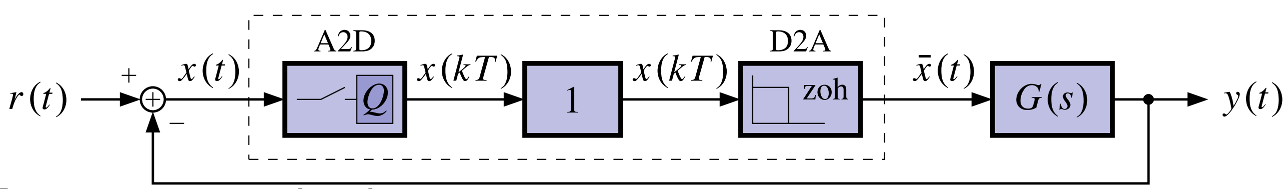
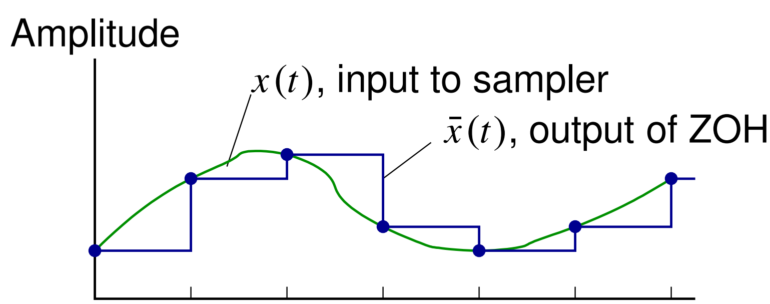
DISCRETE AND HYBRID SYSTEMS

3.1: Laplace transform of ideal sampler

- Digital control systems are a mixture of discrete-time and continuous-time components.
- The interactions between these two parts require that we understand both from a continuous-time point of view.
- We begin by looking at the sampling and reconstruction (A2D, D2A) operations from a continuous-time point of view.
- Consider a simple unity-gain feedback digital control system with proportional controller having unity gain.



- Every T seconds, the sampler records $x(kT)$. The controller multiplies this value by 1. The D2A outputs this value for the entire sample period. *i.e.*, it "holds" the value.



- So,

$$\bar{x}(t) = x(0)\left[1(t) - 1(t-T)\right] + x(T)\left[1(t-T) - 1(t-2T)\right] +$$

$$x(2T)\left[1(t-2T)-1(t-3T)\right]+\cdots$$

$$\bar{X}(s)=x(0)\left[\frac{1}{s}-\frac{e^{-Ts}}{s}\right]+x(T)\left[\frac{e^{-Ts}}{s}-\frac{e^{-2Ts}}{s}\right]+$$

$$x(2T)\left[\frac{e^{-2Ts}}{s}-\frac{e^{-3Ts}}{s}\right]+\cdots$$

$$=\left[\frac{1-e^{-Ts}}{s}\right]\left(x(0)+x(T)e^{-Ts}+x(2T)e^{-2Ts}+\cdots\right)$$

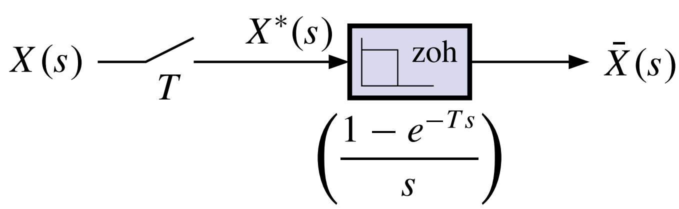
$$=\left[\sum_{k=0}^{\infty}x(kT)e^{-kTs}\right]\left[\frac{1-e^{-Ts}}{s}\right].$$

- First factor is function of input $x(t)$ and sampling period T.
- Second factor is independent of $x(t)$, so can be considered a transfer function.

DEFINE:

$$X^*(s)=\left[\sum_{k=0}^{\infty}x(kT)e^{-kTs}\right].$$

- This is a representation of the A2D/sampling operation. $X^*(s)$ is not a physical signal, but results from factoring $\bar{X}(s)$ into two parts. Then:



- Note, the operation converting $X(s)\to X^*(s)$ is not LTI. (why?) So, it cannot be represented by a transfer function.

Ideal sampling

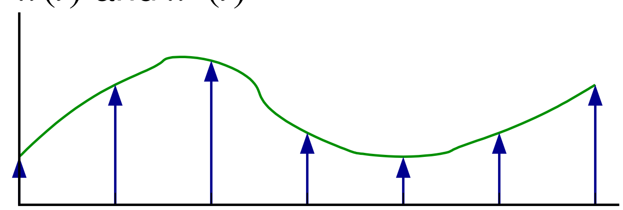
- Take the inverse-Laplace transform of $X^*(s)$ to view sampling in the time-domain.

$$x^*(t) = \mathcal{L}^{-1}\left[X^*(s)\right] = x(0)\delta(t) + x(T)\delta(t-T) + x(2T)\delta(t-2T) + \cdots$$

- So, $x^*(t)$ is an impulse train, with impulse values weighted (multiplied) by the value of $x(t)$ at the sampling instants. That is,

$$x^*(t) = x(t)\underbrace{\sum_{k=0}^{\infty}\delta(t-kT)}_{\delta_T(t)}$$

$$= x(t)\delta_T(t).$$



EXAMPLE: Find $X^*(s)$ for $x(t) = 1(t)$.

$$X^*(s) = \sum_{k=0}^{\infty} x(kT)e^{-kTs}$$

$$= x(0) + x(T)e^{-Ts} + x(2T)e^{-2Ts} + \cdots$$

$$= 1 + e^{-Ts} + e^{-2Ts} + \cdots$$

- Multiply both sides by $(1 - e^{-Ts})$

$$(1 - e^{-Ts})X^*(s) = 1$$

$$X^*(s) = \frac{1}{1 - e^{-Ts}} \qquad \left|e^{-Ts}\right| < 1.$$

- Hmmm. Does this look familiar? Consider z-transform of $1[k]$ with $z = e^{Ts}$.

$$1[k] \leftrightarrow \frac{z}{z-1}$$

$$\text{let } z = e^{Ts} \rightarrow \frac{e^{Ts}}{e^{Ts}-1} = \frac{1}{1-e^{-Ts}}.$$

EXAMPLE: Find $X^*(s)$ when $x(t) = e^{-t}1(t)$.

$$X^*(s) = \sum_{k=0}^{\infty} x(kT)e^{-kTs}$$

$$
\begin{aligned}
&= x(0) + x(T)e^{-Ts} + x(2T)e^{-2Ts} + \cdots \\
&= 1 + e^{-T}e^{-Ts} + e^{-2T}e^{-2Ts} + \cdots \\
&= 1 + \left(e^{-(1+s)T}\right) + \left(e^{-(1+s)T}\right)^2 + \cdots \\
&= \frac{1}{1 - e^{-(1+s)T}} \qquad \left|e^{-(1+s)T}\right| < 1.
\end{aligned}
$$

- Hmmm. Does this look familiar too?

$$
\begin{aligned}
\text{Let } y[k] &= \left(e^{-T}\right)^k \\
Y(z) &= \frac{z}{z - (e^{-T})} \qquad \text{let } z = e^{Ts} \\
&= \frac{e^{Ts}}{e^{Ts} - e^{-T}} = \frac{1}{1 - e^{-(1+s)T}}.
\end{aligned}
$$

- So, $X^*(s) = X(z)|_{z=e^{Ts}}$.

3.2: Properties of $X^*(s)$ and data reconstruction

Property 1: $X^*(s)$ is periodic in s, with period $j\omega_s$.

$$X^*(s + jm\omega_s) = \sum_{k=0}^{\infty} x(kT)e^{-kT(s+jm\omega_s)}$$

$$= \sum_{k=0}^{\infty} x(kT)e^{-kTs}e^{-jkm\omega_s T}.$$

Now, $\omega_s = \dfrac{2\pi}{T}$

$$= \sum_{k=0}^{\infty} x(kT)e^{-kTs}e^{-j2\pi km}.$$

From Euler, $e^{-j2\pi km} = \underbrace{\cos(2\pi km)}_{1,\, km\in\mathbb{Z}} + \underbrace{j\sin(2\pi km)}_{0,\, km\in\mathbb{Z}} = 1$

So, $X^*(s + jm\omega_s) = \displaystyle\sum_{k=0}^{\infty} x(kT)e^{-kTs}$

$$= X^*(s).$$

Property 2: If $X(s)$ has a pole at $s = s_1$, then $X^*(s)$ has poles at $s = s_1 + jm\omega_s, \quad m \in \mathbb{Z}$.

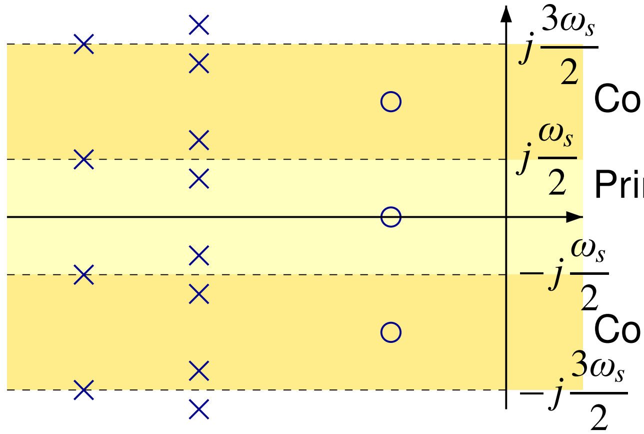
- A relationship between $X(s)$ and $X^*(s)$ may be proven:

$$X^*(s) = \frac{1}{T}\sum_{m=-\infty}^{\infty} X(s + jm\omega_s) + \frac{x(0)}{2}$$

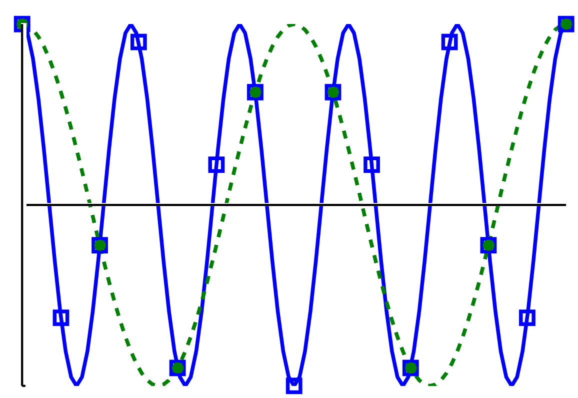
  Then,

$$X^*(s) = \frac{1}{T}(X(s) + X(s + j\omega_s) + X(s + j2\omega_s) + \cdots)$$

  [Zero locations of $X^*(s)$ are periodic (from property 1) but are not uniquely determined by the *zeros* of $X(s)$].

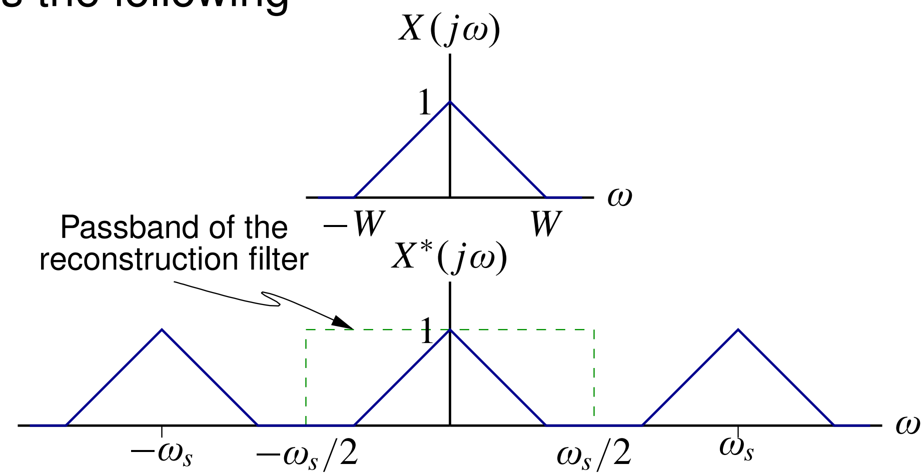


- What these two properties *MEAN* is that two different signals $X_1(s)$ and $X_2(s)$ may have the *SAME* starred transform $X_1^*(s) = X_2^*(s)$. Why?
- One signal is said to alias as the other.
- This is very easy to see in the time domain.
- The (blue) solid line is the analog signal $\cos(2\pi(5)t)$ drawn over one second.
- The (blue) squares show sampling points for a rate of $f_s = 14$ Hz.

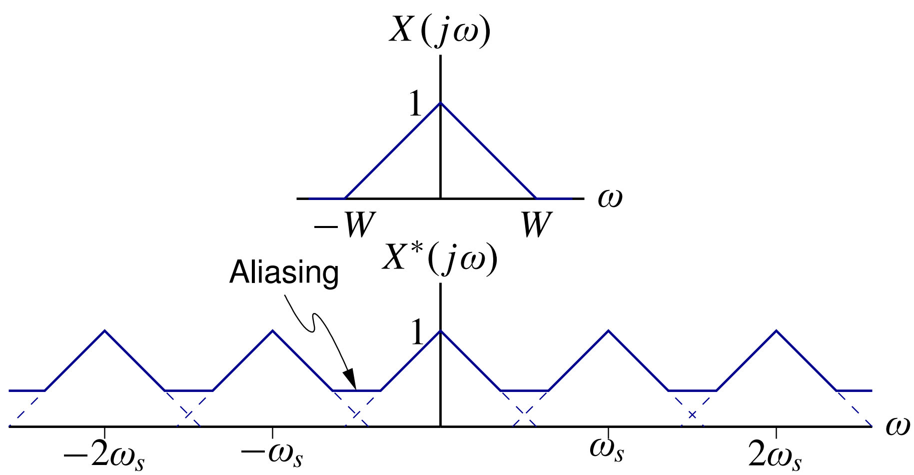


- The (green) dots show sampling times for a rate of $f_s = 7$ Hz.
- The dashed (green) curve is a $\cos(2\pi(2)t)$ signal.
  - Both the $\cos(2\pi(5)t)$ and $\cos(2\pi(2)t)$ curves pass through all the dots so the signal may be assumed to be a 2 Hz sinusoid (aliasing!).
  - Only the $\cos(2\pi(5)t)$ curve passes through all the squares, so the signal must be assumed to be a 5 Hz sinusoid (no aliasing).

- The solution to the problem is easiest to see in the Fourier domain. $\mathcal{F}[x(t)] = \mathcal{L}[x(t)]|_{s=j\omega}$. Since $X^*(s)$ is periodic, $X^*(j\omega)$ is also periodic.
- For $X(j\omega)$ having a spectral bandwidth $W < 2f_s$ the sampled signal spectrum is the following



- If the sampling rate $f_s < 2W$, then spectral overlap called *ALIASING* occurs.

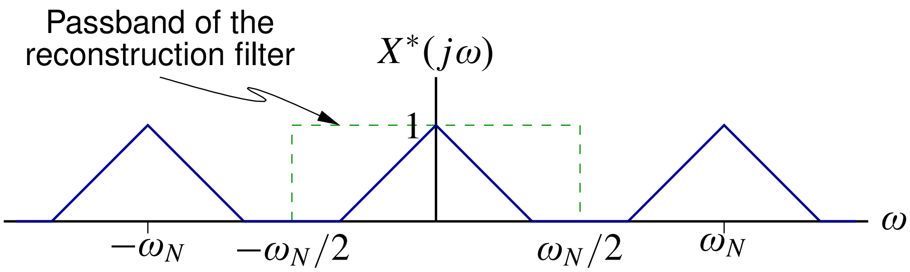


- When aliasing is present, it is impossible to recover the original analog signal from the sampled signal.
- Nyquist/Shannon said that if a signal is sampled at a rate ω_N there will be no aliasing if $\omega_N \geq 2$ times the highest frequency in the signal.

KEY POINT: Our digital control system will need to sample at least twice as fast as the closed-loop bandwidth. Four to six times faster is safer.

Data deconstruction

- If the signal $x(t)$ has been sampled at or above the Nyquist rate ω_N, it is mathematically possible to recover $x(t)$ from $x(kT)$. Surprise? Magic?
- Process $X^*(j\omega)$ by passing it through an ideal low-pass filter with bandwidth $\omega_N/2$.

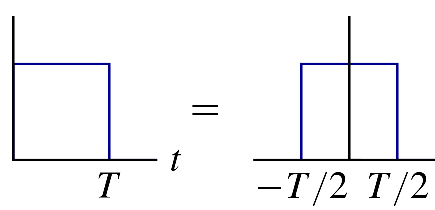


- The filter must have gain T to recover the magnitude lost by sampling.

$$h(t) = \frac{\sin(\pi t/T)}{\pi t/T} = \text{sinc}(t/T).$$

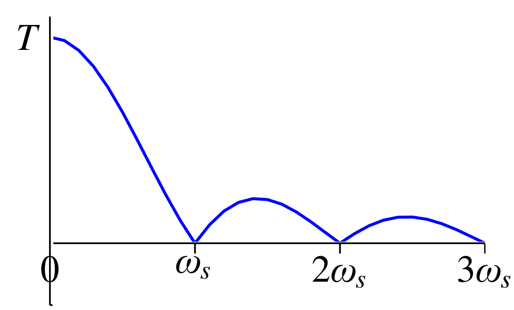
PROBLEM: $h(t)$ is non-causal.

- Solution: "Approximate" $h(t)$ very crudely with a causal filter—the hold circuit.
- The impulse response of the hold circuit is:

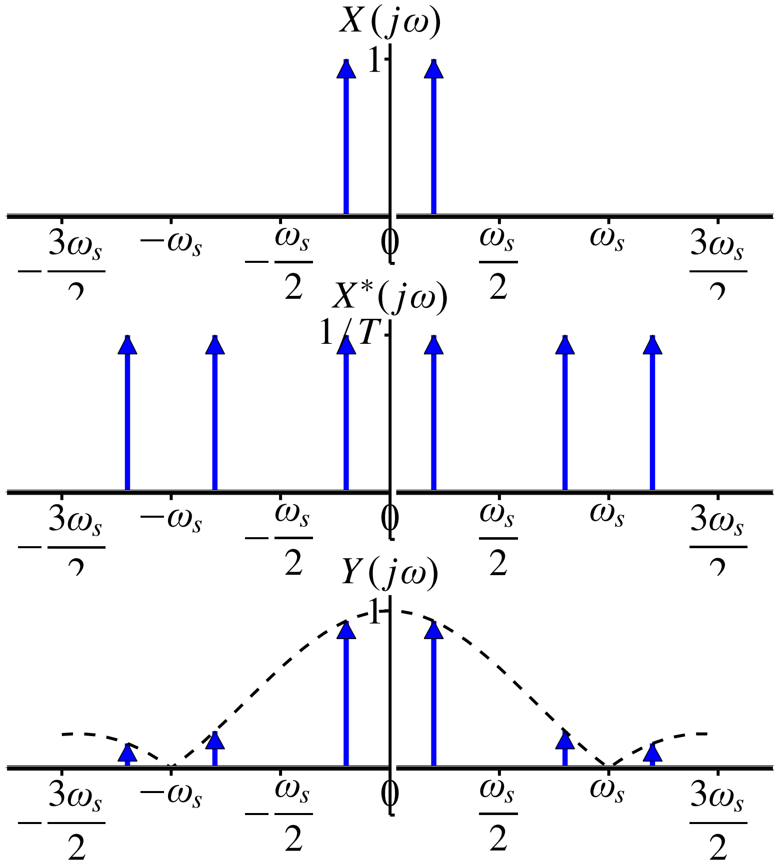


- So, $H(j\omega)$ is

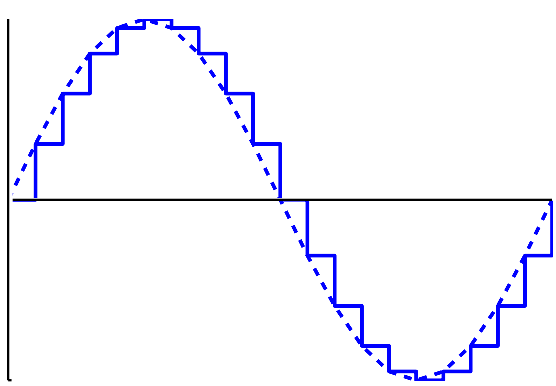
$$H(j\omega) = T\left(\frac{\sin(\omega T/2)}{\omega T/2}\right)e^{-j\omega T/2}$$
$$= T\,\text{sinc}\left(\frac{\omega}{\omega_s}\right)e^{-j\pi\omega/\omega_s}.$$



EXAMPLE: Let $x(t) = 2\sin(\omega_1 t)$ with $\omega_1 < \omega_s/2$.

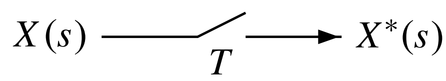


- Notice that the ZOH does not completely filter out the aliased frequencies due to sampling. Since most plants are low-pass devices, this is often okay.
- The high-frequency components look like stair-steps in time-domain. Note: Familiar $T/2$ delay of fundamental frequency.

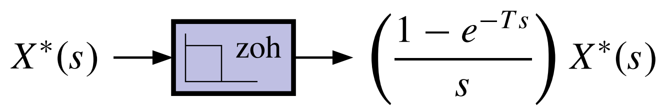


3.3: Open-loop discrete-time systems

- We have seen some properties of the sampling operation and of the hold operation.
- Sampling:



- Hold:



- The z-transform will enable us to put these elements into a system and analyze them a little more easily.
- We first look at open-loop systems, and then at closed-loop systems.

Open-loop discrete-time systems

- We have already seen the relationship between $X^*(s)$ and $X(z)$.
- Easy to prove:

$$\mathcal{Z}[x[k]] = X(z) = x[0] + x[1]z^{-1} + x[2]z^{-2} + \cdots$$

$$X^*(s) = x(0) + x(T)e^{-sT} + x(2T)e^{-s2T} + \cdots$$

$$\text{so, } X(z) = X^*(s)\big|_{e^{sT}=z}$$

$$\text{and } X^*(s) = X(z)|_{z=e^{sT}}.$$

- So, the z-transform is a special case of the Laplace transform for discrete-time systems.
- Sometimes it is easier to find the starred transform $X^*(s)$ from the z-transform $X(z)$!

EXAMPLE: Let

$$X(s) = \frac{1}{(s+1)(s+2)}.$$

- From the z-transform table in Topic 2.7 of your notes:

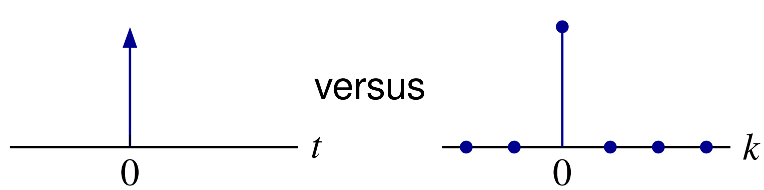
$$\begin{aligned} X(z) &= \frac{z(e^{-T} - e^{-2T})}{(z - e^{-T})(z - e^{-2T})} \\ X^*(s) &= X(z)|_{z=e^{sT}} \\ &= \frac{e^{sT}(e^{-T} - e^{-2T})}{(e^{sT} - e^{-T})(e^{sT} - e^{-2T})}. \end{aligned}$$

NOTE:

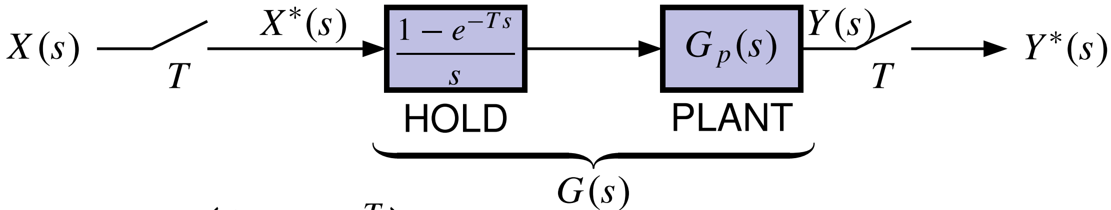
- $X^*(s)$ has ∞ poles and zeros in the s-plane because of its periodic nature.
- $X(z)$ has a single zero at $z = 0$, and two poles at $z = e^{-T}$ and $z = e^{-2T}$.
- Pole-zero analysis is going to be easier in the z-domain.

The discrete-impulse/unit-pulse transfer function

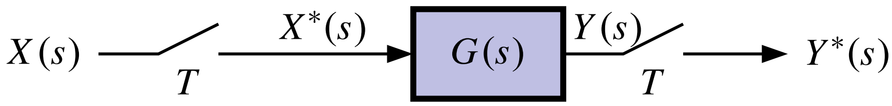
- For continuous-time LTI systems, the impulse response described the output of the system when the input was the impulse/delta function.
- The transfer function is the Laplace transform of the impulse response.
- The same can be done for discrete-time systems, but the input is now a discrete-impulse, often called a unit pulse or simply a pulse.



- We are interested in systems that look like:



- Define $G(s) = \left(\dfrac{1-e^{-sT}}{s}\right) G_p(s)$ to "hide" the mess introduced by the hold operation.



- So, $Y(s) = G(s)X^*(s)$.
- $Y^*(s)$ would occur if we sampled $Y(s)$. $Y^*(s) = [G(s)X^*(s)]^*$.

$$\begin{aligned} Y^*(s) &= \frac{1}{T}\sum_{k=-\infty}^{\infty} Y(s+jk\omega_s) + \frac{y(0)}{2} \\ &= \frac{1}{T}\sum_{k=-\infty}^{\infty} G(s+jk\omega_s)X^*(s+jk\omega_s) + \frac{y(0)}{2}. \end{aligned}$$

- But, since $X^*(s)$ is a sampled signal, it is periodic in $s + jk\omega_s$:

$$X^*(s) = X^*(s+jk\omega_s).$$

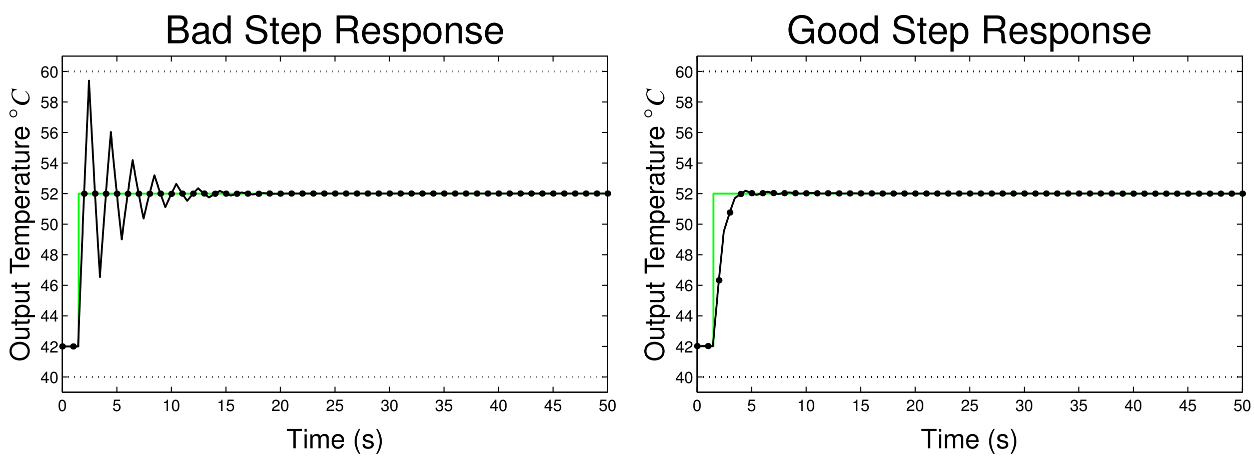
- So,

$$\begin{aligned} Y^*(s) &= \frac{1}{T}\sum_{k=-\infty}^{\infty} G(s+jk\omega_s)X^*(s) + \frac{y(0)}{2} \\ &= X^*(s)\left[\frac{1}{T}\sum_{k=-\infty}^{\infty} G(s+jk\omega_s)\right] + \frac{y(0)}{2} \\ &= X^*(s)G^*(s) + \frac{y(0)}{2} \end{aligned}$$

or, $Y(z) = X(z)G(z) + \dfrac{y(0)}{2}.$

- $G(z)$ is the pulse transfer function between the sampled input and the output *at the sampling instants*.
  - Recall that transfer functions assume initial conditions to be zero.
- It tells us *NO* information about the output between the sampling instants.
  - If we choose a fast sampling rate, then interpolation can be a good approximation.
  - Simulation is a general tool to help determine inter-sample behavior accurately.

EXAMPLE: At sample points, left plot looks better than right plot, but in-between sample points, left plot looks worse than right plot.



3.4: Finding $G(z)$ from $G_p(s)$

- Recall that
  - $G(s) = \left(\frac{1-e^{-sT}}{s}\right) G_p(s)$,
  - $G^*(s)$ is a sampled Laplace transform of $G(s)$, and
  - $G(z) = G^*(s)\big|_{z=e^{sT}}$.
- To find $G(z)$ we must then

1. Find $g(t) = \mathcal{L}^{-1}\left[\left(\frac{1-e^{-sT}}{s}\right) G_p(s)\right]$.
2. Find $g[k] = g(kT)$.
3. Find $G(z) = \mathcal{Z}\left[g(kT)\right]$.

- Notation: $G(z) = \mathcal{Z}\left[\left(\frac{1-e^{-sT}}{s}\right) G_p(s)\right]$.

A shortcut for systems having no time delays:

- We calculate

$$
\begin{aligned}
G(z) &= \mathcal{Z}\left[\left(\frac{1-e^{-sT}}{s}\right) G_p(s)\right] \\
&= \mathcal{Z}\left[\left(1-e^{-sT}\right)\frac{G_p(s)}{s}\right] \\
&= \mathcal{Z}\left[\left(\frac{G_p(s)}{s}\right)\right] - \mathcal{Z}\left[e^{-sT}\frac{G_p(s)}{s}\right].
\end{aligned}
$$

- But: e^{-sT} is the cts-time transfer function of a delay of T seconds.
- In disc-time, z^{-1} is the transfer function of a unit delay, (T seconds).

$$
G(z) = (1-z^{-1})\mathcal{Z}\left[\frac{G_p(s)}{s}\right] = \frac{z-1}{z}\mathcal{Z}\left[\frac{G_p(s)}{s}\right].
$$

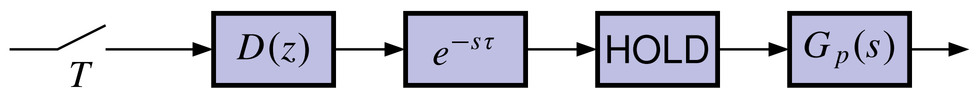
- This last term may often be looked up in a table.
- In MATLAB: Let $G_p(s) = \dfrac{1}{s+1}$.

```
numc=1;
denc=[1 1];
sysc=tf(numc,denc);
sysd=c2d(sysc,T);%  = sampling period.
[numd,dend,T]=tfdata(sysd);
```

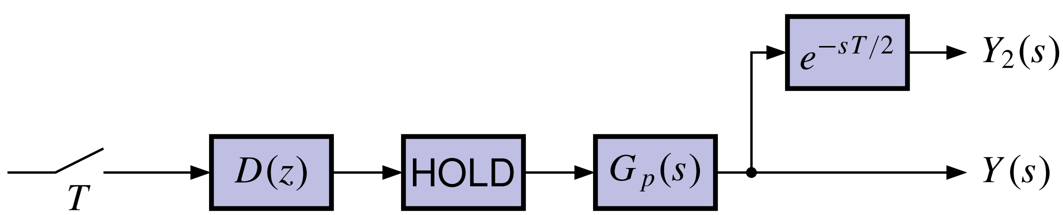
A shortcut for systems having time delays: (The modified z-transform)

- We often need to analyze systems with pure time delays:

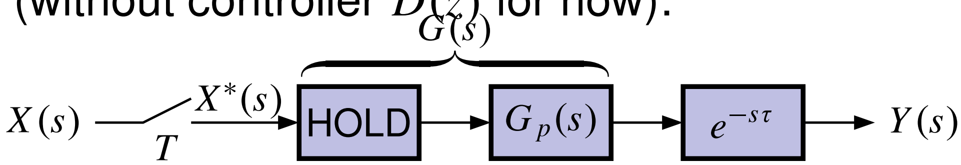
1. Plant is of the form $G_p(s) = e^{-s\tau}\dfrac{\text{num}(s)}{\text{den}(s)}$;
2. Digital controller introduces delay in computation;



3. And/or we want to know plant output between normal sampling times (the ripple).



- Consider (without controller $D(z)$ for now):



$$Y(s) = e^{-s\tau}G(s)X^*(s)$$

$$Y^*(s) = [e^{-s\tau}G(s)]^*X^*(s)$$

$$Y(z) = \mathcal{Z}\left[e^{-s\tau}G(s)\right]X(z)$$

- Let $\tau = kT + \Delta T,\ k \in \mathbb{Z},\ 0 < \Delta < 1$

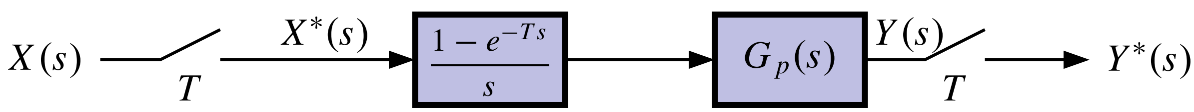
$$Y(z) = z^{-k} \underbrace{\mathcal{Z}\left[e^{-s\Delta T}G(s)\right]}_{\triangleq G(z,m)\text{ where } m=1-\Delta} X(z)$$

$$= z^{-k}G(z,(1-\Delta))X(z).$$

  - You can use modified-z-transform table from Topic 2.7,
  - Or you can work it out by hand,
  - Or... you can use MATLAB

```
sys=tf(num,den);
set(sys,'ioDelay',tau); % tau is total delay, in seconds
sysd=c2d(sys,T);
```

EXAMPLE OF PULSE TRANSFER FUNCTION:



- Let $G_p(s) = \dfrac{1}{s+1}$ and $X(s) = \dfrac{1}{s}$ (a unit step).
- So,

$$Y(s) = \left(\frac{1-e^{-sT}}{s}\right)\left(\frac{1}{s+1}\right)X^*(s) = G(s)X^*(s).$$

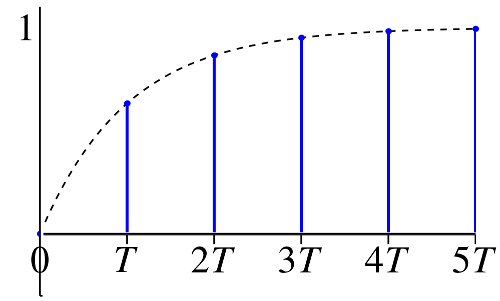
- Also note that

$$G(z) = \mathcal{Z}\left[\left(\frac{1-e^{-sT}}{s}\right)\left(\frac{1}{s+1}\right)\right] = \frac{1-e^{-T}}{z-e^{-T}}$$

and $X(z) = \mathcal{Z}[1(t)] = \dfrac{z}{z-1}$.

- So,

$$Y(z) = \left(\frac{1-e^{-T}}{z-e^{-T}}\right)\left(\frac{z}{z-1}\right)$$
$$= \frac{z}{z-1} - \frac{z}{z-e^{-T}}$$
$$y[k] = \left(1-e^{-kT}\right)1[k].$$



NOTE: A shortcut for this particular example would have been to notice that the sample-and-hold does nothing to the unit-step input!

- This example is also a good one to illustrate dc gain:

$$y_{ss} = \lim_{z\to 1}(z-1)Y(z)$$
$$= \lim_{z\to 1}\frac{(1-e^{-T})z}{(z-e^{-T})}$$
$$= \frac{1-e^{-T}}{1-e^{-T}} = 1.$$

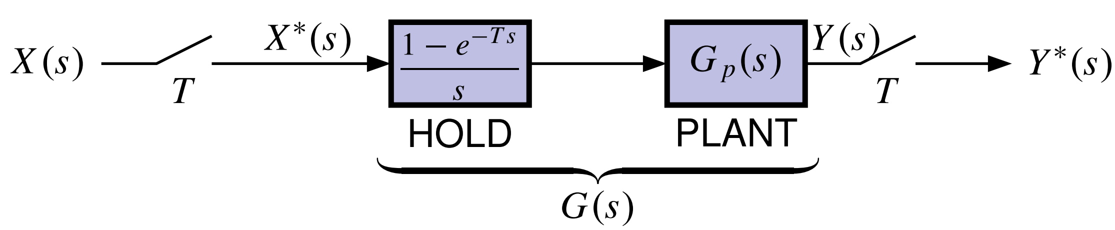
- Equivalently, we can find in general

$$\text{dc gain} = \lim_{z\to 1}(z-1)Y(z)$$
$$= \lim_{z\to 1}(z-1)G(z)1(z)$$
$$= \lim_{z\to 1}G(z)z = G(1).$$

So, $G(1)$ is the dc gain (a.k.a., the final value to a unit-step input).

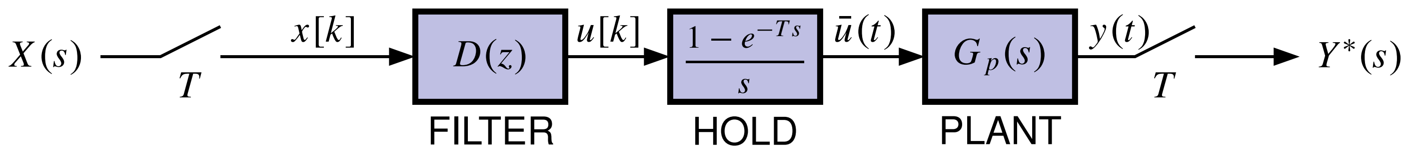
- DC gain $= \lim_{z\to 1}G(z) = \lim_{s\to 0}G_p(s)$.

CONCLUSION: For the following open loop system, we have $Y(z) = G(z)X(z)$.



3.5: Systems with digital filters

- Our digital controllers will not simply sample and hold their input value. They will perform some mathematical operations on the sampled input. "FILTER"
- So, we expand our system of interest:



- Suppose our digital filter has transfer function $D(z)$.
- Then,

$$U(z) = D(z)X(z), \qquad \text{or, via } z = e^{sT}$$

$$U^*(s) = D^*(s)X^*(s).$$

$\bar{u}(t)$ occurs at the output of the hold circuit, so

$$\bar{U}(s) = \left(\frac{1-e^{-sT}}{s}\right)U^*(s), \quad \text{and}$$

$$Y(s) = G_p(s)\bar{U}(s) = G_p(s)\left(\frac{1-e^{-sT}}{s}\right)U^*(s).$$

- Since

$$U^*(s) = D^*(s)X^*(s),$$

$$Y(s) = G_p(s)\left(\frac{1-e^{-sT}}{s}\right)\underbrace{D^*(s)}_{=D(z)|_{z=e^{sT}}}X^*(s)$$

$$Y^*(s) = \left(G_p(s)\left(\frac{1-e^{-sT}}{s}\right)\right)^* D^*(s)X^*(s)$$

$$Y(z) = \mathcal{Z}\left[G_p(s)\left(\frac{1-e^{-sT}}{s}\right)\right]D(z)X(z)$$

$$Y(z) = G(z)D(z)X(z).$$

EXAMPLE: Let the digital controller implement

$$u[k] = 2x[k] - x[k-1]$$

and the plant is:

$$G_p(s) = \frac{1}{s+1}.$$

- Find $D(z)$:

$$U(z) = 2X(z) - z^{-1}X(z)$$

$$= \left(2 - z^{-1}\right)X(z)$$

$$D(z) = \frac{U(z)}{X(z)} = 2 - z^{-1} = \frac{2z-1}{z}.$$

- Find $G(z)$:

$$G(z) = \mathcal{Z}\left\{\frac{1-e^{sT}}{s}\frac{1}{s+1}\right\} = \frac{1-e^{-T}}{z-e^{-T}}.$$

- So,

$$Y(z) = \left(\frac{1-e^{-T}}{z-e^{-T}}\right)\left(\frac{2z-1}{z}\right)X(z).$$

EXAMPLE: Let $x(t) = 1(t)$, and find $Y(z)$.

$$X(z) = \frac{z}{z-1}$$

$$Y(z) = \left(\frac{1-e^{-T}}{z-e^{T}}\right)\left(\frac{2z-1}{z}\right)\left(\frac{z}{z-1}\right) = \frac{(2z-1)\left(1-e^{-T}\right)}{(z-1)\left(z-e^{-T}\right)}.$$

- We can determine $y[k]$ by partial-fraction expansion and inversion of $Y(z)$:

$$\frac{Y(z)}{z} = \frac{(2z-1)(1-e^{-T})}{z(z-1)(z-e^{-T})} = \frac{1-e^{T}}{z} + \frac{1}{z-1} + \frac{e^{T}-2}{z-e^{-T}}.$$

$$Y(z) = (1-e^{T}) + \frac{z}{z-1} + \frac{z(e^{T}-2)}{z-e^{-T}}$$

$$\Updownarrow \qquad \Updownarrow \qquad \Updownarrow$$

$$(1-e^{T})\delta[k] \qquad 1[k] \qquad (e^{T}-2)e^{-kT}1[k]$$

$$y[k] = (1-e^{T})\delta[k] + 1[k] + (e^{T}-2)e^{-kT}1[k].$$

NOTE: At $k=0$,

$$y[0] = (1-e^{T}) + 1 + (e^{T}-2) = 0.$$

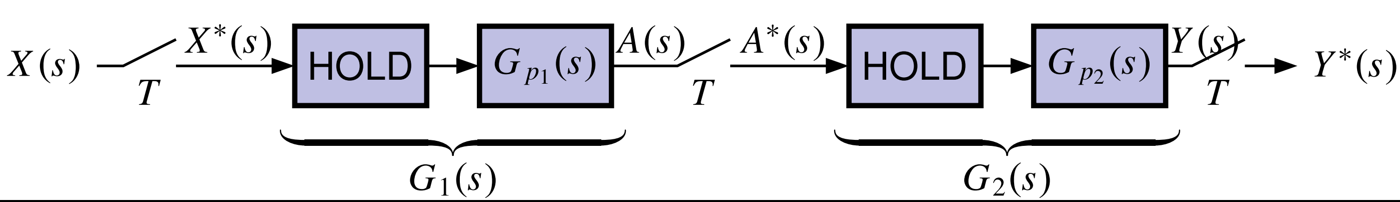
- This could also be observed by noting that $Y(z)$ has higher-order denominator than numerator.

➠ The delay in the *HOLD* means that the output is not instantaneous.

Block diagrams and sampled data

- Control systems exist in great variety.
  - Almost always more complicated than the open-loop system we saw here.
  - To be able to analyze closed-loop and more involved systems, we need to be able to analyze general block diagrams with samplers in them.

EXAMPLE:



$$Y(s) = G_2(s)A^*(s)$$

$$A(s) = G_1(s)X^*(s)$$

$$\left.\begin{aligned} Y^*(s) &= \left[G_2(s)A^*(s)\right]^* \\ A^*(s) &= \left[G_1(s)X^*(s)\right]^* \end{aligned}\right\} \quad \text{can this simplify? (pg. 3–12)}$$

$$Y^*(s) = G_2^*(s)A^*(s)$$

$$A^*(s) = G_1^*(s)X^*(s)$$

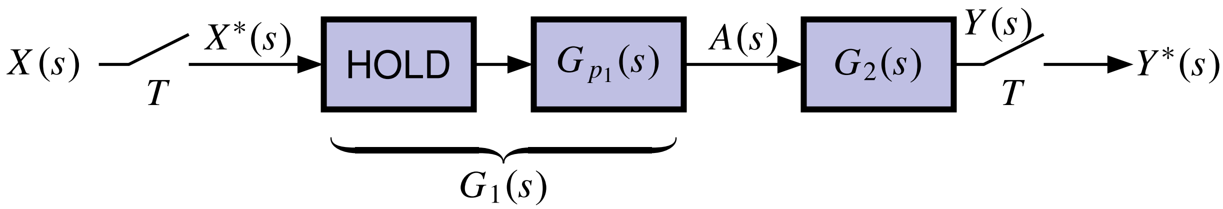
or, $Y^*(s) = G_2^*(s)G_1^*(s)X^*(s)$

$$Y(z) = G_2(z)G_1(z)X(z).$$

- Things are not always this straightforward.

3.6: Some more challenging examples

EXAMPLE: Consider the hybrid system drawn below.



$$Y(s) = G_2(s)A(s) = G_2(s)G_1(s)X^*(s)$$

$$\begin{aligned} Y^*(s) &= [\underbrace{G_2(s)G_1(s)}_{G(s)}X^*(s)]^* \\ &= [G(s)X^*(s)]^* \\ &= G^*(s)X^*(s) \end{aligned}$$

$$Y(z) = G(z)X(z) \quad \text{or,} \quad Y(z) = (\overline{G_2G_1})(z)X(z).$$

- But, is $G^*(s) = G_2^*(s)G_1^*(s)$?
- That is, does $G(z) = G_2(z)G_1(z)$?
- Let:

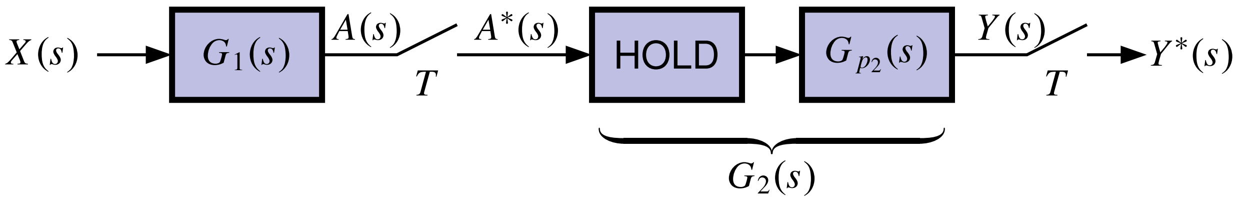
$G_2(s) = 1/s$	$G_1(s) = 1/s$	$G(s) = 1/s^2$
$G_2^*(s) = \dfrac{e^{sT}}{e^{sT}-1}$	$G_1^*(s) = \dfrac{e^{sT}}{e^{sT}-1}$	$G^*(s) = \dfrac{Te^{sT}}{(e^{sT}-1)^2}$

$$G_1^*(s)G_2^*(s) = \frac{e^{2sT}}{(e^{sT}-1)^2} \neq \frac{Te^{sT}}{(e^{sT}-1)^2}.$$

KEY CONCLUSION: $[G_1(s)G_2(s)]^* \neq G_1^*(s)G_2^*(s)$ in general.

REASON: Sampling is time-varying, so system is not LTI. In fact, in some cases we may not even be able to find transfer functions. . .

EXAMPLE: Consider the hybrid system drawn below.



$$Y(s) = G_2(s)A^*(s)$$

$$A(s) = G_1(s)X(s)$$

$$A^*(s) = [G_1(s)X(s)]^* \neq G_1^*(s)X^*(s)$$

$$Y(s) = G_2(s)[G_1(s)X(s)]^*$$

$$Y^*(s) = G_2^*(s)[G_1(s)X(s)]^*$$

$$Y(z) = G_2(z)[\overline{G_1X}](z).$$

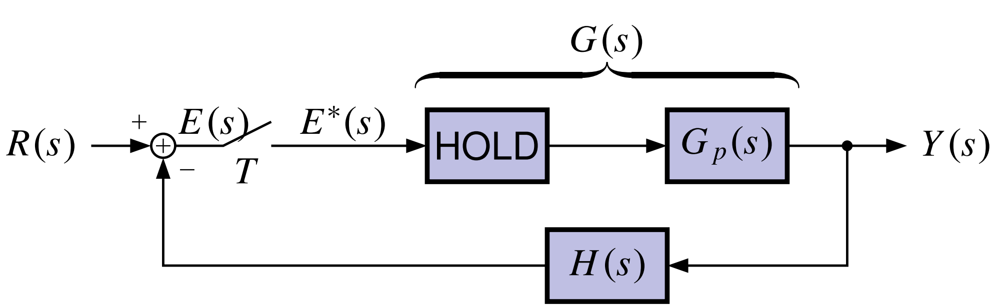
NOTE: We cannot separate $[\overline{G_1X}](z)$ to create a transfer function $Y(z)/X(z)$ (!)

- ($a(t)$ is a function of all previous values of $x(t)$, not just the values of $x[kT]$, at the sampling instants.)

Closing the loop—Feedback

- Ultimately, we are more interested in closed-loop than in open-loop systems.

EXAMPLE:



$$E(s) = R(s) - H(s)Y(s)$$

$$Y(s) = G(s)E^*(s)$$

$$\Rightarrow E(s) = R(s) - H(s)[G(s)E^*(s)] = R(s) - [H(s)G(s)]E^*(s)$$

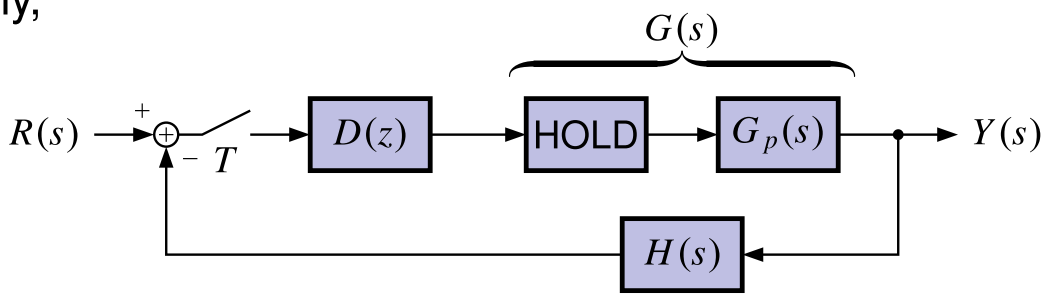
$$E^*(s) = R^*(s) - [H(s)G(s)]^*E^*(s)$$

$$= \frac{R^*(s)}{1 + \left(\overline{GH}\right)^*(s)}$$

$$Y^*(s) = \left[\frac{G^*(s)}{1 + \left(\overline{GH}\right)^*(s)}\right] R^*(s)$$

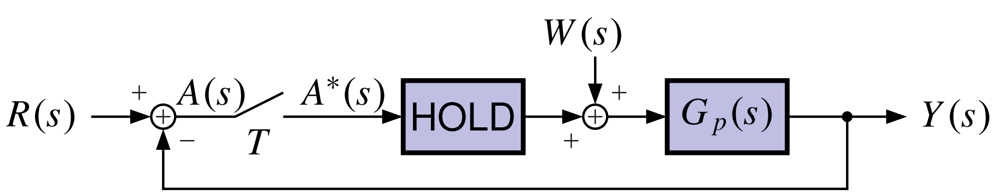
$$Y(z) = \left[\frac{G(z)}{1 + \overline{GH}(z)}\right] R(z).$$

- Similarly,



$$\frac{Y(z)}{R(z)} = \frac{D(z)G(z)}{1 + D(z)\overline{GH}(z)}.$$

The problem of disturbance



$$Y(s) = G_p(s)W(s) + G(s)A^*(s)$$

$$A(s) = R(s) - Y(s) = R(s) - G_p(s)W(s) - G(s)A^*(s)$$

$$A^*(s) = R^*(s) - \left(\overline{G_pW}\right)^*(s) - G^*(s)A^*(s)$$

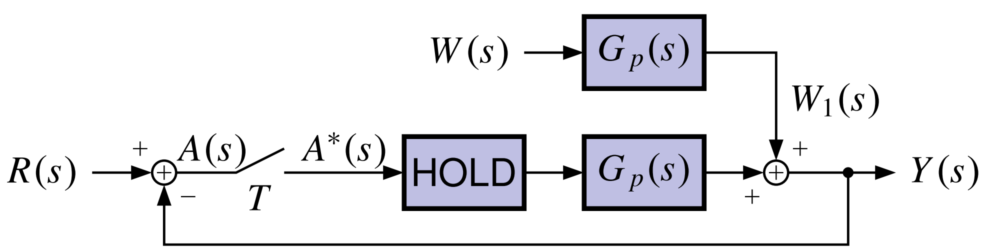
$$= \frac{R^*(s)}{1+G^*(s)} - \frac{\left(\overline{G_pW}\right)^*(s)}{1+G^*(s)}$$

$$Y(s) = G_p(s)W(s)\left[\frac{1+G^*(s)}{1+G^*(s)}\right] + \frac{G(s)R^*(s)}{1+G^*(s)} - \frac{\left(\overline{G_pW}\right)^*(s)G(s)}{1+G^*(s)}$$

$$Y^*(s) = \left[\frac{G^*(s)}{1+G^*(s)}\right]R^*(s) + \left[\frac{1}{1+G^*(s)}\right]\left(\overline{G_pW}\right)^*(s)$$

$$Y(z) = \left[\frac{G(z)}{1+G(z)}\right]R(z) + \left[\frac{1}{1+G(z)}\right](\overline{G_pW})(z).$$

- The transfer function from $R(s)$ to $Y(s)$ is $\dfrac{G(z)}{1+G(z)}\ldots$ no surprise.
- *No transfer function exists* between $W(z)$ and $Y(z)$ (!)
- Solution (?)



$$Y(z) = \left[\frac{G(z)}{1+G(z)}\right]R(z) + \left[\frac{1}{1+G(z)}\right]W_1(z).$$

Where to from here?

- We now understand the hybrid nature of digital control systems and are able to analyze such hybrid systems.
- We're almost ready for controller design—but first, we look at stability analysis for discrete-time systems.