STABILITY ANALYSIS TECHNIQUES

4.1: Bilinear transformation

- Three main aspects to control-system design:
  1. Stability,
  2. Steady-state response,
  3. Transient response.
- Here, we look at determining system stability using various methods.

DEFINITION: A system is BIBO stable iff a bounded input produces a bounded output.

- Check by first writing system input–output relationship as

$$Y(z) = \frac{G(z)}{1+\overline{GH}(z)}R(z) = \frac{K\prod^m(z-z_i)}{\prod^n(z-p_i)}R(z).$$

- Assume for now that all the poles $\{p_i\}$ are distinct and different from the poles in $R(z)$. Then,

$$Y(z) = \underbrace{\frac{k_1 z}{z-p_1} + \cdots + \frac{k_n z}{z-p_n}}_{\text{Response to initial conditions}} + \underbrace{Y_R(z).}_{\text{Response to } R(z)}$$

- If the system is stable, the response to initial conditions must decay to zero as time progresses.

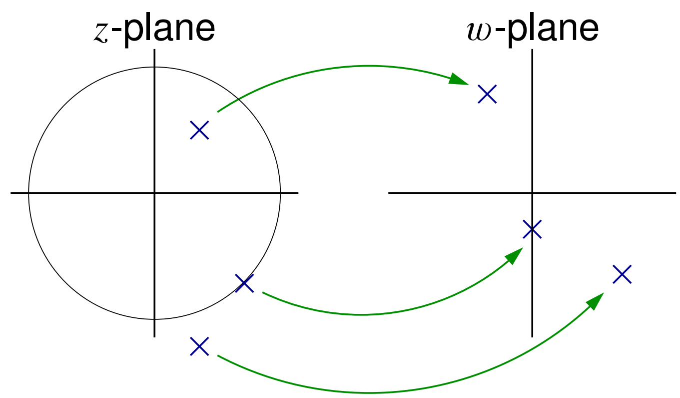
$$\mathcal{Z}^{-1}\left[\frac{k_i z}{z-p_i}\right] = k_i(p_i)^k 1[k].$$

So, the system is stable if $|p_i| < 1$.

- $\{p_i\}$ are the roots of $1 + \overline{GH}(z) = 0$. So, the roots of $1 + \overline{GH}(z) = 0$ must lie within the unit circle of the z-plane.
  - Same result even if poles are repeated, but harder to show.
- If the magnitude of a pole $|p_i| = 1$, then the system is marginally stable. The unforced response does not decay to zero but also does not increase to ∞. *However*, it is possible to drive the system with a bounded input and have the output go to ∞. Therefore, a marginally stable system is *unstable*.

Bilinear transformation

- The stability criteria for a discrete-time system is that all its poles lie within the unit circle on the z-plane.
- Stability criteria for cts.-time systems is that the poles be in the LHP.
  - Simple tool to test for continuous-time stability—Routh test.
- Can we use the Routh test to determine stability of a discrete-time system (either directly or indirectly)?
- To use the Routh test, we need to do a z-plane to s-plane conversion that retains stability information. The s-plane version of the z-plane system does NOT need to correspond in any other way.
- That is,
  - The frequency responses may be different
  - The step responses may be different . . .



- Since only stability properties are maintained by the transform, it is not accurate to label the destination plane the s-plane. It is often called the w-plane, and the transformation between the z-plane and the w-plane is called *the w-Transform*.
- A transform that satisfies these requirements is the bilinear transform. Recall:

$$H(w) = H(z)\big|_{z=\frac{1+(T/2)w}{1-(T/2)w}} \qquad \text{and} \qquad H(z) = H(w)\big|_{w=\frac{2}{T}\frac{z-1}{z+1}}.$$

- Three things to check:

1. Unit circle in z-plane $\mapsto$ $j\omega$-axis in w-plane.
2. Inside unit circle in z-plane $\mapsto$ LHP in w-plane.
3. Outside unit circle in z-plane $\mapsto$ RHP in w-plane.

- If true,

1. Take $H(z) \mapsto H(w)$ via the bilinear transform.
2. Perform Routh test on $H(w)$.

CHECK: Let $z = re^{j\omega T}$. Then, z is on the unit circle if $r = 1$, z is inside the unit circle if $|r| < 1$ and z is outside the unit circle if $|r| > 1$.

$$\begin{aligned} z &= re^{j\omega T} \\ w &= \frac{2}{T}\frac{z-1}{z+1}\bigg|_{z=re^{j\omega T}} = \frac{2}{T}\frac{re^{j\omega T}-1}{re^{j\omega T}+1}. \end{aligned}$$

- Expand $e^{j\omega T} = \cos(\omega T) + j\sin(\omega T)$ and use the shorthand $c \triangleq \cos(\omega T)$ and $s \triangleq \sin(\omega T)$. Also note that $s^2 + c^2 = 1$.

$$w = \frac{2}{T}\left[\frac{rc + jrs - 1}{rc + jrs + 1}\right]$$

$$= \frac{2}{T}\left[\frac{(rc-1)+jrs}{(rc+1)+jrs}\right]\left[\frac{(rc+1)-jrs}{(rc+1)-jrs}\right]$$

$$= \frac{2}{T}\left[\frac{(r^2c^2-1)+j(rs)(rc+1)-j(rs)(rc-1)+r^2s^2}{(rc+1)^2+(rs)^2}\right]$$

$$= \frac{2}{T}\left[\frac{r^2-1}{r^2+2rc+1}\right]+j\frac{2}{T}\left[\frac{2rs}{r^2+2rc+1}\right].$$

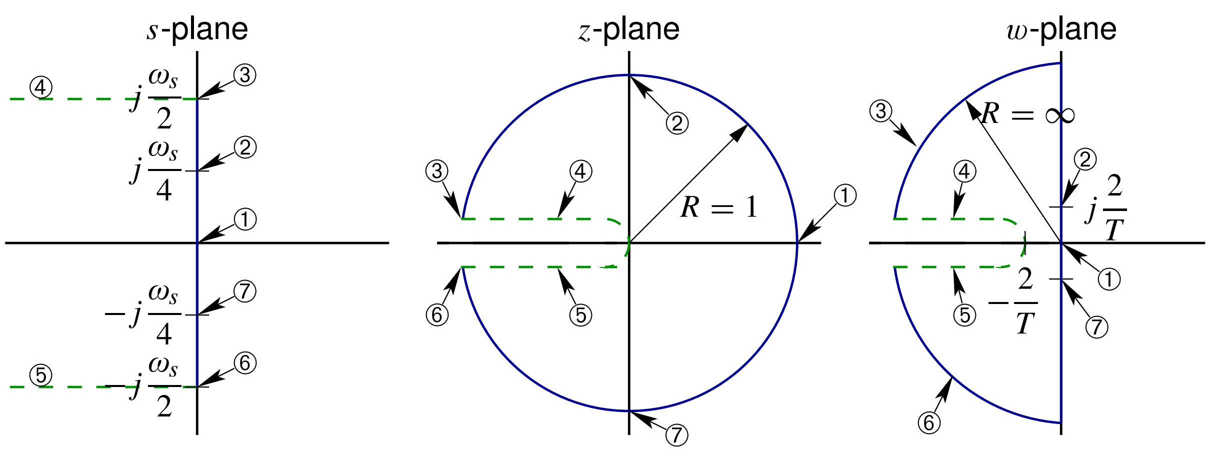
Notice that the real part of w is 0 when $r=1$ (w is on the imaginary axis), the real part of w is negative when $|r|<1$ (w in LHP), and that the real part of w is positive when $|r|>1$ (w in RHP). Therefore, the bilinear transformation does exactly what we want.

- When $r=1$,

$$w = j\frac{2}{T}\frac{2\sin(\omega T)}{2+2\cos(\omega T)} = j\frac{2}{T}\tan\left(\frac{\omega T}{2}\right),$$

which will be useful to know.

- The following diagram summarizes the relationship between the s-plane, z-plane, and w-plane:



4.2: Discrete-time stability via Routh–Hurwitz test

- Review of Routh test.

Let $H(w) = \dfrac{b(w)}{a(w)} \ldots a(w)$ is the characteristic polynomial.

$$a(w) = a_n w^n + a_{n-1} w^{n-1} + \cdots + a_1 w + a_0.$$

Case 0: If any of the a_n are negative then the system is unstable (unless *ALL* are negative).

Case 1: Form Routh array:

$$\begin{array}{c|cccc} w^n & a_n & a_{n-2} & a_{n-4} & \cdots \\ w^{n-1} & a_{n-1} & a_{n-3} & a_{n-5} & \cdots \\ w^{n-2} & b_1 & b_2 & \cdots & \\ w^{n-3} & c_1 & c_2 & \cdots & \\ \vdots & & & & \\ w^1 & j_1 & & & \\ w^0 & k_1 & & & \end{array}$$

$$b_1 = \frac{-1}{a_{n-1}} \begin{vmatrix} a_n & a_{n-2} \\ a_{n-1} & a_{n-3} \end{vmatrix} \qquad b_2 = \frac{-1}{a_{n-1}} \begin{vmatrix} a_n & a_{n-4} \\ a_{n-1} & a_{n-5} \end{vmatrix} \qquad \cdots$$

$$c_1 = \frac{-1}{b_1} \begin{vmatrix} a_{n-1} & a_{n-3} \\ b_1 & b_2 \end{vmatrix} \qquad c_2 = \frac{-1}{b_1} \begin{vmatrix} a_{n-1} & a_{n-5} \\ b_1 & b_3 \end{vmatrix} \qquad \cdots$$

TEST: Number of RHP roots = number of sign changes in left column.

Case 2: If one of the left column entries is zero, replace it with ϵ as $\epsilon \to 0$.

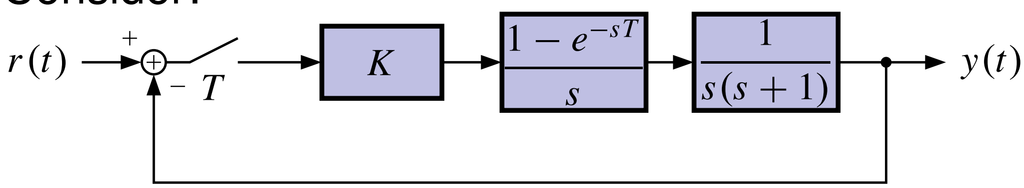
Case 3: Suppose an entire row of the Routh array is zero, the w^{i-1}th row. The w^ith row, right above it, has coefficients $\alpha_1, \alpha_2, \ldots$

Then, form the auxiliary equation:

$$\alpha_1 w^i + \alpha_2 w^{i-2} + \alpha_3 w^{i-4} + \cdots = 0.$$

This equation is a factor of the characteristic equation and must be tested for RHP roots (it *WILL* have non-LHP roots—we might want to know how many are RHP).

EXAMPLE: Consider:



$$G(s) = \left(\frac{1 - e^{-Ts}}{s}\right)\left(\frac{1}{s(s+1)}\right).$$

- From z-transform tables:

$$\begin{aligned} G(z) &= \left(\frac{z-1}{z}\right)\mathcal{Z}\left[\frac{1}{s^2(s+1)}\right] \\ &= \left(\frac{z-1}{z}\right)\left(\frac{(e^{-T}+T-1)z^2 + (1-e^{-T}-Te^{-T})z}{(z-1)^2(z-e^{-T})}\right). \end{aligned}$$

Let $T = 0.1$ s.

$$= \frac{0.00484z + 0.00468}{(z-1)(z-0.905)}.$$

- Perform the bilinear transform

$$\begin{aligned} G(w) &= G(z)\big|_{z=\frac{1+(T/2)w}{1-(T/2)w}} \\ &= G(z)\big|_{z=\frac{1+0.05w}{1-0.05w}} \\ &= \frac{-0.00016w^2 - 0.1872w + 3.81}{3.81w^2 + 3.80w}. \end{aligned}$$

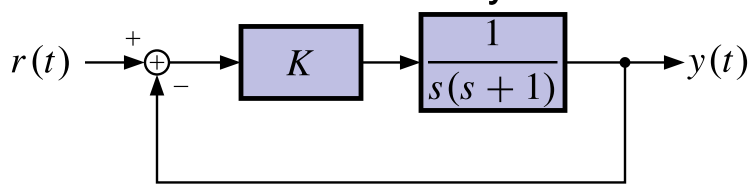
- The characteristic equation is:

$$0 = 1 + KG(w)$$
$$= (3.81 - 0.00016K)w^2 + (3.80 - 0.1872K)w + 3.81K.$$

$$\begin{array}{c|ll} w^2 & (3.81 - 0.00016K) & 3.81K \\ w^1 & (3.80 - 0.1872K) & \\ w^0 & 3.81K & \end{array} \quad \begin{array}{l} \Rightarrow K < 23,813 \\ \Rightarrow K < 20.3 \\ \Rightarrow K > 0 \end{array}$$

- So, for stability, $0 < K < 20.3$.

NOTE: The "equivalent" continuous-time system is:



$$T(s) = \frac{KG(s)}{1 + KG(s)}.$$

- Characteristic equation: $s(s + 1) + K = 0$.

$$\begin{array}{c|ll} s^2 & 1 & K \\ s^1 & 1 & \\ s^0 & K & \end{array}$$

- Stable for all $K > 0$ ➠ sample and hold destabilizes the system.

EXAMPLE: Let's do the same example, but with $T = 1$ s (not 0.1 s).

- (math happens)

$$0 = 1 + KG(w)$$
$$= (1 - 0.0381K)w^2 + (0.924 - 0.86K)w + 0.924K.$$

$$\begin{array}{c|ll} w^2 & (1 - 0.0381K) & 0.924K \\ w^1 & (0.924 - 0.386K) & \\ w^0 & 0.924K & \end{array} \quad \begin{array}{l} \Rightarrow K < 26.2 \\ \Rightarrow K < 2.39 \\ \Rightarrow K > 0 \end{array}$$

- So, for stability, $0 < K < 2.39$.
- This is a much more restrictive range than when $T = 0.1$ s ➠ slow sampling really destabilizes a system.

4.3: Jury's stability test

- $H(z) \mapsto H(w) \mapsto$ Routh is complicated and error-prone.
- Jury made a direct test on $H(z)$ for stability.
- Disadvantage (?) ... another test to learn.
- Let $T(z) = \dfrac{b(z)}{a(z)}$, $\quad a(z) =$"characteristic polynomial."
- $a(z) = a_n z^n + a_{n-1}z^{n-1} + \cdots + a_1 z + a_0 = 0, \qquad a_n > 0.$
- Form Jury array:

z^0	z^1	z^2	$\cdots$	z^{n-k}	$\cdots$	z^{n-1}	z^n
a_0	a_1	a_2	$\cdots$	a_{n-k}	$\cdots$	a_{n-1}	a_n
a_n	a_{n-1}	a_{n-2}	$\cdots$	a_k	$\cdots$	a_1	a_0
b_0	b_1	b_2	$\cdots$	b_{n-k}	$\cdots$	b_{n-1}	
b_{n-1}	b_{n-2}	b_{n-3}	$\cdots$	b_{k-1}	$\cdots$	b_0	
c_0	c_1	c_2	$\cdots$	c_{n-k}	$\cdots$		
c_{n-2}	c_{n-3}	c_{n-4}	$\cdots$	c_{k-2}	$\cdots$		
$\vdots$	$\vdots$	$\vdots$		$\vdots$			
l_0	l_1	l_2	l_3				
l_3	l_2	l_1	l_0				
m_0	m_1	m_2					

- Quite different from Routh array.
  - Every row is duplicated ... in reverse order.
  - Final row in table has three entries (always).
  - Elements are calculated differently.

$$b_k = \begin{vmatrix} a_0 & a_{n-k} \\ a_n & a_k \end{vmatrix} \qquad c_k = \begin{vmatrix} b_0 & b_{n-1-k} \\ b_{n-1} & b_k \end{vmatrix} \qquad d_k = \begin{vmatrix} c_0 & c_{n-2-k} \\ c_{n-2} & c_k \end{vmatrix} \qquad \cdots$$

- Stability criteria is different.

$$a(z)|_{z=1} > 0$$

$$(-1)^n\, a(z)|_{z=-1} > 0 \qquad\qquad n = \text{order of } a(z)$$

$$|a_0| < a_n$$

$$|b_0| > |b_{n-1}|$$

$$|c_0| > |c_{n-2}|$$

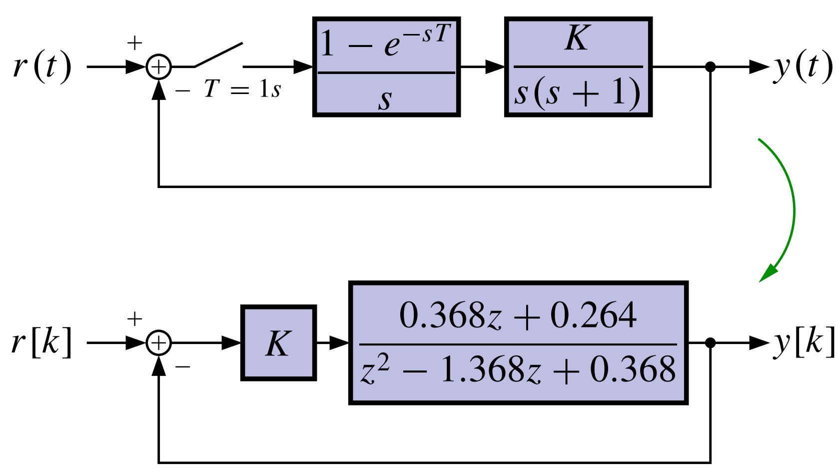
$$|d_0| > |d_{n-3}|$$

$$\vdots$$

$$|m_0| > |m_2|\,.$$

- First, check that $a(1) > 0$, $(-1)^n a(-1) > 0$ and $|a_0| < a_n$. (relatively few calculations). If not satisfied, *stop*.
- Next, construct array. *Stop* if any condition not satisfied.

EXAMPLE:



- Characteristic equation:

$$0 = 1 + KG(z) = 1 + K\frac{(0.368z + 0.264)}{z^2 - 1.368z + 0.368}$$

$$= z^2 + (0.368K - 1.368)z + (0.368 + 0.264K).$$

- The Jury array is:

z^0	z^1	z^2
$0.368 + 0.264K$	$0.368K - 1.368$	1

- The constraint $a(1) > 0$ yields

$$1 + 0.368K - 1.368 + 0.368 + 0.264K = 0.632K > 0 \quad \Rightarrow \quad K > 0.$$

- The constraint $(-1)^2 a(-1) > 0$ yields

$$1 - 0.368K + 1.368 + 0.368 + 0.264K = -0.104K + 2.736 > 0 \quad \Rightarrow \quad K < 26.3.$$

- The constraint $|a_0| < a_2$ yields

$$0.368 + 0.264K < 1 \quad \Rightarrow \quad K < \frac{0.632}{0.264} = 2.39.$$

- So, $0 < K < 2.39$. (Same result as on pg. 4–8 using bilinear rule.)

EXAMPLE: Suppose that the characteristic equation for a closed-loop discrete-time system is given by the expression:

$$a(z) = z^3 - 1.8z^2 + 1.05z - 0.20 = 0.$$

- $a(1) = 1 - 1.8 + 1.05 - 0.2 = 0.05 > 0$ ✓
- $(-1)^3 a(-1) = -[-1 - 1.8 - 1.05 - 0.2] > 0$ ✓
- $|a_0| = 0.2 < a_3 = 1$ ✓
- Jury array:

z^0	z^1	z^2	z^3
-0.2	1.05	-1.8	1
1	-1.8	1.05	-0.2
-0.96	1.59	-0.69	

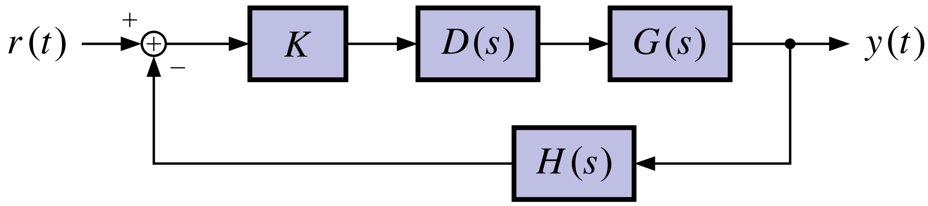
$$b_0 = \begin{vmatrix} -0.2 & 1 \\ 1 & -0.2 \end{vmatrix} = -0.96 \quad b_1 = \begin{vmatrix} -0.2 & -1.8 \\ 1 & 1.05 \end{vmatrix} = 1.59$$

$$b_2 = \begin{vmatrix} -0.2 & 1.05 \\ 1 & -1.8 \end{vmatrix} = -0.69$$

- $|b_0| = 0.96 > |b_2| = 0.69$ ✓
  - ➠ The system is stable.

4.4: Root-locus and Nyquist tests

- For cts.-time control, we examined the locations of the roots of the closed-loop system as a function of the loop gain K ➠ Root locus.



$$T(s) = \frac{KD(s)G(s)}{1 + KD(s)G(s)H(s)}.$$

- Let $L(s) = D(s)G(s)H(s)$. (The "loop transfer function").
- Developed rules for plotting the roots of the equation

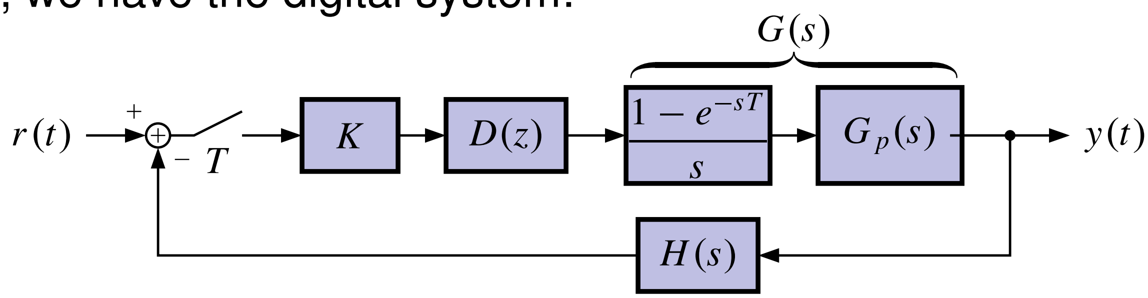
$$1 + K\frac{b(s)}{a(s)} = 0.$$

"Root Locus Drawing Rules."

- Applied them to plotting roots of

$$1 + KL(s) = 0.$$

Now, we have the digital system:



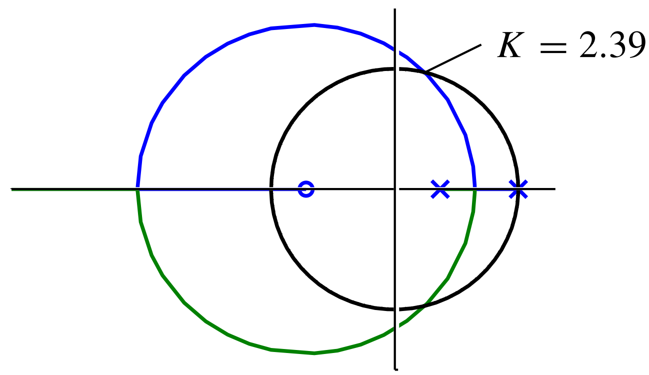
$$T(z) = \frac{KD(z)G(z)}{1 + KD(z)\overline{GH}(z)},$$

- So, we let $L(z) = D(z)\overline{GH}(z)$.
- Poles are roots of $1 + KL(z) = 0$.

- This is *exactly the same form* as the Laplace-transform root locus. Plot roots in exactly the same way.

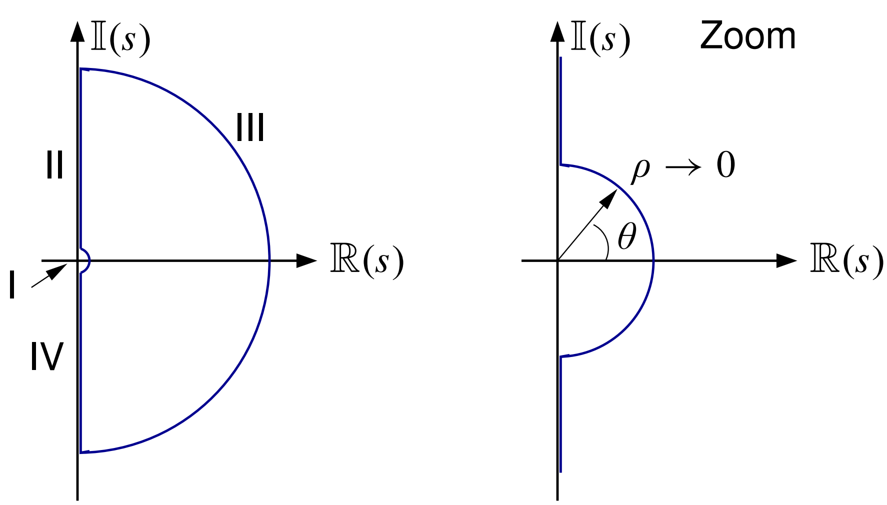
EXAMPLE:
$$\overline{GH}(z) = \frac{0.368(z+0.717)}{(z-1)(z-0.368)} \qquad D(z) = 1.$$

```
numd=0.368*[1 0.717];
dend=conv([1 -1],[1 -0.368]);
d=tf(numd,dend,-1);
rlocus(d);
```



The Nyquist test

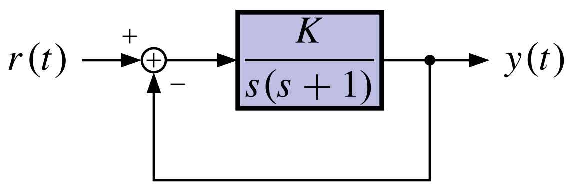
- In continuous-time control we also used the Nyquist test to assess stability.



- The Nyquist "D" path encircles the entire (unstable) RHP.
- The Nyquist plot is a polar plot of $L(s)$ evaluated on the "D" path.
- Adjustments to "D" shape are made if pole on the $j\omega$-axis.

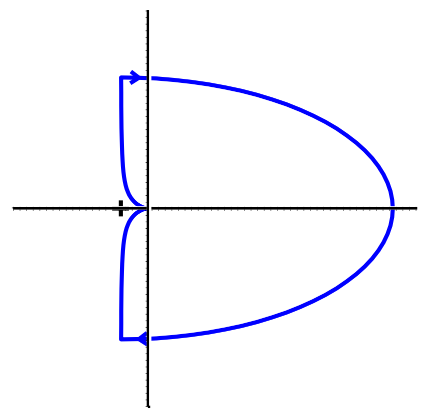
- The Nyquist test evaluated stability by looking at the Nyquist plot.
  - N=No. of CW encirclements of -1 in Nyquist plot.
  - P=No. of open-loop unstable poles (poles inside "D" shape).
  - Z=No. of closed-loop unstable poles.
  - $Z = N + P, \qquad Z = 0$ for stable closed-loop system.

EXAMPLE:



- This gives:

$$L(s) = \frac{1}{s(s+1)}$$



- Pole at origin: Need detour $s = \rho e^{j\theta}$, $\rho \ll 1$.
- Resulting Nyquist map has infinite radius. Cannot draw to scale.
- No poles inside modified-"D" curve: $P = 0$.
- $Z = N + P = 0$ ➠ Stable system.
- Note that increasing the gain "K" only magnifies the entire plot. The -1 point is not encircled for $K > 0$ (infinite gain margin).

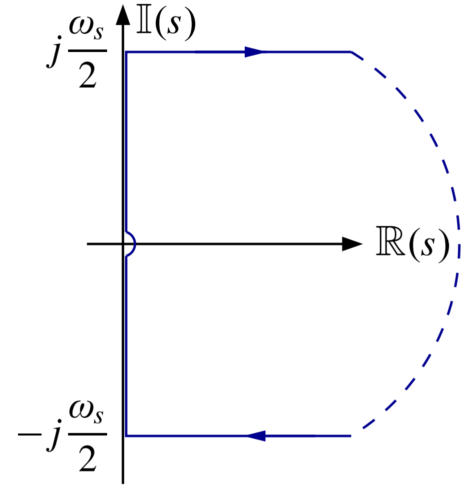
Nyquist test for discrete systems

- Three different ways to do the Nyquist test for discrete systems.
- Based on three different representations of the characteristic eqn.

1. $1 + L^*(s) = 0. \qquad L = D\overline{GH}$
2. $1 + L(z) = 0.$
3. $1 + L(w) = 0.$

1. $1 + L^*(s) = 0$.

- We know that $L^*(s)$ is periodic in $j\omega_s$. Therefore, the "D" curve does not need to encircle the entire RHP to encircle all unstable poles. [If there were any, there would be an infinite number.]
- Modify "D" curve to be:

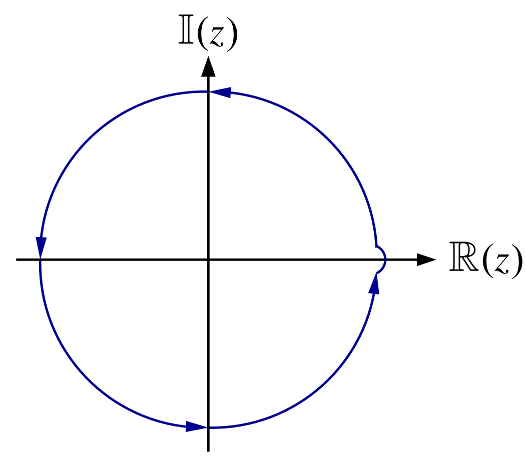


- Evaluate $L^*(s)$ on new contour and plot polar plot. Same Nyquist test as before.

2. $1 + L(z) = 0$.

- We can do the Nyquist test directly using z-transforms. The stable region is the unit circle. The z-domain Nyquist plot is done using a Nyquist curve which is the unit circle.
- Nyquist *test* changes because we are now encircling the *STABLE* region (albeit CCW).

  - $Z =$ # closed-loop unstable poles.
  - $P =$ # open-loop unstable poles.
  - $N =$ # *CCW* encirclements of -1 in Nyquist plot.
  - $Z = P - N$.



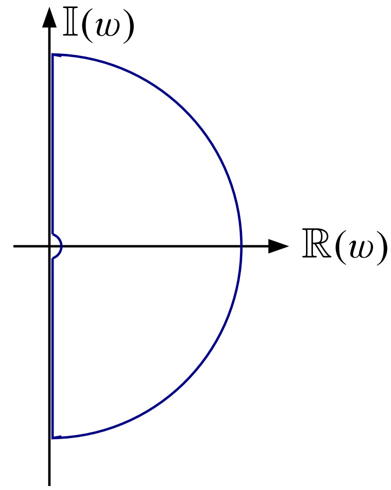
- Probably difficult to evaluate $L(z)|_{z=e^{j\theta}}$ for $-\pi \le \theta \le \pi$ unless using a digital computer.

3. $1 + L(w) = 0$.

- Now, we convert $L(z) \mapsto L(w)$

$$L(w) = L(z)\big|_{z=\frac{1+(T/2)w}{1-(T/2)w}}.$$

- Bilinear transform maps unit circle to $j\omega$-axis in w-plane.
- Use standard continuous-time test in w-plane.

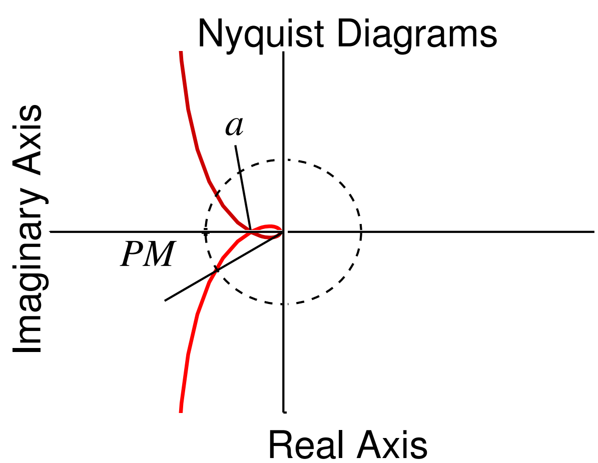


- Summary:

Open-loop fn.	Range of variable		Rule
$\overline{GH}^*(s)$	$s = j\omega,$	$-\omega_s/2 \le \omega \le \omega_s/2$	$Z = P + N_{cw}$
$\overline{GH}(z)$	$z = e^{j\omega T},$	$-\pi \le \omega T \le \pi$	$Z = P - N_{ccw} = P + N_{cw}$
$\overline{GH}(w)$	$w = j\omega_w,$	$-\infty \le \omega_w \le \infty$	$Z = P + N_{cw}$

- All three methods produce *identical* Nyquist plots.

- Note that the sampled system does not have ∞ gain margin ($a = 0.418$, $GM = 2.39$) and has smaller PM than cts.-time system.



4.5: Bode methods

- Bode plots are an extremely important tool for analyzing and designing control systems.
- They provide a critical link between continuous-time and discrete-time control design methods.
- Recall:
  - Bode plots are plots of frequency response of a system: Magnitude and Phase.
  - In s-plane, $H(s)|_{s=j\omega}$ is frequency response for $0 \le \omega < \infty$.
  - In z-plane, $H(z)|_{z=e^{j\omega T}}$ is frequency response for $0 \le \omega \le \omega_s/2$.
- Straight-line tools of s-plane analysis *DON'T WORK!* They are based on geometry and geometry has changed—$j\omega$-axis to z-unit circle.
- *BUT* in w-plane, $H(w)|_{w=j\omega_w}$ is the frequency response for $0 \le \omega_w < \infty$. Straight-line tools work, but frequency axis is warped.

PROCEDURE:

1. Convert $H(z)$ to $H(w)$ by $H(w) = H(z)|_{z=\frac{1+(T/2)w}{1-(T/2)w}}$.
2. Simplify expression to rational-polynomial in w.
3. Factor into zeros and poles in standard "Bode Form" (Refer to review notes).
4. Plot the response exactly the same way as an s-plane Bode plot. Note: Plots are versus $\log_{10}\omega_w \quad \ldots \quad \omega_w = \dfrac{2}{T}\tan\left(\dfrac{\omega T}{2}\right)$. Can re-scale axis in terms if ω if we want.

EXAMPLE: Example seen before with $T = 1$ second.

$$\text{Let } G(z) = \frac{0.368z + 0.264}{z^2 - 1.368z + 0.368}.$$

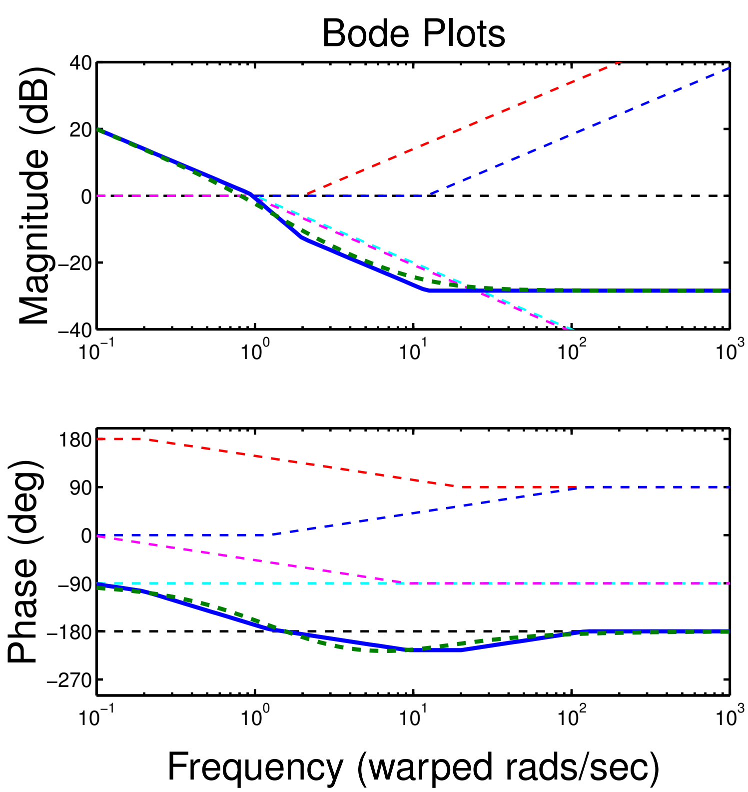
(1,2)

$$\begin{aligned}
G(w) &= \frac{0.368\left[\frac{1+0.5w}{1-0.5w}\right] + 0.264}{\left[\frac{1+0.5w}{1-0.5w}\right]^2 - 1.368\left[\frac{1+0.5w}{1-0.5w}\right] + 0.368} \\
&= \frac{0.368(1+0.5w)(1-0.5w) + 0.264(1-0.5w)^2}{(1+0.5w)^2 - 1.368(1+0.5w)(1-0.5w) + 0.368(1-0.5w)^2} \\
&= \frac{-0.0381(w-2)(w+12.14)}{w(w+0.924)}.
\end{aligned}$$

(3)

$$G(j\omega_w) = \frac{-\left(j\frac{\omega_w}{2} - 1\right)\left(j\frac{\omega_w}{12.14} + 1\right)}{j\omega_w\left(j\frac{\omega_w}{0.924} + 1\right)}.$$

(4)



- Gain margin and phase margin work the *SAME* way we expect.

WAIT!

- We have discussed frequency-response methods without verifying that discrete-time frequency response means the same thing as continuous-time frequency response.
- Verify

$$X(z) \longrightarrow G(z) \longrightarrow Y(z)$$

- Let $x[k] = \sin(\omega k T)$ $\quad\ldots\quad$ $X(z) = \dfrac{z \sin \omega T}{(z - e^{j\omega T})(z - e^{-j\omega T})}$.

$$\begin{aligned} Y(z) &= G(z)X(z) \\ &= \frac{G(z) z \sin \omega T}{(z - e^{j\omega T})(z - e^{-j\omega T})}. \end{aligned}$$

- Do partial-fraction expansion

$$\frac{Y(z)}{z} = \frac{k_1}{z - e^{j\omega T}} + \frac{k_2}{z - e^{-j\omega T}} + Y_g(z).$$

- $Y_g(z)$ is the response due to the poles of $G(z)$. *IF* the system is stable, the response due to $Y_g(z) \to 0$ as $t \to \infty$.
- So, as $t \to \infty$ we say

$$\begin{aligned} \frac{Y_{ss}(z)}{z} &= \frac{k_1}{z - e^{j\omega T}} + \frac{k_2}{z - e^{-j\omega T}} \\ k_1 &= \left.\frac{G(z) \sin \omega T}{z - e^{-j\omega T}}\right|_{z = e^{j\omega T}} \\ &= \frac{G(e^{j\omega T}) \sin \omega T}{e^{j\omega T} - e^{-j\omega T}} \\ &= \frac{G(e^{j\omega T})}{2j} \\ &= \frac{|G(e^{j\omega T})| e^{j\angle G(e^{j\omega T})}}{2j}. \end{aligned}$$

- Similarly,

$$k_2 = \frac{|G(e^{j\omega T})|e^{-j\angle G(e^{j\omega T})}}{2(-j)} = -\frac{|G(e^{j\omega T})|e^{-j\angle G(e^{j\omega T})}}{2j}.$$

- Combining and solving for $y_{ss}[k]$

$$\begin{aligned} y_{ss}[k] &= k_1(e^{j\omega T})^k + k_2(e^{-j\omega T})^k \\ &= |G(e^{j\omega T})|\frac{e^{j\omega kT + j\angle G(e^{j\omega T})} - e^{-j\omega kT - j\angle G(e^{j\omega T})}}{2j} \\ &= |G(e^{j\omega T})| \sin(\omega kT + \angle G(e^{j\omega T})). \end{aligned}$$

- Sure enough, $|G(e^{j\omega T})|$ is magnitude response to sinusoid, and $\angle G(e^{j\omega T})$ is phase response to sinusoid.

Closed-loop frequency response

- We have looked at open-loop concepts and how they apply to closed loop systems . . . our end product.
- Closed-loop frequency response usually calculated by computer: $\dfrac{G(z)}{1 + G(z)}$, for example.
- In general, if $|G(e^{j\omega T})|$ large, $|T(e^{j\omega T})| \approx 1$. If $|G(e^{j\omega T})|$ small, $|T(e^{j\omega T})| \approx |G(e^{j\omega T})|$.
- Closed-loop bandwidth similar to open-loop bandwidth.
  - If $PM = 90^\circ$, then C.L. BW = O.L. BW.
  - If $PM = 45^\circ$, then C.L. BW $= 2\times$O.L. BW.