3.1: Laplace transform of ideal sampler

Digital control systems are a mixture of discrete-time and continuous-time components.

The interactions between these two parts require that we understand both from a continuous-time point of view.

We begin by looking at the sampling and reconstruction (A2D, D2A) operations from a continuous-time point of view.

Consider a simple unity-gain feedback digital control system with proportional controller having unity gain.

\[ r(t) \xrightarrow{\oplus} x(t) \xrightarrow{A2D} x(kT) \xrightarrow{1} x(kT) \xrightarrow{D2A} \bar{x}(t) \xrightarrow{G(s)} y(t) \]

Every \( T \) seconds, the sampler records \( x(kT) \). The controller multiplies this value by 1. The D2A outputs this value for the entire sample period. *i.e.*, it “holds” the value.

So,

\[ \bar{x}(t) = x(0) [1(t) - 1(t - T)] + x(T) [1(t - T) - 1(t - 2T)] + \]

Amplitude

\[ x(t), \text{ input to sampler} \]

\[ \bar{x}(t), \text{ output of ZOH} \]
\[ x(2T) \left[ 1(t - 2T) - 1(t - 3T) \right] + \cdots \]

\[
\tilde{X}(s) = x(0) \left[ \frac{1}{s} - \frac{e^{-Ts}}{s} \right] + x(T) \left[ \frac{e^{-Ts}}{s} - \frac{e^{-2Ts}}{s} \right] + \\
x(2T) \left[ \frac{e^{-2Ts}}{s} - \frac{e^{-3Ts}}{s} \right] + \cdots \\
= \left[ \frac{1 - e^{-Ts}}{s} \right] \left( x(0) + x(T)e^{-Ts} + x(2T)e^{-2Ts} + \cdots \right) \\
= \left[ \sum_{k=0}^{\infty} x(kT)e^{-kTs} \right] \left[ \frac{1 - e^{-Ts}}{s} \right].
\]

- First factor is function of input \( x(t) \) and sampling period \( T \).
- Second factor is independent of \( x(t) \), so can be considered a transfer function.

**DEFINE:**

\[
X^*(s) = \left[ \sum_{k=0}^{\infty} x(kT)e^{-kTs} \right].
\]

- This is a representation of the A2D/sampling operation. \( X^*(s) \) is not a physical signal, but results from factoring \( \tilde{X}(s) \) into two parts. Then:

\[
\begin{aligned}
X(s) &\xrightarrow{T} X^*(s) &\xrightarrow{\text{zoh}} &\tilde{X}(s) \\
&\left( \frac{1 - e^{-Ts}}{s} \right)
\end{aligned}
\]

- Note, the operation converting \( X(s) \rightarrow X^*(s) \) is not LTI. (why?) So, it cannot be represented by a transfer function.

**Ideal sampling**

- Take the inverse-Laplace transform of \( X^*(s) \) to view sampling in the time-domain.
\[ x^*(t) = \mathcal{L}^{-1} \left[ X^*(s) \right] = x(0)\delta(t) + x(T)\delta(t - T) + x(2T)\delta(t - 2T) + \cdots \]

- So, \( x^*(t) \) is an impulse train, with impulse values weighted (multiplied) by the value of \( x(t) \) at the sampling instants. That is,

\[
x^*(t) = x(t) \sum_{k=0}^{\infty} \delta(t - kT) \delta_T(t)
\]

\[ = x(t)\delta_T(t). \]

**EXAMPLE:** Find \( X^*(s) \) for \( x(t) = 1(t) \).

\[
X^*(s) = \sum_{k=0}^{\infty} x(kT)e^{-kTs} = x(0) + x(T)e^{-Ts} + x(2T)e^{-2Ts} + \cdots
\]

\[ = 1 + e^{-Ts} + e^{-2Ts} + \cdots \]

- Multiply both sides by \((1 - e^{-Ts})\)

\[
(1 - e^{-Ts})X^*(s) = 1
\]

\[
X^*(s) = \frac{1}{1 - e^{-Ts}} \quad |e^{-Ts}| < 1.
\]

- Hmmm. Does this look familiar? Consider \( z \)-transform of \( 1[k] \) with \( z = e^{Ts} \).

\[
1[k] \leftrightarrow \frac{z}{z - 1} \quad \text{let } z = e^{Ts} \rightarrow \frac{e^{Ts}}{e^{Ts} - 1} = \frac{1}{1 - e^{-Ts}}.
\]

**EXAMPLE:** Find \( X^*(s) \) when \( x(t) = e^{-t}1(t) \).

\[
X^*(s) = \sum_{k=0}^{\infty} x(kT)e^{-kTs}
\]
\[ x(0) + x(T)e^{-Ts} + x(2T)e^{-2Ts} + \cdots \]
\[ = 1 + e^{-T}e^{-Ts} + e^{-2T}e^{-2Ts} + \cdots \]
\[ = 1 + (e^{-(1+s)T}) + (e^{-(1+s)T})^2 + \cdots \]
\[ = \frac{1}{1 - e^{-(1+s)T}} \quad \left| e^{-(1+s)T} \right| < 1. \]

- Hmm. Does this look familiar too?

Let \( y[k] = (e^{-T})^k \)

\[ Y(z) = \frac{z}{z - (e^{-T})} \quad \text{let } z = e^{Ts} \]
\[ = \frac{e^{Ts}}{e^{Ts} - e^{-T}} = \frac{1}{1 - e^{-(1+s)T}}. \]

- So, \( X^*(s) = X(z)|_{z=e^{Ts}} \).
3.2: Properties of $X^*(s)$ and data reconstruction

**Property 1:** $X^*(s)$ is periodic in $s$, with period $j\omega_s$.

\[ X^*(s + jm\omega_s) = \sum_{k=0}^{\infty} x(kT)e^{-kT(s+jm\omega_s)} \]

\[ = \sum_{k=0}^{\infty} x(kT)e^{-kTs}e^{-jkm\omega_sT}. \]

Now, $\omega_s = \frac{2\pi}{T}$

\[ = \sum_{k=0}^{\infty} x(kT)e^{-kTs}e^{-j2\pi km}. \]

From Euler, $e^{-j2\pi km} = \cos(2\pi km) + j\sin(2\pi km) = 1$

So, $X^*(s + jm\omega_s) = \sum_{k=0}^{\infty} x(kT)e^{-kTs}$

\[ = X^*(s). \]

**Property 2:** If $X(s)$ has a pole at $s = s_1$, then $X^*(s)$ has poles at $s = s_1 + jm\omega_s, \ m \in \mathbb{Z}$.

- A relationship between $X(s)$ and $X^*(s)$ may be proven:

\[ X^*(s) = \frac{1}{T} \sum_{m=-\infty}^{\infty} X(s + jm\omega_s) + \frac{x(0)}{2} \]

Then,

\[ X^*(s) = \frac{1}{T}(X(s) + X(s + j\omega_s) + X(s + j2\omega_s) + \cdots) \]

[Zero locations of $X^*(s)$ are periodic (from property 1) but are not uniquely determined by the zeros of $X(s)$].
What these two properties *MEAN* is that two different signals $X_1(s)$ and $X_2(s)$ may have the *SAME* starred transform $X_1^*(s) = X_2^*(s)$. Why?

- One signal is said to alias as the other.
- This is very easy to see in the time domain.

- The (blue) solid line is the analog signal $\cos(2\pi (5)t)$ drawn over one second.

- The (blue) squares show sampling points for a rate of $f_s = 14 \text{ Hz}$.

- The (green) dots show sampling times for a rate of $f_s = 7 \text{ Hz}$.

- The dashed (green) curve is a $\cos(2\pi (2)t)$ signal.

  - Both the $\cos(2\pi (5)t)$ and $\cos(2\pi (2)t)$ curves pass through all the dots so the signal may be assumed to be a 2 Hz sinusoid (aliasing!).
  - Only the $\cos(2\pi (5)t)$ curve passes through all the squares, so the signal must be assumed to be a 5 Hz sinusoid (no aliasing).
The solution to the problem is easiest to see in the Fourier domain.
\[ \mathcal{F}[x(t)] = \mathcal{L}[x(t)]|_{s=j\omega}. \] Since \( X^*(s) \) is periodic, \( X^*(j\omega) \) is also periodic.

For \( X(j\omega) \) having a spectral bandwidth \( W < 2f_s \), the sampled signal spectrum is the following

![Diagram of X(jω) and X*(jω) with passband and reconstruction filter](image)

- If the sampling rate \( f_s < 2W \), then spectral overlap called *aliasing* occurs.

![Diagram of X(jω) and X*(jω) with aliasing](image)

- When aliasing is present, it is impossible to recover the original analog signal from the sampled signal.

- Nyquist/Shannon said that if a signal is sampled at a rate \( \omega_N \) there will be no aliasing if \( \omega_N \geq 2 \) times the highest frequency in the signal.

**KEY POINT:** Our digital control system will need to sample at least twice as fast as the closed-loop bandwidth. Four to six times faster is safer.
Data deconstruction

- If the signal \( x(t) \) has been sampled at or above the Nyquist rate \( \omega_N \), it is mathematically possible to recover \( x(t) \) from \( x(kT) \). Surprise? Magic?

- Process \( X^*(j\omega) \) by passing it through an ideal low-pass filter with bandwidth \( \omega_N/2 \).

- The filter must have gain \( T \) to recover the magnitude lost by sampling.

\[
h(t) = \frac{\sin(\pi t/T)}{\pi t/T} = \text{sinc}(t/T).
\]

**PROBLEM:** \( h(t) \) is non-causal.

- Solution: “Approximate” \( h(t) \) very crudely with a causal filter—the hold circuit.

- The impulse response of the hold circuit is:

\[
\text{hold circuit}
\]

- So, \( H(j\omega) \) is

\[
H(j\omega) = T \left( \frac{\sin(\omega T/2)}{\omega T/2} \right) e^{-j\omega T/2}
\]

\[
= T \text{sinc} \left( \frac{\omega}{\omega_s} \right) e^{-j\pi \omega/\omega_s}.
\]
**EXAMPLE:** Let \( x(t) = 2 \sin(\omega_1 t) \) with \( \omega_1 < \omega_s/2 \).

Notice that the ZOH does not completely filter out the aliased frequencies due to sampling. Since most plants are low-pass devices, this is often okay.

- The high-frequency components look like stair-steps in time-domain. Note: Familiar \( T/2 \) delay of fundamental frequency.
3.3: Open-loop discrete-time systems

- We have seen some properties of the sampling operation and of the hold operation.

- Sampling:

  \[ X(s) \xrightarrow{T} X^*(s) \]

- Hold:

  \[ X^*(s) \xrightarrow{\text{zoh}} \left( \frac{1 - e^{-Ts}}{s} \right) X^*(s) \]

- The \( z \)-transform will enable us to put these elements into a system and analyze them a little more easily.

- We first look at open-loop systems, and then at closed-loop systems.

**Open-loop discrete-time systems**

- We have already seen the relationship between \( X^*(s) \) and \( X(z) \).

- Easy to prove:

  \[ \mathcal{Z}[x[k]] = X(z) = x[0] + x[1]z^{-1} + x[2]z^{-2} + \cdots \]

  \[ X^*(s) = x(0) + x(T)e^{-sT} + x(2T)e^{-2sT} + \cdots \]

  so, \( X(z) = X^*(s) \mid_{esT = z} \)

  and \( X^*(s) = X(z) \mid_{z = esT} \).

- So, the \( z \)-transform is a special case of the Laplace transform for discrete-time systems.

- Sometimes it is easier to find the starred transform \( X^*(s) \) from the \( z \)-transform \( X(z) \)!
EXAMPLE: Let

\[ X(s) = \frac{1}{(s + 1)(s + 2)}. \]

- From the \(z\)-transform table in Topic 2.7 of your notes:

\[ X(z) = \frac{z(e^{-T} - e^{-2T})}{(z - e^{-T})(z - e^{-2T})} \]

\[ X^*(s) = X(z)|_{z = e^{sT}} = \frac{e^{sT}(e^{-T} - e^{-2T})}{(e^{sT} - e^{-T})(e^{sT} - e^{-2T})}. \]

NOTE:

- \(X^*(s)\) has \(\infty\) poles and zeros in the \(s\)-plane because of its periodic nature.

- \(X(z)\) has a single zero at \(z = 0\), and two poles at \(z = e^{-T}\) and \(z = e^{-2T}\).

- Pole-zero analysis is going to be easier in the \(z\)-domain.

The discrete-impulse/unit-pulse transfer function

- For continuous-time LTI systems, the impulse response described the output of the system when the input was the impulse/delta function.

- The transfer function is the Laplace transform of the impulse response.

- The same can be done for discrete-time systems, but the input is now a discrete-impulse, often called a unit pulse or simply a pulse.
We are interested in systems that look like:

\[ X(s) \xrightarrow{T} X^*(s) \xrightarrow{\frac{1-e^{-Ts}}{s}} G_p(s) Y(s) \xrightarrow{T} Y^*(s) \]

\[ \text{HOLD} \]

\[ \text{PLANT} \]

\[ G(s) \]

- Define \( G(s) = \left( \frac{1 - e^{-sT}}{s} \right) G_p(s) \) to "hide" the mess introduced by the hold operation.

\[ X(s) \xrightarrow{T} X^*(s) \xrightarrow{G(s)} Y(s) \xrightarrow{T} Y^*(s) \]

- So, \( Y(s) = G(s)X^*(s) \).

- \( Y^*(s) \) would occur if we sampled \( Y(s) \). \( Y^*(s) = [G(s)X^*(s)]^* \).

\[
Y^*(s) = \frac{1}{T} \sum_{k=-\infty}^{\infty} Y(s + jk\omega_s) + \frac{y(0)}{2}
\]

\[
= \frac{1}{T} \sum_{k=-\infty}^{\infty} G(s + jk\omega_s)X^*(s + jk\omega_s) + \frac{y(0)}{2}.
\]

- But, since \( X^*(s) \) is a sampled signal, it is periodic in \( s + jk\omega_s \):

\[ X^*(s) = X^*(s + jk\omega_s). \]

- So,

\[
Y^*(s) = \frac{1}{T} \sum_{k=-\infty}^{\infty} G(s + jk\omega_s)X^*(s) + \frac{y(0)}{2}
\]

\[
= X^*(s) \left[ \frac{1}{T} \sum_{k=-\infty}^{\infty} G(s + jk\omega_s) \right] + \frac{y(0)}{2}
\]

\[
= X^*(s)G^*(s) + \frac{y(0)}{2}
\]

or, \( Y(z) = X(z)G(z) + \frac{y(0)}{2} \).
- $G(z)$ is the pulse transfer function between the sampled input and the output at the sampling instants.
  - Recall that transfer functions assume initial conditions to be zero.
- It tells us NO information about the output between the sampling instants.
  - If we choose a fast sampling rate, then interpolation can be a good approximation.
  - Simulation is a general tool to help determine inter-sample behavior accurately.

**EXAMPLE:** At sample points, left plot looks better than right plot, but in-between sample points, left plot looks worse than right plot.
3.4: Finding $G(z)$ from $G_p(s)$

- Recall that
  - $G(s) = \left(\frac{1 - e^{-sT}}{s}\right) G_p(s)$,
  - $G^*(s)$ is a sampled Laplace transform of $G(s)$, and
  - $G(z) = G^*(s) \bigg|_{z=e^{sT}}$.

- To find $G(z)$ we must then
  1. Find $g(t) = \mathcal{L}^{-1} \left[ \left(\frac{1 - e^{-sT}}{s}\right) G_p(s) \right]$.
  2. Find $g[k] = g(kT)$.
  3. Find $G(z) = \mathcal{Z} \left[ g(kT) \right]$.

- Notation: $G(z) = \mathcal{Z} \left[ \left(\frac{1 - e^{-sT}}{s}\right) G_p(s) \right]$.

A shortcut for systems having no time delays:

- We calculate
  \[
  G(z) = \mathcal{Z} \left[ \left(\frac{1 - e^{-sT}}{s}\right) G_p(s) \right] = \mathcal{Z} \left[ (1 - e^{-sT}) \frac{G_p(s)}{s} \right] = \mathcal{Z} \left[ \left(\frac{G_p(s)}{s}\right) \right] - \mathcal{Z} \left[ e^{-sT} \frac{G_p(s)}{s} \right].
  \]

- But: $e^{-sT}$ is the cts-time transfer function of a delay of $T$ seconds.

- In disc-time, $z^{-1}$ is the transfer function of a unit delay, $(T$ seconds).
  \[
  G(z) = (1 - z^{-1}) \mathcal{Z} \left[ \frac{G_p(s)}{s} \right] = \frac{z - 1}{z} \mathcal{Z} \left[ \frac{G_p(s)}{s} \right].
  \]
This last term may often be looked up in a table.

In MATLAB: Let \( G_p(s) = \frac{1}{s + 1} \).

\[
\text{num} = 1; \\
\text{denc} = [1 1]; \\
\text{sysc} = \text{tf(num,denc);} \\
\text{sysd} = \text{c2d(sysc, T); % = sampling period.} \\
\text{[numd, dend, T] = tfdata(sysd);} \\
\]

A shortcut for systems having time delays: (The modified \( z \)-transform)

- We often need to analyze systems with pure time delays:
  1. Plant is of the form \( G_p(s) = e^{-s\tau} \frac{\text{num}(s)}{\text{den}(s)} \);
  2. Digital controller introduces delay in computation;
  3. And/or we want to know plant output between normal sampling times (the ripple).

Consider (without controller \( D(z) \) for now):

\[
Y(s) = e^{-s\tau} G(s) X^*(s) \\
Y^*(s) = [e^{-s\tau} G(s)]^* X^*(s) \\
Y(z) = \mathcal{Z} [e^{-s\tau} G(s)] X(z)
\]
Let \( \tau = kT + \Delta T, \ k \in \mathbb{Z}, \ 0 < \Delta < 1 \)

\[
Y(z) = z^{-k} \mathcal{Z} \left[ e^{-s\Delta T} G(s) \right] X(z)
\]

\[
= z^{-k} G(z, (1 - \Delta)) X(z).
\]

- You can use modified-\(z\)-transform table from Topic 2.7,
- Or you can work it out by hand,
- Or... you can use MATLAB

```matlab
sys=tf(num,den);
set(sys,'ioDelay',tau); \% tau is total delay, in seconds
sysd=c2d(sys,T);
```

**EXAMPLE OF PULSE TRANSFER FUNCTION:**

\[
X(s) \xrightarrow{T} X*(s) \xrightarrow{\frac{1-e^{-Ts}}{s}} G_p(s) \xrightarrow{T} Y*(s)
\]

- Let \( G_p(s) = \frac{1}{s+1} \) and \( X(s) = \frac{1}{s} \) (a unit step).
- So,

\[
Y(s) = \left( \frac{1 - e^{-sT}}{s} \right) \left( \frac{1}{s+1} \right) X^*(s) = G(s)X^*(s).
\]

- Also note that

\[
G(z) = \mathcal{Z} \left[ \left( \frac{1 - e^{-sT}}{s} \right) \left( \frac{1}{s+1} \right) \right] = \frac{1-e^{-T}}{z-e^{-T}}
\]

and \( X(z) = \mathcal{Z} [1(t)] = \frac{z}{z-1} \).

- So,
\[ Y(z) = \left( \frac{1 - e^{-T}}{z - e^{-T}} \right) \left( \frac{z}{z - 1} \right) \]
\[ = \frac{z}{z - 1} - \frac{z}{z - e^{-T}} \]
\[ y[k] = (1 - e^{-kT}) 1[k]. \]

**NOTE:** A shortcut for this particular example would have been to notice that the sample-and-hold does nothing to the unit-step input!

- This example is also a good one to illustrate dc gain:

\[ y_{ss} = \lim_{z \to 1} (z - 1)Y(z) \]
\[ = \lim_{z \to 1} \frac{(1 - e^{-T})z}{z - e^{-T}} \]
\[ = \frac{1 - e^{-T}}{1 - e^{-T}} = 1. \]

- Equivalently, we can find in general

\[ \text{dc gain} = \lim_{z \to 1} (z - 1)Y(z) \]
\[ = \lim_{z \to 1} (z - 1)G(z)1(z) \]
\[ = \lim_{z \to 1} G(z)z = G(1). \]

So, \( G(1) \) is the dc gain (a.k.a., the final value to a unit-step input).

- **DC gain**

\[ \lim_{z \to 1} G(z) = \lim_{s \to 0} G_p(s). \]

**CONCLUSION:** For the following open loop system, we have

\[ Y(z) = G(z)X(z). \]
3.5: Systems with digital filters

- Our digital controllers will not simply sample and hold their input value. They will perform some mathematical operations on the sampled input. “FILTER”

- So, we expand our system of interest:

\[
\begin{align*}
X(s) & \xrightarrow{T} x[k] & & \xrightarrow{D(z)} u[k] & & \xrightarrow{1 - e^{-Ts}} \tilde{u}(t) & & \xrightarrow{G_p(s)} y(t) & & \xrightarrow{T} Y^*(s)
\end{align*}
\]

- Suppose our digital filter has transfer function \(D(z)\).

- Then,

\[
U(z) = D(z)X(z), \quad \text{or, via } z = e^{sT}
\]

\[
U^*(s) = D^*(s)X^*(s).
\]

\(\tilde{u}(t)\) occurs at the output of the hold circuit, so

\[
\tilde{U}(s) = \left(\frac{1 - e^{-sT}}{s}\right)U^*(s), \quad \text{and}
\]

\[
Y(s) = G_p(s)\tilde{U}(s)
\]

\[
= G_p(s)\left(\frac{1 - e^{-sT}}{s}\right)U^*(s).
\]

- Since

\[
U^*(s) = D^*(s)X^*(s),
\]

\[
Y(s) = G_p(s)\left(\frac{1 - e^{-sT}}{s}\right)D^*(s)X^*(s)
\]

\[
= D(z)|_{z = e^{sT}}
\]

\[
Y^*(s) = \left(G_p(s)\left(\frac{1 - e^{-sT}}{s}\right)\right)^* D^*(s)X^*(s)
\]
\[ Y(z) = \mathcal{Z} \left[ G_p(s) \left( \frac{1 - e^{-st}}{s} \right) \right] D(z) X(z) \]
\[ Y(z) = G(z) D(z) X(z). \]

**EXAMPLE:** Let the digital controller implement

\[ u[k] = 2x[k] - x[k - 1] \]

and the plant is:

\[ G_p(s) = \frac{1}{s + 1}. \]

- **Find** \( D(z) \):

\[ U(z) = 2X(z) - z^{-1}X(z) = (2 - z^{-1}) X(z) \]
\[ D(z) = \frac{U(z)}{X(z)} = 2 - z^{-1} = \frac{2z - 1}{z}. \]

- **Find** \( G(z) \):

\[ G(z) = \mathcal{Z} \left[ \frac{1 - e^{sT}}{s} \right] \left( \frac{1}{s + 1} \right) = \frac{1 - e^{-T}}{z - e^{-T}}. \]

- **So,**

\[ Y(z) = \left( \frac{1 - e^{-T}}{z - e^{-T}} \right) \left( \frac{2z - 1}{z} \right) X(z). \]

**EXAMPLE:** Let \( x(t) = 1(t) \), and find \( Y(z) \).

\[ X(z) = \frac{z}{z - 1} \]
\[ Y(z) = \left( \frac{1 - e^{-T}}{z - e^{-T}} \right) \left( \frac{2z - 1}{z} \right) \left( \frac{z}{z - 1} \right) = \frac{(2z - 1)(1 - e^{-T})}{(z - 1)(z - e^{-T})}. \]
We can determine \( y[k] \) by partial-fraction expansion and inversion of \( Y(z) \):

\[
\frac{Y(z)}{z} = \frac{(2z - 1)(1 - e^{-T})}{z(z - 1)(z - e^{-T})} = \frac{1 - e^T}{z} + \frac{1}{z - 1} + \frac{e^T - 2}{z - e^{-T}}.
\]

\[
Y(z) = (1 - e^T) + \frac{z}{z - 1} + \frac{z(e^T - 2)}{z - e^{-T}}
\]

\[\Downarrow\]

\[\Downarrow\]

\[\Downarrow\]

\( (1 - e^T)\delta[k] \quad 1[k] \quad (e^T - 2)e^{-kT} 1[k] \)

\[
y[k] = (1 - e^T)\delta[k] + 1[k] + (e^T - 2)e^{-kT} 1[k].
\]

**NOTE:** At \( k = 0 \),

\[
y[0] = (1 - e^T) + 1 + (e^T - 2) = 0.
\]

This could also be observed by noting that \( Y(z) \) has higher-order denominator than numerator.

The delay in the HOLD means that the output is not instantaneous.

**Block diagrams and sampled data**

Control systems exist in great variety.

- Almost always more complicated than the open-loop system we saw here.
- To be able to analyze closed-loop and more involved systems, we need to be able to analyze general block diagrams with samplers in them.

**EXAMPLE:**

\[
\begin{array}{c}
X(s) \xrightarrow{T X^*(s)} \text{HOLD} \xrightarrow{G_{p1}(s) A(s)} \text{HOLD} \xrightarrow{G_{p2}(s)} Y(s) \xrightarrow{T Y^*(s)}
\end{array}
\]

\[
\begin{array}{c}
G_1(s) \quad \text{HOLD} \quad G_2(s)
\end{array}
\]
\[ Y(s) = G_2(s)A^*(s) \]
\[ A(s) = G_1(s)X^*(s) \]

\[
\begin{align*}
Y^*(s) &= \left[ G_2(s)A^*(s) \right]^* \\
A^*(s) &= \left[ G_1(s)X^*(s) \right]^*
\end{align*}
\]

\[ Y^*(s) = G_2^*(s)A^*(s) \]
\[ A^*(s) = G_1^*(s)X^*(s) \]

or, \[ Y^*(s) = G_2^*(s)G_1^*(s)X^*(s) \]

\[ Y(z) = G_2(z)G_1(z)X(z). \]

- Things are not always this straightforward.
3.6: Some more challenging examples

EXAMPLE: Consider the hybrid system drawn below.

\[
\begin{align*}
X(s) & \xrightarrow{T} X^*(s) & \text{HOLD} & \xrightarrow{G_1(s)} G_{p1}(s) & \xrightarrow{A(s)} G_2(s) & \xrightarrow{T} Y(s) & \xrightarrow{Y^*(s)} \\
Y(s) & = G_2(s)A(s) = G_2(s)G_1(s)X^*(s) \\
Y^*(s) & = [G_2(s)G_1(s)X^*(s)]^* \\
& = [G(s)X^*(s)]^* \\
& = G^*(s)X^*(s) \\
Y(z) & = G(z)X(z) \quad \text{or,} \quad Y(z) = (G_2G_1)(z)X(z).
\end{align*}
\]

- But, is \(G^*(s) = G_2^*(s)G_1^*(s)\)?
- That is, does \(G(z) = G_2(z)G_1(z)\)?
- Let:

\[
\begin{align*}
G_2(s) & = \frac{1}{s} & G_1(s) & = \frac{1}{s^2} & G(s) & = \frac{2s}{(e^{sT} - 1)^2} \\
G_2^*(s) & = \frac{e^{sT}}{e^{sT} - 1} & G_1^*(s) & = \frac{e^{sT}}{e^{sT} - 1} & G^*(s) & = \frac{T e^{sT}}{(e^{sT} - 1)^2} \\
G_1^*(s)G_2^*(s) & = \frac{e^{2sT}}{(e^{sT} - 1)^2} \neq \frac{T e^{sT}}{(e^{sT} - 1)^2}.
\end{align*}
\]

KEY CONCLUSION: \([G_1(s)G_2(s)]^* \neq G_1^*(s)G_2^*(s)\) in general.

REASON: Sampling is time-varying, so system is not LTI. In fact, in some cases we may not even be able to find transfer functions...
EXAMPLE: Consider the hybrid system drawn below.

\[
Y(s) = G_2(s)A^*(s) \\
A(s) = G_1(s)X(s) \\
A^*(s) = [G_1(s)X(s)]^* \neq G_1^*(s)X^*(s) \\
Y(s) = G_2(s)[G_1(s)X(s)]^* \\
Y^*(s) = G_2^*(s)[G_1(s)X(s)]^* \\
Y(z) = G_2(z)[G_1X](z).
\]

NOTE: We cannot separate \([G_1X](z)\) to create a transfer function \(Y(z)/X(z)\) (!)

- \(a(t)\) is a function of all previous values of \(x(t)\), not just the values of \(x[kT]\), at the sampling instants.

Closing the loop—Feedback

- Ultimately, we are more interested in closed-loop than in open-loop systems.

EXAMPLE:
\[ E(s) = R(s) - H(s)Y(s) \]

\[ Y(s) = G(s)E^*(s) \]

\[ \Rightarrow E(s) = R(s) - H(s)[G(s)E^*(s)] = R(s) - [H(s)G(s)]E^*(s) \]

\[ E^*(s) = R^*(s) - [H(s)G(s)]^*E^*(s) \]

\[ = \frac{R^*(s)}{1 + (GH)^*(s)} \]

\[ Y^*(s) = \left[ \frac{G^*(s)}{1 + (GH)^*(s)} \right] R^*(s) \]

\[ Y(z) = \left[ \frac{G(z)}{1 + GH(z)} \right] R(z). \]

- Similarly,

\[ \frac{Y(z)}{R(z)} = \frac{D(z)G(z)}{1 + D(z)GH(z)}. \]

**The problem of disturbance**

\[ Y(s) = G_p(s)W(s) + G(s)A^*(s) \]

\[ A(s) = R(s) - Y(s) = R(s) - G_p(s)W(s) - G(s)A^*(s) \]
\[ A^*(s) = R^*(s) - (G_p W)^*(s) - G^*(s) A^*(s) \]

\[ = \frac{R^*(s)}{1 + G^*(s)} - \frac{(G_p W)^*(s)}{1 + G^*(s)} \]

\[ Y(s) = G_p(s) W(s) \left[ \frac{1 + G^*(s)}{1 + G^*(s)} \right] + \frac{G(s) R^*(s)}{1 + G^*(s)} - \frac{(G_p W)^*(s) G(s)}{1 + G^*(s)} \]

\[ Y^*(s) = \left[ \frac{G^*(s)}{1 + G^*(s)} \right] R^*(s) + \left[ \frac{1}{1 + G^*(s)} \right] (G_p W)^*(s) \]

\[ Y(z) = \left[ \frac{G(z)}{1 + G(z)} \right] R(z) + \frac{1}{1 + G(z)} W(z). \]

- The transfer function from \( R(s) \) to \( Y(s) \) is \( \frac{G(z)}{1 + G(z)} \) . . . no surprise.

- **No transfer function exists** between \( W(z) \) and \( Y(z) \) (!)

- Solution (?)

\[ Y(z) = \left[ \frac{G(z)}{1 + G(z)} \right] R(z) + \frac{1}{1 + G(z)} W_1(z). \]

**Where to from here?**

- We now understand the hybrid nature of digital control systems and are able to analyze such hybrid systems.

- We’re almost ready for controller design—but first, we look at stability analysis for discrete-time systems.