STABILITY ANALYSIS TECHNIQUES

4.1: Bilinear transformation

- Three main aspects to control-system design:
  1. Stability,
  2. Steady-state response,
  3. Transient response.

- Here, we look at determining system stability using various methods.

**DEFINITION:** A system is BIBO stable iff a bounded input produces a bounded output.

- Check by first writing system input–output relationship as

\[ Y(z) = \frac{G(z)}{1 + GH(z)}R(z) = \frac{K \prod_{i=1}^{m} (z - z_i)}{\prod_{i=1}^{n} (z - p_i)}R(z). \]

- Assume for now that all the poles \( \{p_i\} \) are distinct and different from the poles in \( R(z) \). Then,

\[
Y(z) = \frac{k_1 z}{z - p_1} + \cdots + \frac{k_n z}{z - p_n} \quad + \quad Y_R(z).
\]

- If the system is stable, the response to initial conditions must decay to zero as time progresses.

\[
\mathcal{Z}^{-1} \left[ \frac{k_i z}{z - p_i} \right] = k_i (p_i)^k 1[k].
\]

So, the system is stable if \( |p_i| < 1 \).
• \{p_i\} are the roots of \(1 + \overline{GH}(z) = 0\). So, the roots of \(1 + \overline{GH}(z) = 0\) must lie within the unit circle of the \(z\)-plane.
  
  • Same result even if poles are repeated, but harder to show.
  
  • If the magnitude of a pole \(|p_i| = 1\), then the system is marginally stable. The unforced response does not decay to zero but also does not increase to \(\infty\). However, it is possible to drive the system with a bounded input and have the output go to \(\infty\). Therefore, a marginally stable system is \textit{unstable.}

**Bilinear transformation**

• The stability criteria for a discrete-time system is that all its poles lie within the unit circle on the \(z\)-plane.
  
  • Stability criteria for cts.-time systems is that the poles be in the LHP.
    
    • Simple tool to test for continuous-time stability—Routh test.
  
  • Can we use the Routh test to determine stability of a discrete-time system (either directly or indirectly)?
  
  • To use the Routh test, we need to do a \(z\)-plane to \(s\)-plane conversion that retains stability information. The \(s\)-plane version of the \(z\)-plane system does NOT need to correspond in any other way.
  
  • That is,
    
    • The frequency responses may be different
    
    • The step responses may be different...
Since only stability properties are maintained by the transform, it is not accurate to label the destination plane the $s$-plane. It is often called the $w$-plane, and the transformation between the $z$-plane and the $w$-plane is called the $w$-Transform.

A transform that satisfies these requirements is the bilinear transform. Recall:

$$H(w) = H(z) \big|_{z = \frac{1+(T/2)w}{1-(T/2)w}} \quad \text{and} \quad H(z) = H(w) \big|_{w = \frac{z-1}{T(z+1)}}.$$  

Three things to check:

1. Unit circle in $z$-plane $\leftrightarrow j\omega$-axis in $w$-plane.
2. Inside unit circle in $z$-plane $\leftrightarrow$ LHP in $w$-plane.
3. Outside unit circle in $z$-plane $\leftrightarrow$ RHP in $w$-plane.

If true,

1. Take $H(z) \leftrightarrow H(w)$ via the bilinear transform.
2. Perform Routh test on $H(w)$.

CHECK: Let $z = re^{j\omega T}$. Then, $z$ is on the unit circle if $r = 1$, $z$ is inside the unit circle if $|r| < 1$ and $z$ is outside the unit circle if $|r| > 1$.

$$z = re^{j\omega T}$$

$$w = \frac{2z-1}{Tz+1} \bigg|_{z=re^{j\omega T}} = \frac{2re^{j\omega T} - 1}{Tre^{j\omega T} + 1}.$$  

Expand $e^{j\omega T} = \cos(\omega T) + j\sin(\omega T)$ and use the shorthand $c \triangleq \cos(\omega T)$ and $s \triangleq \sin(\omega T)$. Also note that $s^2 + c^2 = 1$.

$$w = \frac{2}{T} \left[ \frac{rc + jrs - 1}{rc + jrs + 1} \right]$$
\[
\begin{align*}
\frac{2}{T} \left[ \frac{(rc - 1) + jrs}{(rc + 1) + jrs} \right] \left[ \frac{(rc + 1) - jrs}{(rc + 1) - jrs} \right] \\
= \frac{2}{T} \left[ \frac{(r^2c^2 - 1) + j(rs)(rc + 1) - j(rs)(rc - 1) + r^2s^2}{(rc + 1)^2 + (rs)^2} \right] \\
= \frac{2}{T} \left[ \frac{r^2 - 1}{r^2 + 2rc + 1} \right] + j \frac{2}{T} \left[ \frac{2rs}{r^2 + 2rc + 1} \right].
\end{align*}
\]

Notice that the real part of \( w \) is 0 when \( r = 1 \) (\( w \) is on the imaginary axis), the real part of \( w \) is negative when \( |r| < 1 \) (\( w \) in LHP), and that the real part of \( w \) is positive when \( |r| > 1 \) (\( w \) in RHP). Therefore, the bilinear transformation does exactly what we want.

- When \( r = 1 \),
  \[ w = j \frac{2}{T} \frac{2 \sin(\omega T)}{2 + 2 \cos(\omega T)} = j \frac{2}{T} \tan \left( \frac{\omega T}{2} \right), \]
  which will be useful to know.

- The following diagram summarizes the relationship between the \( s \)-plane, \( z \)-plane, and \( w \)-plane:
4.2: Discrete-time stability via Routh–Hurwitz test

- Review of Routh test.

Let \( H(w) = \frac{b(w)}{a(w)} \ldots a(w) \) is the characteristic polynomial.

\[
a(w) = a_n w^n + a_{n-1} w^{n-1} + \cdots + a_1 w + a_0.
\]

Case 0: If any of the \( a_n \) are negative then the system is unstable (unless ALL are negative).

Case 1: Form Routh array:

| \( w^n \) | \( a_n \) | \( a_{n-2} \) | \( a_{n-4} \) | \( \cdots \) |
| \( w^{n-1} \) | \( a_{n-1} \) | \( a_{n-3} \) | \( a_{n-5} \) | \( \cdots \) |
| \( w^{n-2} \) | \( b_1 \) | \( b_2 \) | \( \cdots \) |
| \( w^{n-3} \) | \( c_1 \) | \( c_2 \) | \( \cdots \) |
| \( \vdots \) | \( j_1 \) | \( \vdots \) |
| \( w^1 \) | \( k_1 \) |
| \( w^0 \) | \( k_1 \) |

\[
b_1 = \frac{-1}{a_{n-1}} \quad b_2 = \frac{-1}{a_{n-1}} \quad c_1 = \frac{-1}{b_1} \quad c_2 = \frac{-1}{b_1}
\]

**TEST:** Number of RHP roots = number of sign changes in left column.

Case 2: If one of the left column entries is zero, replace it with \( \varepsilon \) as \( \varepsilon \to 0 \).

Case 3: Suppose an entire row of the Routh array is zero, the \( w^{i-1} \)th row. The \( w^i \)th row, right above it, has coefficients \( a_1, a_2, \ldots \)
Then, form the auxiliary equation:

\[ a_1 w^i + a_2 w^{i-2} + a_3 w^{i-4} + \cdots = 0. \]

This equation is a factor of the characteristic equation and must be tested for RHP roots (it WILL have non-LHP roots—we might want to know how many are RHP).

**EXAMPLE:** Consider:

![Control System Diagram](image)

\[ G(s) = \left( \frac{1 - e^{-Ts}}{s} \right) \left( \frac{1}{s(s+1)} \right). \]

- From \( z \)-transform tables:

  \[ G(z) = \left( \frac{z - 1}{z} \right) Z \left[ \frac{1}{s^2(s+1)} \right] \]
  \[ = \left( \frac{z - 1}{z} \right) \left( \frac{(e^{-T} + T - 1)z^2 + (1 - e^{-T} - Te^{-T})z}{(z - 1)^2(z - e^{-T})} \right). \]

  Let \( T = 0.1 \) s.

  \[ = \frac{0.00484z + 0.00468}{(z - 1)(z - 0.905)}. \]

- Perform the bilinear transform

  \[ G(w) = G(z) \bigg|_{z=\frac{1+(T/2)w}{1-(T/2)w}} \]
  \[ = G(z) \bigg|_{z=\frac{1+0.05w}{1-0.05w}} \]
  \[ = -0.00016w^2 - 0.1872w + 3.81 \]
  \[ \quad = \frac{3.81w^2 + 3.80w}{3.81w^2 + 3.80w}. \]

- The characteristic equation is:
\[ 0 = 1 + KG(w) \]
\[ = (3.81 - 0.00016K)w^2 + (3.80 - 0.1872K)w + 3.81K. \]

\[
\begin{array}{c|cc}
\text{w}^2 & (3.81 - 0.00016K) & 3.81K \\
\text{w}^1 & (3.80 - 0.1872K) & \Rightarrow K < 20.3 \\
\text{w}^0 & 3.81K & \Rightarrow K > 0 \\
\end{array}
\]

- So, for stability, \(0 < K < 20.3\).

**NOTE:** The “equivalent” continuous-time system is:

\[
T(s) = \frac{KG(s)}{1 + KG(s)}.
\]

- Characteristic equation: \(s(s + 1) + K = 0\).

\[
\begin{array}{c|cc}
\text{s}^2 & 1 & K \\
\text{s}^1 & 1 & \Rightarrow \text{K} < 2.39 \\
\text{s}^0 & K & \Rightarrow \text{K} > 0 \\
\end{array}
\]

- Stable for all \(K > 0\) \(\Rightarrow\) sample and hold destabilizes the system.

**EXAMPLE:** Let’s do the same example, but with \(T = 1\) s (not 0.1 s).

- (math happens)

\[
0 = 1 + KG(w) \\
= (1 - 0.0381K)w^2 + (0.924 - 0.86K)w + 0.924K.
\]

\[
\begin{array}{c|cc}
\text{w}^2 & (1 - 0.0381K) & 0.924K \\
\text{w}^1 & (0.924 - 0.386K) & \Rightarrow K < 26.2 \\
\text{w}^0 & 0.924K & \Rightarrow K > 0 \\
\end{array}
\]
- So, for stability, $0 < K < 2.39$.

- This is a much more restrictive range than when $T = 0.1$ s → slow sampling really destabilizes a system.
4.3: Jury’s stability test

- \( H(z) \leftrightarrow H(w) \leftrightarrow \) Routh is complicated and error-prone.
- Jury made a direct test on \( H(z) \) for stability.
- Disadvantage (?) ... another test to learn.
- Let \( T(z) = \frac{b(z)}{a(z)} \), \( a(z) = \) “characteristic polynomial.”
- \( a(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_1 z + a_0 = 0, \quad a_n > 0. \)
- Form Jury array:

<table>
<thead>
<tr>
<th>( z^0 )</th>
<th>( z^1 )</th>
<th>( z^2 )</th>
<th>\cdots</th>
<th>( z^{n-k} )</th>
<th>\cdots</th>
<th>( z^{n-1} )</th>
<th>( z^n )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( a_0 )</td>
<td>( a_1 )</td>
<td>( a_2 )</td>
<td>\cdots</td>
<td>( a_{n-k} )</td>
<td>\cdots</td>
<td>( a_{n-1} )</td>
<td>( a_n )</td>
</tr>
<tr>
<td>( a_n )</td>
<td>( a_{n-1} )</td>
<td>( a_{n-2} )</td>
<td>\cdots</td>
<td>( a_k )</td>
<td>\cdots</td>
<td>( a_1 )</td>
<td>( a_0 )</td>
</tr>
<tr>
<td>( b_0 )</td>
<td>( b_1 )</td>
<td>( b_2 )</td>
<td>\cdots</td>
<td>( b_{n-k} )</td>
<td>\cdots</td>
<td>( b_{n-1} )</td>
<td></td>
</tr>
<tr>
<td>( b_{n-1} )</td>
<td>( b_{n-2} )</td>
<td>( b_{n-3} )</td>
<td>\cdots</td>
<td>( b_{k-1} )</td>
<td>\cdots</td>
<td>( b_0 )</td>
<td></td>
</tr>
<tr>
<td>( c_0 )</td>
<td>( c_1 )</td>
<td>( c_2 )</td>
<td>\cdots</td>
<td>( c_{n-k} )</td>
<td>\cdots</td>
<td></td>
<td></td>
</tr>
<tr>
<td>( c_{n-2} )</td>
<td>( c_{n-3} )</td>
<td>( c_{n-4} )</td>
<td>\cdots</td>
<td>( c_{k-2} )</td>
<td>\cdots</td>
<td></td>
<td></td>
</tr>
<tr>
<td>( l_0 )</td>
<td>( l_1 )</td>
<td>( l_2 )</td>
<td>( l_3 )</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>( l_3 )</td>
<td>( l_2 )</td>
<td>( l_1 )</td>
<td>( l_0 )</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>( m_0 )</td>
<td>( m_1 )</td>
<td>( m_2 )</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

- Quite different from Routh array.
  - Every row is duplicated ... in reverse order.
  - Final row in table has three entries (always).
  - Elements are calculated differently.

\[
b_k = \begin{vmatrix} a_0 & a_{n-k} \\ a_n & a_k \end{vmatrix}, \quad c_k = \begin{vmatrix} b_0 & b_{n-1-k} \\ b_{n-1} & b_k \end{vmatrix}, \quad d_k = \begin{vmatrix} c_0 & c_{n-2-k} \\ c_{n-2} & c_k \end{vmatrix} \cdots
\]

• Stability criteria is different.

\[ a(z)|_{z=1} > 0 \]
\[ (-1)^n \ a(z)|_{z=-1} > 0 \quad n = \text{order of } a(z) \]
\[ |a_0| < a_n \]
\[ |b_0| > |b_{n-1}| \]
\[ |c_0| > |c_{n-2}| \]
\[ |d_0| > |d_{n-3}| \]
\[ \vdots \]
\[ |m_0| > |m_2|. \]

• First, check that \( a(1) > 0 \), \((-1)^n a(-1) > 0\) and \( |a_0| < a_n \). (relatively few calculations). If not satisfied, \textit{stop}.

• Next, construct array. \textit{Stop} if any condition not satisfied.

**EXAMPLE:**

\[ r(t) \rightarrow K \frac{1 - e^{-st}}{s} \frac{K}{s(s + 1)} \rightarrow y(t) \]
\[ r[k] \rightarrow K \frac{0.368z + 0.264}{z^2 - 1.368z + 0.368} \rightarrow y[k] \]

• Characteristic equation:

\[ 0 = 1 + KG(z) = 1 + K \frac{(0.368z + 0.264)}{z^2 - 1.368z + 0.368} \]
\[ z^2 + (0.368K - 1.368)z + (0.368 + 0.264K). \]

- The Jury array is:

<table>
<thead>
<tr>
<th>( z^0 )</th>
<th>( z^1 )</th>
<th>( z^2 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.368 + 0.264K</td>
<td>0.368K - 1.368</td>
<td>1</td>
</tr>
</tbody>
</table>

- The constraint \( a(1) > 0 \) yields

\[ 1 + 0.368K - 1.368 + 0.368 + 0.264K = 0.632K > 0 \implies K > 0. \]

- The constraint \((-1)^2a(-1) > 0 \) yields

\[ 1 - 0.368K + 1.368 + 0.368 + 0.264K = -0.104K + 2.736 > 0 \implies K < 26.3. \]

- The constraint \(|a_0| < a_2 \) yields

\[ 0.368 + 0.264K < 1 \implies K < \frac{0.632}{0.264} = 2.39. \]

- So, \( 0 < K < 2.39. \) (Same result as on pg. 4–8 using bilinear rule.)

**EXAMPLE:** Suppose that the characteristic equation for a closed-loop discrete-time system is given by the expression:

\[ a(z) = z^3 - 1.8z^2 + 1.05z - 0.20 = 0. \]

- \( a(1) = 1 - 1.8 + 1.05 - 0.2 = 0.05 > 0 \)
- \((-1)^3a(-1) = -[-1 - 1.8 - 1.05 - 0.2] > 0 \)
- \(|a_0| = 0.2 < a_3 = 1 \)

Jury array:

<table>
<thead>
<tr>
<th>( z^0 )</th>
<th>( z^1 )</th>
<th>( z^2 )</th>
<th>( z^3 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>-0.2</td>
<td>1.05</td>
<td>-1.8</td>
<td>1</td>
</tr>
<tr>
<td>1</td>
<td>-1.8</td>
<td>1.05</td>
<td>-0.2</td>
</tr>
<tr>
<td>-0.96</td>
<td>1.59</td>
<td>-0.69</td>
<td></td>
</tr>
</tbody>
</table>
\[ b_0 = \begin{vmatrix} -0.2 & 1 \\ 1 & -0.2 \end{vmatrix} = -0.96 \quad b_1 = \begin{vmatrix} -0.2 & -1.8 \\ 1 & 1.05 \end{vmatrix} = 1.59 \]

\[ b_2 = \begin{vmatrix} -0.2 & 1.05 \\ 1 & -1.8 \end{vmatrix} = -0.69 \]

\[ |b_0| = 0.96 > |b_2| = 0.69 \quad \checkmark \]

\[ \Rightarrow \text{The system is stable.} \]
4.4: Root-locus and Nyquist tests

- For cts.-time control, we examined the locations of the roots of the closed-loop system as a function of the loop gain $K \Rightarrow$ Root locus.

\[
T(s) = \frac{K D(s) G(s)}{1 + K D(s) G(s) H(s)}.
\]

- Let $L(s) = D(s) G(s) H(s)$. (The “loop transfer function”).

- Developed rules for plotting the roots of the equation

\[
1 + K \frac{b(s)}{a(s)} = 0.
\]

“Root Locus Drawing Rules.”

- Applied them to plotting roots of

\[
1 + K L(s) = 0.
\]

Now, we have the digital system:

\[
T(z) = \frac{K D(z) G(z)}{1 + K D(z) G H(z)},
\]

- So, we let $L(z) = D(z) G H(z)$.

- Poles are roots of $1 + K L(z) = 0$. 

This is exactly the same form as the Laplace-transform root locus. Plot roots in exactly the same way.

**EXAMPLE:**

\[
\frac{G(s)}{H(s)} = \frac{0.368(z + 0.717)}{(z - 1)(z - 0.368)} \quad D(z) = 1.
\]

```matlab
numd = 0.368*[1 0.717];
dend = conv([1 -1], [1 -0.368]);
d = tf(numd, dend, -1);
rlocus(d);
```

The Nyquist test

- In continuous-time control we also used the Nyquist test to assess stability.

- The Nyquist "D" path encircles the entire (unstable) RHP.
- The Nyquist plot is a polar plot of \(L(s)\) evaluated on the "D" path.
- Adjustments to "D" shape are made if pole on the \(j\omega\)-axis.
The Nyquist test evaluated stability by looking at the Nyquist plot.

- $N =$ No. of CW encirclements of $-1$ in Nyquist plot.
- $P =$ No. of open-loop unstable poles (poles inside “D” shape).
- $Z =$ No. of closed-loop unstable poles.
- $Z = N + P, \quad Z = 0$ for stable closed-loop system.

**EXAMPLE:**

This gives:

$$L(s) = \frac{1}{s(s + 1)}$$

- Pole at origin: Need detour $s = \rho e^{j\theta}, \rho \ll 1$.
- Resulting Nyquist map has infinite radius. Cannot draw to scale.
- No poles inside modified-“D” curve: $P = 0$.
- $Z = N + P = 0 \Rightarrow$ Stable system.

Note that increasing the gain “$K$” only magnifies the entire plot. The $-1$ point is not encircled for $K > 0$ (infinite gain margin).

**Nyquist test for discrete systems**

- Three different ways to do the Nyquist test for discrete systems.

Based on three different representations of the characteristic eqn.

1. $1 + L^*(s) = 0.$ \hspace{1cm} $L = \overline{DGH}$
2. $1 + L(z) = 0.$
3. $1 + L(w) = 0.$
1. \( 1 + L^*(s) = 0 \).

- We know that \( L^*(s) \) is periodic in \( j\omega \).
  Therefore, the “D” curve does not need to encircle the entire RHP to encircle all unstable poles. [If there were any, there would be an infinite number.]
- Modify “D” curve to be:

- Evaluate \( L^*(s) \) on new contour and plot polar plot. Same Nyquist test as before.

2. \( 1 + L(z) = 0 \).

- We can do the Nyquist test directly using \( z \)-transforms. The stable region is the unit circle. The \( z \)-domain Nyquist plot is done using a Nyquist curve which is the unit circle.
- Nyquist test changes because we are now encircling the \textit{STABLE} region (albeit CCW).
  - \( Z \) = \# closed-loop unstable poles.
  - \( P \) = \# open-loop unstable poles.
  - \( N \) = \# \textit{CCW} encirclements of \(-1\) in Nyquist plot.
  - \( Z = P - N \).

- Probably difficult to evaluate \( L(z)|_{z = e^{j\theta}} \) for \(-\pi \leq \theta \leq \pi\) unless using a digital computer.
3. \(1 + L(w) = 0\).

- Now, we convert \(L(z) \mapsto L(w)\)

\[
L(w) = L(z)|_{z=\frac{1+(T/2)w}{1-(T/2)w}}.
\]

- Bilinear transform maps unit circle to \(j\omega\)-axis in \(w\)-plane.

- Use standard continuous-time test in \(w\)-plane.

**Summary:**

<table>
<thead>
<tr>
<th>Open-loop fn.</th>
<th>Range of variable</th>
<th>Rule</th>
</tr>
</thead>
<tbody>
<tr>
<td>(\overline{GH}(s))</td>
<td>(s = j\omega, -\omega_s/2 \leq \omega \leq \omega_s/2)</td>
<td>(Z = P + N_{cw})</td>
</tr>
<tr>
<td>(\overline{GH}(z))</td>
<td>(z = e^{j\omega T}, -\pi \leq \omega T \leq \pi)</td>
<td>(Z = P - N_{cw} = P + N_{cw})</td>
</tr>
<tr>
<td>(\overline{GH}(w))</td>
<td>(w = j\omega_w, -\infty \leq \omega_w \leq \infty)</td>
<td>(Z = P + N_{cw})</td>
</tr>
</tbody>
</table>

- All three methods produce identical Nyquist plots.

- Note that the sampled system does not have \(\infty\) gain margin \((a = 0.418, GM = 2.39)\) and has smaller \(PM\) than cts.-time system.
4.5: Bode methods

- Bode plots are an extremely important tool for analyzing and designing control systems.

- They provide a critical link between continuous-time and discrete-time control design methods.

- Recall:
  - Bode plots are plots of frequency response of a system: Magnitude and Phase.
  - In $s$-plane, $H(s)|_{s=j\omega}$ is frequency response for $0 \leq \omega < \infty$.
  - In $z$-plane, $H(z)|_{z=e^{j\omega T}}$ is frequency response for $0 \leq \omega \leq \omega_s/2$.

- Straight-line tools of $s$-plane analysis DON'T WORK! They are based on geometry and geometry has changed—$j\omega$-axis to $z$-unit circle.

- BUT in $w$-plane, $H(w)|_{w=j\omega_w}$ is the frequency response for $0 \leq \omega_w < \infty$. Straight-line tools work, but frequency axis is warped.

PROCEDURE:

1. Convert $H(z)$ to $H(w)$ by $H(w) = H(z)|_{z=\frac{1+(T/2)w}{1-(T/2)w}}$.

2. Simplify expression to rational-polynomial in $w$.

3. Factor into zeros and poles in standard “Bode Form” (Refer to review notes).

4. Plot the response exactly the same way as an $s$-plane Bode plot.

Note: Plots are versus $\log_{10} \omega_w \ldots \omega_w = \frac{2}{T} \tan \left( \frac{\omega T}{2} \right)$. Can re-scale axis in terms if $\omega$ if we want.

EXAMPLE: Example seen before with $T = 1$ second.
Let \( G(z) = \frac{0.368z + 0.264}{z^2 - 1.368z + 0.368} \).

\((1,2)\)

\[
G(w) = \frac{0.368 \left[ \frac{1+0.5w}{1-0.5w} \right] + 0.264}{\left[ \frac{1+0.5w}{1-0.5w} \right]^2 - 1.368 \left[ \frac{1+0.5w}{1-0.5w} \right] + 0.368} = \frac{0.368(1 + 0.5w)(1 - 0.5w) + 0.264(1 - 0.5w)^2}{(1 + 0.5w)^2 - 1.368(1 + 0.5w)(1 - 0.5w) + 0.368(1 - 0.5w)^2} = -0.0381(w - 2)(w + 12.14) / w(w + 0.924).
\]

\((3)\)

\[
G(j\omega) = \frac{-\left( j\frac{\omega}{2} - 1 \right) \left( j\frac{\omega}{12.14} + 1 \right)}{j\omega \left( j\frac{\omega}{0.924} + 1 \right)}.
\]

\((4)\)

Gain margin and phase margin work the \textit{SAME} way we expect.
We have discussed frequency-response methods without verifying that discrete-time frequency response means the same thing as continuous-time frequency response.

Verify

\[ X(z) \rightarrow G(z) \rightarrow Y(z) \]

Let \( x[k] = \sin(\omega kT) \) . . . \( X(z) = \frac{z \sin \omega T}{(z - e^{j\omega T})(z - e^{-j\omega T})} \).

\[ Y(z) = G(z)X(z) = \frac{G(z)z \sin \omega T}{(z - e^{j\omega T})(z - e^{-j\omega T})}. \]

Do partial-fraction expansion

\[ \frac{Y(z)}{z} = \frac{k_1}{z - e^{j\omega T}} + \frac{k_2}{z - e^{-j\omega T}} + Y_g(z). \]

\( Y_g(z) \) is the response due to the poles of \( G(z) \). IF the system is stable, the response due to \( Y_g(z) \) → 0 as \( t \rightarrow \infty \).

So, as \( t \rightarrow \infty \) we say

\[ \frac{Y_{ss}(z)}{z} = \frac{k_1}{z - e^{j\omega T}} + \frac{k_2}{z - e^{-j\omega T}} \]

\[ k_1 = \frac{G(z) \sin \omega T}{z - e^{-j\omega T}} \bigg|_{z = e^{j\omega T}} = \frac{G(e^{j\omega T}) \sin \omega T}{e^{j\omega T} - e^{-j\omega T}} \]

\[ = \frac{G(e^{j\omega T})}{2j} \]

\[ = \frac{|G(e^{j\omega T})|e^{j\angle G(e^{j\omega T})}}{2j}. \]
Similarly,
\[ k_2 = \frac{|G(e^{j\omega T})|e^{-j\angle G(e^{j\omega T})}}{2(-j)} = \frac{-|G(e^{j\omega T})|e^{-j\angle G(e^{j\omega T})}}{2j}. \]

Combining and solving for \( y_{ss}[k] \)
\[ y_{ss}[k] = k_1(e^{j\omega T})^k + k_2(e^{-j\omega T})^k \]
\[ = |G(e^{j\omega T})|e^{j\omega kT + j\angle G(e^{j\omega T})} - e^{-j\omega kT - j\angle G(e^{j\omega T})} \]
\[ = \frac{1}{2j} |G(e^{j\omega T})|\sin(\omega kT + \angle G(e^{j\omega T})). \]

Sure enough, \( |G(e^{j\omega T})| \) is magnitude response to sinusoid, and \( \angle G(e^{j\omega T}) \) is phase response to sinusoid.

**Closed-loop frequency response**

We have looked at open-loop concepts and how they apply to closed loop systems . . . our end product.

Closed-loop frequency response usually calculated by computer: \( \frac{G(z)}{1 + G(z)} \), for example.

In general, if \( |G(e^{j\omega T})| \) large, \( |T(e^{j\omega T})| \approx 1 \). If \( |G(e^{j\omega T})| \) small, \( |T(e^{j\omega T})| \approx |G(e^{j\omega T})| \).

Closed-loop bandwidth similar to open-loop bandwidth.

- If \( PM = 90^\circ \), then C.L. BW = O.L. BW.
- If \( PM = 45^\circ \), then C.L. BW = 2 × O.L. BW.