable control in the feasible region

- Consider the reaction jet problem,

$$\begin{bmatrix} \dot{\theta} \\ \dot{\omega} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \theta \\ \omega \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{1}{J} \end{bmatrix} T$$

where the torque generated by the reaction jets is limited to the range $-T_m \le T \le T_m$

MINIMUM TIME PROBLEM

$$J = \int_{t_0}^{t_f} dt$$

with terminal constraints $\Rightarrow$ $\boldsymbol{x}(t_f) = 0$

$$\bar{J} = \boldsymbol{\nu}^T \boldsymbol{x}(t_f) + \int_{t_0}^{t_f} \left\{ 1 + \boldsymbol{\lambda}^T \left(A\boldsymbol{x} + B\boldsymbol{u} - \dot{\boldsymbol{x}} \right) \right\} dt$$

$$\dot{\boldsymbol{\lambda}}^T = -\frac{\partial H}{\partial \boldsymbol{x}} = -\boldsymbol{\lambda}^T A \qquad \lambda^T(t_f) = \frac{\partial \Phi}{\partial x(t_f)} = \nu^T$$

$$\dot{\boldsymbol{x}} = A\boldsymbol{x} + B\boldsymbol{u} \qquad \boldsymbol{x}(0) = \boldsymbol{x}_0$$

$$H(t_f) = 1 + \boldsymbol{\lambda}^T(t_f)\left\{A\boldsymbol{x}(t_f) + B\boldsymbol{u}(t_f)\right\} = -\frac{\partial \Phi}{\partial t_f} = 0$$

$$\boldsymbol{x}(t_f) = 0$$

And

$$\frac{\partial H}{\partial \boldsymbol{u}} = 0 \quad (???)$$

But,

$$\frac{\partial H}{\partial \boldsymbol{u}} = \boldsymbol{\lambda}^T B \neq 0 \quad (unless\ \boldsymbol{\lambda}^T = 0)$$

So what does this mean? What do we do?

- The dilemma that has arisen is a direct result of the linear nature of the problem that we've set up
  - if the control were unconstrained, this condition $(\partial H/\partial \boldsymbol{u} = 0)$ would suggest that there is always a $\delta \boldsymbol{u}$ that can be selected to reduce the cost
  - but remember, our control is constrained
- To identify the proper $\boldsymbol{u}$, we'll use Pontryagin's Principle,

$$\min_{\boldsymbol{u}} H = \min_{\boldsymbol{u}} \left\{1 + \boldsymbol{\lambda}^T A\boldsymbol{x} + \boldsymbol{\lambda}^T B\boldsymbol{u}\right\}$$

  - I can't do anything about $1 + \boldsymbol{\lambda}^T A\boldsymbol{x}$, but I can choose $\boldsymbol{u}$ so that $\boldsymbol{\lambda}^T B\boldsymbol{u}$ is as small as possible:

$$\boldsymbol{u} = -T_m \operatorname{sgn}\left(\boldsymbol{\lambda}^T A\right)$$

  - $\boldsymbol{\lambda}^T A$ is a function of time and is often referred to as the "switching function" since it will determine when $\boldsymbol{u} = +T_m$ and $\boldsymbol{u} = -T_m$
- Additional details of the solution can now be obtained by examining the remaining equations:

$$\dot{\boldsymbol{\lambda}}^T = -\boldsymbol{\lambda}^T A \quad \Rightarrow \quad \begin{aligned} \dot{\lambda}_1 &= 0 & \Rightarrow \lambda_1 &= \text{constant} \\ \dot{\lambda}_2 &= -\lambda_1 & \Rightarrow \lambda_2 &= \lambda_2(0) - \lambda_1 t \end{aligned}$$

$$\lambda^T B = \frac{\lambda_2}{J} \quad \Rightarrow \quad u(t) \text{ depends on } \lambda_2(t)$$

– note that since $\lambda_2(t)$ is linear, $\boldsymbol{\lambda}^T A$ can change signs at most once $\Rightarrow$ the control, $u(t)$, will switch at most one time

– can we get more information about the control? YES

$$H(t_f) = 0 \quad \Rightarrow \quad \left\{ \frac{\lambda_2(t_f)}{J} \right\} u(t_f) = -1$$

NOTE: $\boldsymbol{x}(t_f) = 0$ by the forced constraints

$\Rightarrow$

$$\lambda_2(t_f) = \frac{-1}{T_m} \quad \text{when } u(t_f) = T_m$$

$$\lambda_2(t_f) = \frac{1}{T_m} \quad \text{when } u(t_f) = -T_m$$

- Based on the results above, what possible optimal control strategies exist?

1. $u = -T_m$ for all $t \geq 0$
2. $u = -T_m$ for $t < t_f$; switches to $u = +T_m$ for $t \geq t_f$
3. $u = +T_m$ for $t < t_f$; switches to $u = -T_m$ for $t \geq t_f$
4. $u = +T_m$ for all $t \geq 0$

– which of these strategies do we use? it depends on θ and ω

FEEDBACK

$$\dot{\theta} = \omega$$

$$\dot{\omega} = \frac{u}{J}$$

$\Rightarrow$

$$\frac{d\omega}{d\theta} = \frac{u}{J\omega}$$

$$\omega d\omega = \frac{u}{J} d\theta$$

$$\omega^2 = \frac{2u}{J}\theta + c_0$$

$\Rightarrow$

$$\omega^2 = \pm\frac{2T_m}{J}\theta + c_0$$

which gives a family of parabolas

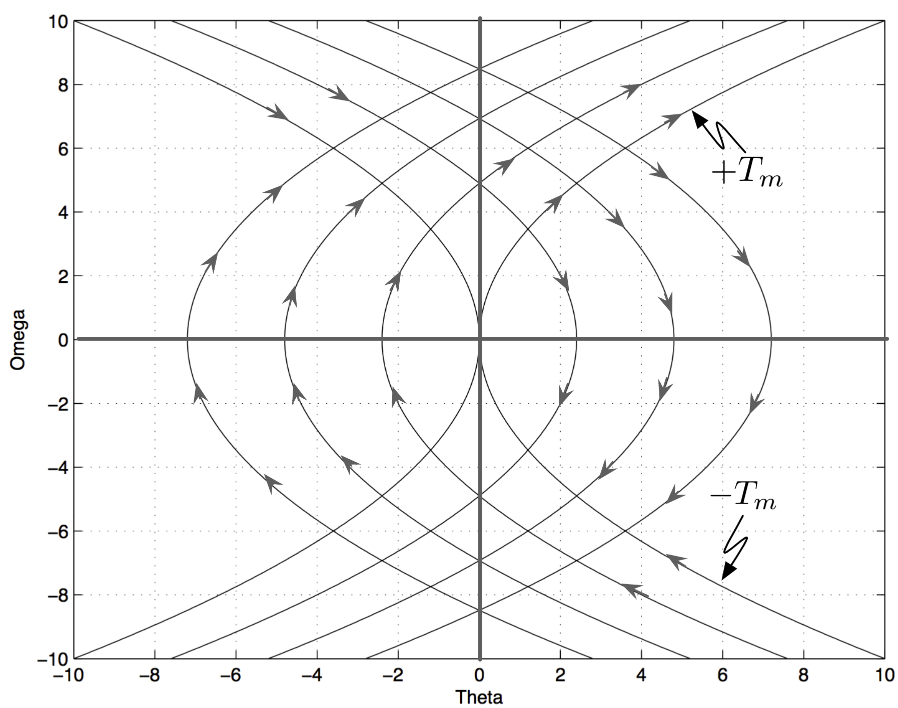

Figure 5.10 Family of Parabolas

- Two of these parabolas go through the origin and these are the ones we want to get on in order to get to $\boldsymbol{x}(t_f) = 0$

1. if $\omega < 0$, $\theta = \dfrac{J}{2T_m}\omega^2$
2. if $\omega > 0$, $\theta = -\dfrac{J}{2T_m}\omega^2$

$$\theta = -\frac{J}{2T_m}\omega\,|\omega| \quad \text{switching curve}$$

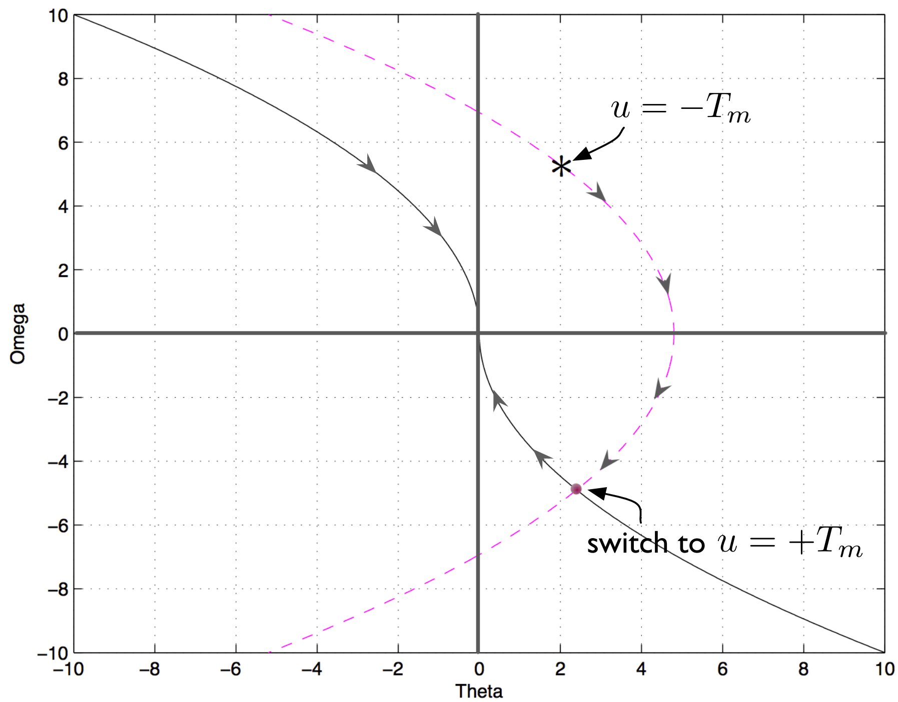

Figure 5.11 Switching Curve

- The FEEDBACK LAW is:
  1. $u = -T_m$ when $\theta > -\dfrac{J}{2T_m}\omega\,|\omega|$ or $\theta = -\dfrac{J}{2T_m}\omega\,|\omega|$ and $\theta < 0$
  2. $u = +T_m$ when $\theta < -\dfrac{J}{2T_m}\omega\,|\omega|$ or $\theta = -\dfrac{J}{2T_m}\omega\,|\omega| > 0$
- Can we solve for the times required to perform this maneuver?
  – YES, but using the dynamics of the satellite and knowledge of the control strategy

– assume we start at $t = 0$ with some initial conditions $(\theta_0,\ \omega_0)$

– since u is constant from 0 to t_s,

$$\theta(t_s) = \theta_0 + \omega_0 t_s + \frac{u}{J}\frac{t_s^2}{2}$$

$$\omega(t_s) = \omega_0 + \frac{u}{J}t_s$$

$$u = \pm T_m$$

– at t_s^- (just before the switch),

$$\theta(t_s) = -\frac{J}{2u}\omega^2(t_s)$$

$$\theta_0 + \omega_0 t_s + \frac{u}{J}\frac{t_s^2}{2} = -\frac{J}{2u}\left\{\omega_0^2 + \frac{2u}{J}\omega_0 t_s + \left(\frac{u}{J}\right)^2 t_s^2\right\}$$

$$-\frac{J}{2u}\omega_0^2 - \omega_0 t_s - \frac{u}{2J}t_s^2 = \theta_0 + \omega_0 t_s + \frac{u}{2J}t_s^2$$

$$\frac{u}{J}t_s^2 + 2\omega_0 t_s + \theta_0 + \frac{J}{2u}\omega_0^2 = 0$$

$$t_s^2 + \frac{2J}{u}\omega_0 t_s + \left(\frac{J}{u}\right)^2\omega_0^2 - \left(\frac{J}{u}\right)^2\omega_0^2 + \frac{J}{u}\theta_0 + \frac{1}{2}\left(\frac{J}{u}\right)^2\omega_0^2 = 0$$

$$\left(t_s + \frac{J}{u}\omega_0\right)^2 = \left(\frac{J}{u}\right)^2\left\{\frac{\omega_0^2}{2} - \frac{u}{J}\theta_0\right\}$$

$$\Rightarrow \quad t_s = -\frac{J}{u}\left\{\omega_0 \pm \sqrt{\frac{\omega_0^2}{2} - \frac{u}{J}\theta_0}\right\}$$

– after the switch at t_s,

$$\omega(t_f) = \omega(t_s) - \frac{u}{J}\left(t_f - t_s\right) = 0$$

◦ but $\omega(t_f) = \omega_0 + \dfrac{u}{J} t_s \;\Rightarrow$

$$t_f = 2t_s + \frac{J}{u}\omega_0$$

MINIMUM FUEL PROBLEM:

- Same as above except

$$J = \int_{t_0}^{t_f} |u|\, dt$$

Note: t_f is specified and $t_f > t_{min}$!

MINIMUM ENERGY PROBLEM:

- Same as above except

$$J = \int_{t_0}^{t_f} u^2 dt$$

– because J is quadratic in u, the standard optimization techniques may be applied to solve this problem

$$\frac{\partial H}{\partial u} = 0 \quad \Rightarrow \quad \text{solve for } u$$

◦ but remember, we must ensure that u doesn't violate its constraints

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