$\mathscr{H}_\infty$ OUTPUT-FEEDBACK CONTROLLER

9.1: Setting up the problem

- $\mathscr{H}_\infty$ output-feedback control (or simply $\mathscr{H}_\infty$ control) uses partial state measurements, corrupted by disturbances, to generate the control.
  - We might consider combining $\mathscr{H}_\infty$ full-information controller with $\mathscr{H}_\infty$ estimation, but this is not correct since the $\mathscr{H}_\infty$ estimator depends on what linear combination of states is being estimated.
  - That is, the estimator will depend on the controller(!).
  - $\mathscr{H}_\infty$ control comprises an estimated-state feedback part and an estimator part, but these two parts are not independent (no separation principle).
- We use the plant model

$$\dot{x}(t) = Ax(t) + \left[\begin{array}{c:c} B_u & B_w \end{array}\right] \left[\begin{array}{c} u(t) \\ \hdashline w(t) \end{array}\right];$$

$$\left[\begin{array}{c} m(t) \\ \hdashline y(t) \end{array}\right] = \left[\begin{array}{c} C_m \\ \hdashline C_y \end{array}\right] x(t) + \left[\begin{array}{c:c} 0 & D_{mw} \\ \hdashline D_{yu} & 0 \end{array}\right] \left[\begin{array}{c} u(t) \\ \hdashline w(t) \end{array}\right].$$

- The matrices B_w and D_{mw} are assumed to satisfy the conditions:

$$D_{mw} B_w^T = 0 \qquad \text{and} \qquad D_{mw} D_{mw}^T = I.$$

- These conditions require:

- That plant disturbance and measurement noise be distinct, and
- That measurement noise be normalized in equations.

- The matrices C_y and D_{yu} are assumed to satisfy the conditions:

$$D_{yu}^T C_y = 0 \qquad \text{and} \qquad D_{yu}^T D_{yu} = I.$$

- These conditions require:
  - The reference output consists of an output dependent only on the state and a distinct output depending only on the control input.
  - The part that depends on the control input is normalized.
- The plant is also assumed controllable and observable.
- The suboptimal $\mathscr{H}_\infty$ control problem is to find an output-feedback controller for the above plant such that the ∞-norm of the closed-loop system is bounded:

$$\left\| G_{yw} \right\|_{\infty\,[0,t_f]} = \sup_{\|w(t)\|_{2,[0,t_f]}\neq 0} \frac{\|y(t)\|_{2,[0,t_f]}}{\|w(t)\|_{2,[0,t_f]}} < \gamma.$$

- The closed-loop system is also required to be internally stable.

The problem. . .

- We want to approximate $u(t) = -K(t)x(t)$ without measuring $x(t)$.
- That is, we want to find $\hat{u}(t) \approx u(t)$ when measuring output only, not state.
- In LQG, a Kalman filter estimated $\hat{u}(t) = -K(t)\hat{x}(t)$ because the solution to the Kalman filter does not depend on what linear combination of states is desired in the estimate:

$$\hat{u}(t) = \mathbb{E}[-K(t)x(t) \mid \mathbb{Z}(t)] = -K(t)\mathbb{E}[x(t) \mid \mathbb{Z}(t)] = -K(t)\hat{x}(t),$$

where $\mathbb{Z}(t)$ is the set of all measurements until time t.

- The $\mathscr{H}_\infty$ estimator does depend on the linear combination of states being estimated, as demonstrated in the example at the end of the last chapter of notes.
- Therefore, the $\mathscr{H}_\infty$ estimator must estimate $\hat{u}(t)$ as an estimate of the signal $-K(t)x(t)$ together.
- The control-design procedure we follow is:
  - Design a full-information $\mathscr{H}_\infty$ controller for the plant;
  - From this, we have a Riccati equation for $P(t)$, we know $K(t) = B_u^T P(t)$, and so altogether we have $u(t) = -K(t)x(t)$.
  - Design an $\mathscr{H}_\infty$ estimator for $-K(t)x(t)$. Estimator output is $\hat{u}(t)$.
  - Use $\hat{u}(t)$ as the plant control input.
- Because of the different plant model assumptions for the full-information controller/estimator and the model above, we must modify our full-information control design somewhat.
- The *control* problem is still to minimize

$$J_\gamma = \|y(t)\|^2_{2,[0,t_f]} - \gamma^2 \|w(t)\|^2_{2,[0,t_f]} \le -\varepsilon^2 \|w(t)\|^2_{2,[0,t_f]}.$$

- However, this cost function is not useful when designing the estimator.
  - It is not a function of $u(t)$. We would like to see $\|u(t) - \hat{u}(t)\|$ in cost function instead of $\|y(t)\|$.
- We must modify it given our knowledge of the Riccati equation developed for the suboptimal full-information controller. This will give us a cost function more suitable for designing the estimator.

9.2: $\mathscr{H}_\infty$ output-feedback control as estimation problem

- We start by asserting a non-obvious identity that will be used to "complete the square"

$$\int_0^{t_f} \frac{\mathrm{d}x^T(t)P(t)x(t)}{\mathrm{d}t}\,\mathrm{d}t = x^T(t_f)\underbrace{P(t_f)}_{0}x(t_f) - \underbrace{x^T(0)}_{0}P(0)\underbrace{x(0)}_{0} = 0$$

  since $x(0) = 0$ and $P(t_f) = 0$ by definition in the $\mathscr{H}_\infty$ problem.

- We insert this identity into the cost function J_γ

$$\begin{aligned} J_\gamma &= \int_0^{t_f} \Big\{x^T(t)C_y^TC_yx(t) + u^T(t)u(t) - \gamma^2 w^T(t)w(t) \\ &\qquad + \frac{\mathrm{d}x^T(t)P(t)x(t)}{\mathrm{d}t}\Big\}\,\mathrm{d}t \\ &= \int_0^{t_f} \Big\{x^T(t)C_y^TC_yx(t) + u^T(t)u(t) - \gamma^2 w^T(t)w(t) \\ &\qquad + \dot{x}^T(t)P(t)x(t) + x^T(t)\dot{P}(t)x(t) + x^T(t)P(t)\dot{x}(t)\Big\}\,\mathrm{d}t. \end{aligned}$$

- To continue, note that $\dot{x}(t) = Ax(t) + B_uu(t) + B_ww(t)$. Regroup and drop dependence on t for clarity

$$\begin{aligned} J_\gamma = \int_0^{t_f} \Big\{&x^T\Big[\dot{P} + C_y^TC_y + A^TP + PA\Big]x + u^Tu - \gamma^2w^Tw \\ &+ (B_uu + B_ww)^TPx + x^TP(B_uu + B_ww)\Big\}\,\mathrm{d}t. \end{aligned}$$

- Now, substitute from the Riccati equation for $\dot{P} + C_y^TC_y + A^TP + PA$

$$\begin{aligned} J_\gamma &= \int_0^{t_f} \Big\{x^TP\Big[B_uB_u^T - \gamma^{-2}B_wB_w^T\Big]Px + u^Tu - \gamma^2w^Tw \\ &\qquad + (B_uu + B_ww)^TPx + x^TP(B_uu + B_ww)\Big\}\,\mathrm{d}t \\ &= \int_0^{t_f} [u + B_u^TPx]^T[u + B_u^TPx] \end{aligned}$$

$$-\gamma^2[w-\gamma^{-2}B_w^T Px]^T[w-\gamma^{-2}B_w^T Px]\,\mathrm{d}t$$

$$= \|u(t)+B_u^T P(t)x(t)\|_{2,[0,t_f]}^2 - \gamma^2\|w(t)-\gamma^{-2}B_w^T P(t)x(t)\|_{2,[0,t_f]}^2.$$

- Combining this with the inequality we started with, we get a cost function suitable for a two-player game optimization:

$$J_\gamma = \left\|u(t)+B_u^T P(t)x(t)\right\|_{2,[0,t_f]}^2 - \gamma^2\left\|w(t)-\gamma^{-2}B_w^T P(t)x(t)\right\|_{2,[0,t_f]}^2$$
$$\le -\varepsilon^2\|w(t)\|_{2,[0,t_f]}^2.$$

- The cost function is identical to what we started with, but now written in terms of $u(t)$.
- To ease notation, define $\Delta w = w(t)-\gamma^{-2}B_w^T P(t)x(t)$.
- Then, the *control* problem of minimizing J_γ is equivalent to minimizing

$$\|G_\Delta\|_\infty = \sup_{\Delta w\neq 0}\frac{\left\|u(t)+B_u^T P(t)x(t)\right\|_{2,[0,t_f]}}{\|\Delta w\|_{2,[0,t_f]}} < \gamma.$$

- Keep this *control* objective in mind, and now consider the task of designing an $\mathcal{H}_\infty$ *estimator* to estimate the linear combination of states

$$u(t) = -B_u^T P(t)x(t),$$

given measurements $m(t)$ such that the ∞-norm of the transfer function between disturbance input Δw and the estimation error is bounded.

- Then, we are minimizing

$$\|G_\Delta\|_\infty = \sup_{\Delta w(t)\neq 0}\frac{\|u(t)-\hat{u}(t)\|_{2,[0,t_f]}}{\|\Delta w\|_{2,[0,t_f]}}$$

$$= \sup_{\Delta w(t)\neq 0} \frac{\left\| -B_u^T P(t)x(t) - \hat{u}(t)\right\|_{2,[0,t_f]}}{\|\Delta w\|_{2,[0,t_f]}} < \gamma.$$

- This is the same problem as the control-design problem we just derived!
- The goal is to find a $u(t)$ that approximates $-B_u^T P(t)x(t)$: That is, we know $u(t)$ should be $u(t) = -B_u^T P(t)x(t)$ but we will try to approximate as closely as possible without measuring $x(t)$ by minimizing this gain.
- We need to be careful how we apply the $\mathscr{H}_\infty$ estimation, however, since our definition of inputs has changed.
  - Our disturbance input is now $\Delta w(t)$, not $w(t)$.

$$\begin{aligned}\dot{x}(t) &= Ax(t) + B_u u(t) + B_w w(t)\\ &\qquad + \underbrace{\gamma^{-2}B_w B_w^T P(t)x(t) - \gamma^{-2}B_w B_w^T P(t)x(t)}_{0}\\ &= [A + \gamma^{-2}B_w B_w^T P(t)]x(t) + B_u u(t) + B_w \Delta w(t).\end{aligned}$$

- The measurement equation may be similarly generated, noting that $\gamma^{-2}D_{mw}B_w^T P(t)x(t) = 0$.

$$\begin{aligned}m(t) &= C_m x(t) + D_{mw}w(t) - \gamma^{-2}D_{mw}B_w^T P(t)x(t)\\ &= C_m x(t) + D_{mw}\Delta w(t).\end{aligned}$$

- The resulting plant model may be used with our previous results on $\mathscr{H}_\infty$ estimation.
  - We will see how to do so in the next topic.

9.3: Finite-time and steady-state control

Finite-time control

- With the plant model reformulated, we can now address the finite-time control problem and the steady-state control problem.
- The finite-time control problem results in a Riccati equation with the modified A matrix from above.
- Let $A_m(t) = A + \gamma^{-2} B_w B_w^T P(t)$. The Riccati equation is then:

$$\begin{aligned}\dot{Q}_m(t) = Q_m(t)A_m^T(t) + A_m(t)Q_m(t) + B_w B_w^T \\ - Q_m(t)[C_m^T C_m - \gamma^{-2} P(t) B_u B_u^T P(t)] Q_m(t),\end{aligned}$$

  with initial condition $Q_m(0) = 0$.
- The suboptimal estimator is then

$$\begin{aligned}\dot{\hat{x}}(t) &= A_m(t)\hat{x}(t) + B_u u(t) + L(t)[m(t) - C_m \hat{x}(t)] \\ \hat{u}(t) &= -B_u^T P(t)\hat{x}(t),\end{aligned}$$

  where the gain $L(t) = Q_m(t) C_m^T$.
- This estimator becomes our controller by letting $u(t) = \hat{u}(t)$. Then,

$$\begin{aligned}\dot{\hat{x}}(t) &= [A_m - B_u B_u^T P(t) - Q_m(t) C_m^T C_m]\hat{x}(t) + Q_m(t) C_m^T m(t) \\ u(t) &= -B_u^T P(t)\hat{x}(t).\end{aligned}$$

- The modified Riccati equation also leads to a Hamiltonian system (see text for a lot of omitted math), which is:

$$\begin{aligned}\dot{\tilde{x}}(t) &= \left[\begin{array}{c|c} (A + \gamma^{-2} B_w B_w^T P(t))^T & \gamma^{-2} P(t) B_u B_u^T P(t) - C_m^T C_m \\ \hline -B_w B_w^T & -(A + \gamma^{-2} B_w B_w^T P(t)) \end{array}\right] \tilde{x}(t) \\ &= \tilde{\mathscr{Y}}_\infty(t)\tilde{x}(t).\end{aligned}$$

- Note that this Hamiltonian matrix is time-varying, with $P(t)$ obeying

$$\dot{P}(t) = -P(t)A - A^T P(t) - P(t)(B_u B_u^T - \gamma^{-2} B_w B_w^T) P(t) - C_y^T C_y.$$

- Therefore, for an output-feedback controller to exist, a solution to the full-information suboptimal control must exist.
- Also, we can find that $Q_m(t) = Q(t)[I - \gamma^{-2} P(t) Q(t)]^{-1}$, so a solution to the $\mathscr{H}_\infty$ state estimator must also exist, and furthermore

$$\rho[P(t)Q(t)] < \gamma^2,$$

where "spectral radius" $\rho[\cdot]$ is max. absolute eigenvalue of matrix.

  - These three tests may be used in bisection algorithm on γ to find best suboptimal output-feedback controller.

Steady-state control

- The conditions for existence of a suboptimal steady-state controller, with performance bound γ, are:
  - There is a solution $P \geq 0$ for

$$PA + A^T P - P(B_u B_u^T - \gamma^{-2} B_w B_w^T) P + C_y^T C_y = 0,$$

  such that $A - (B_u B_u^T - \gamma^{-2} B_w B_w^T) P$ is stable.
  - There is a solution $Q \geq 0$ for

$$AQ + QA^T - Q(C_m^T C_m - \gamma^{-2} C_y^T C_y) Q + B_w B_w^T = 0,$$

  such that $A - Q(C_m^T C_m - \gamma^{-2} C_y^T C_y)$ is stable.
  - The spectral radius $\rho[PQ] < \gamma^2$.
- Note that the two algebraic Riccati equations may be solved via Hamiltonian methods developed in the last chapter.

EXAMPLE 10.1: Consider a satellite tracking antenna, modeled as

$$\begin{bmatrix} \dot\theta(t) \\ \ddot\theta(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & -0.1 \end{bmatrix}\begin{bmatrix} \theta(t) \\ \dot\theta(t) \end{bmatrix} + \begin{bmatrix} 0 \\ 0.001 \end{bmatrix} u(t) + \begin{bmatrix} 0 \\ 0.001 W_b \end{bmatrix} w_1(t)$$

where $\theta(t)$ is the pointing angle error in degrees, $u(t)$ is the control torque, and $w_1(t)$ is the normalized wind torque such that $|w_1(t)| \le 1$ and $w(t) = W_b w_1(t)$.

- The pointing error is measured as

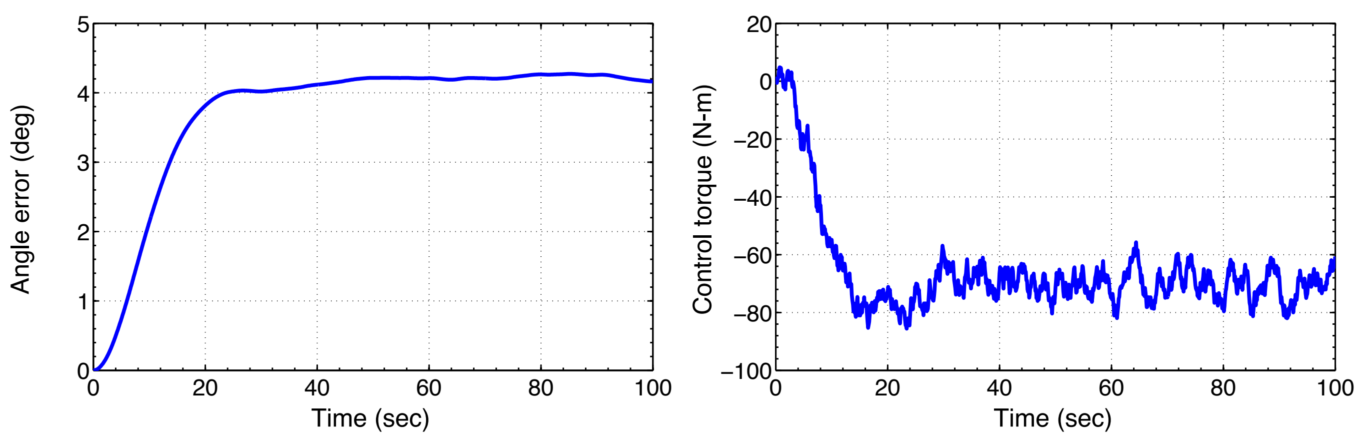
$$m(t) = \begin{bmatrix} 1 & 0 \end{bmatrix}\begin{bmatrix} \theta(t) \\ \dot\theta(t) \end{bmatrix} + v(t)$$

  where $|v(t)| \le 1$ degree.

- The outputs of interest are the pointing error and the control input

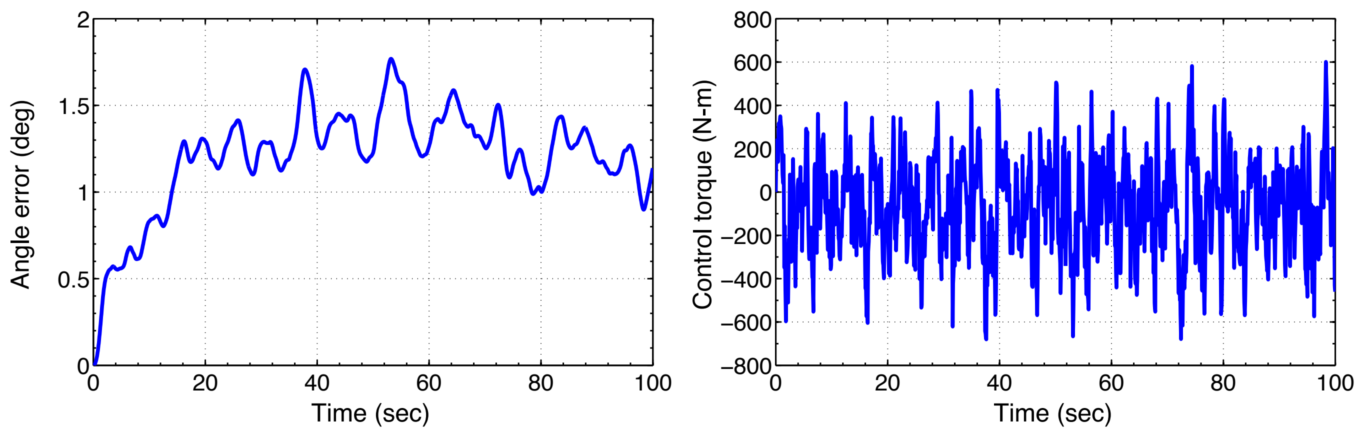
$$y(t) = \begin{bmatrix} W_\theta & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} \theta(t) \\ \dot\theta(t) \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t).$$

- The weight W_θ is used to tune the controller.
- Baseline case is for $W_b = 70$ and $W_\theta = 13$; Simulation used $w(t) = 70$ and $v(t)$ as sampled white uniform noise between -1 and 1.

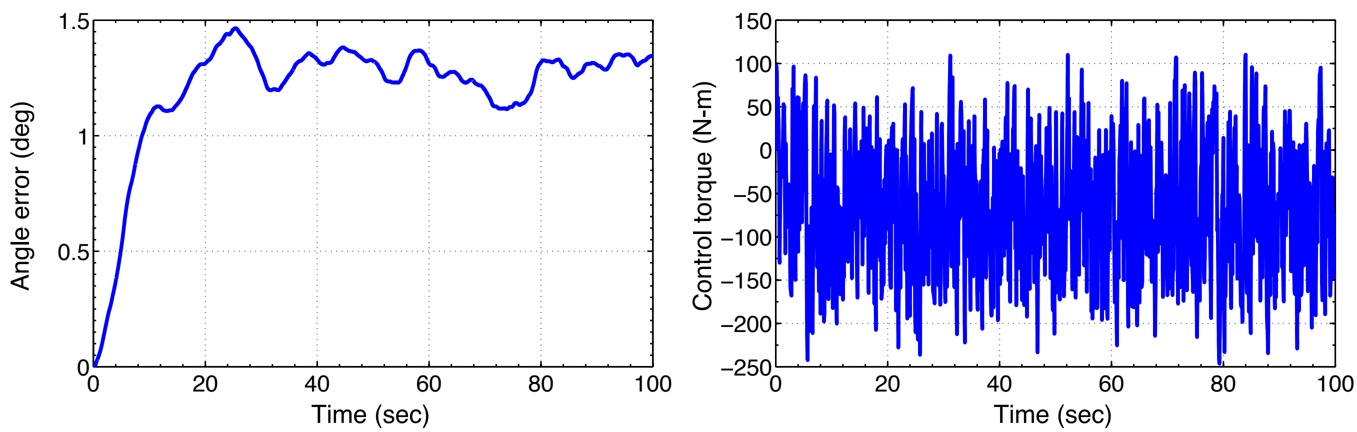


- Steady-state angle error has a dc component due to disturbance torque and a random component due to measurement error.

- Next case *designed* for $W_b = 700$ and $W_\theta = 13$, but *used* $W_b = 70$.
- Increasing bound on disturbance torque in design gives response that uses more control, decreases dc angle error due to disturbance, and increases random component due to measurement error.



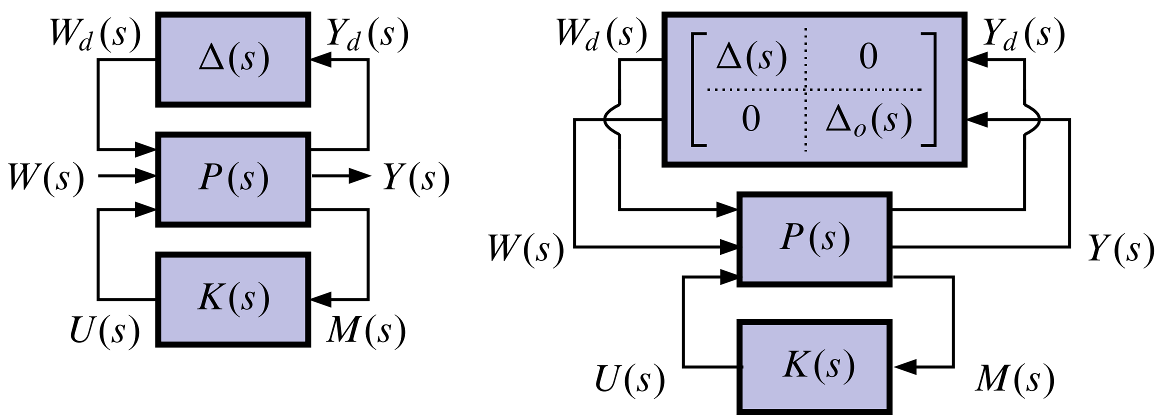
- The final case designed for and used $W_b = 70$ and $W_\theta = 130$.
- Increasing angle-error weighting also yields response that uses more control, decreases dc error due to disturbance, and increases random component of angle error due to measurement noise.



- Note that the text also discusses how to approximate integral control in an $\mathscr{H}_\infty$ setting, and how to approximate LTR for an indirect robust control design method.

9.4: μ-Synthesis

- Nothing that we have done to date is any guarantee of robustness of the control design.
- However, we can formulate the robust control problem as an $\mathscr{H}_\infty$ optimization.
- Recall that robust performance may be cast as a structured robust stability problem.



- For a system in standard form, robust performance is guaranteed if

$$J = \sup_{\omega}\{\mu_{\bar{\Delta}}[N(j\omega)]\} < 1.$$

- The μ-synthesis design procedure optimizes this cost function.
- Recall that the direct computation of $\mu_{\bar{\Delta}}$ is intractable. Therefore, we bound

$$\mu_{\bar{\Delta}}[N(j\omega)] \le \min_{\substack{\{d_1, d_2, \cdots, d_n\} \\ d_i \in (0,\infty)}} \bar{\sigma}(\mathscr{D}_R(j\omega)N(j\omega)\mathscr{D}_L^{-1}(j\omega)).$$

- The $\mathscr{D}$-scales that minimize this relationship are sets of numbers at different frequency samples.
- These values must be fitted by a transfer function for the design procedures to work.

- There are least-squares techniques to do this, discussed later.
- To achieve robust performance, start with the cost function

$$J = \sup_{\omega} \min_{\substack{\{d_1,d_2,\cdots,d_n\} \\ d_i \in (0,\infty)}} \bar{\sigma}(\mathscr{D}_R(j\omega)N(j\omega)\mathscr{D}_L^{-1}(j\omega)).$$

- This cost function is still intractable, but for a given set of $\mathscr{D}$s, we have

$$J_{\mathscr{D}} = \sup_{\omega} \bar{\sigma}[\mathscr{D}_R(j\omega)N(j\omega)\mathscr{D}_L^{-1}(j\omega)] = \left\| \mathscr{D}_R N \mathscr{D}_L^{-1} \right\|_\infty .$$

- The solution to the $\mathscr{H}_\infty$ output-feedback controller for this cost function is valid assuming that the $\mathscr{D}$-scales are described in the state model.
- Unfortunately, the $\mathscr{D}$-scales are not fixed, since they depend on the closed-loop model $N(s)$ which, in turn, depends on the controller $K(s)$. The $\mathscr{D}$-$\mathscr{K}$ iteration seeks to overcome this problem by iteratively performing ∞-norm optimization and $\mathscr{D}$-scale optimization.

1. Model the plant. Include disturbance inputs, control inputs, reference outputs, measured outputs, and perturbations. Append the performance block to the uncertainty matrix.
2. Generate an $\mathscr{H}_\infty$ output-feedback controller to minimize the ∞-norm of the transfer function from the augmented perturbation input to the augmented perturbation output.
3. Compute the structured singular values for the closed-loop system (with both uncertainty and performance blocks). Save the $\mathscr{D}$-scales used in computing the structured singular value.
4. Fit a low-order transfer function to each frequency-dependent $\mathscr{D}$-scale.

5. Append these transfer functions to the plant. The rational transfer function approximations for the $\mathscr{D}$-scales and the inverse $\mathscr{D}$-scales are appended to the nominal closed-loop system. You may augment the plant dynamics with state equations for these $\mathscr{D}$-scales and inverse $\mathscr{D}$-scales.
6. For the augmented plant, generate an $\mathscr{H}_\infty$ output-feedback controller for the augmented plant model to minimize the ∞-norm of the transfer function from the augmented perturbation input to the augmented perturbation output.
7. Return to step 3, until the structured singular value of the closed-loop system fails to improve.

- This algorithm usually converges in a few iterations. However, it is not guaranteed to converge to the global minimum of the cost function.

9.5: Fitting a transfer function to $\mathscr{D}$-scales and inverse $\mathscr{D}$-scales

- Generally, the $\mathscr{D}$-scales found comprise a set of numbers $\{d_1, \cdots, d_n\}$ numerically optimized at each frequency point.
- These numbers fill a block-diagonal matrix, which again varies with frequency ➠ a MIMO transfer function.
- We wish to approximate this transfer function with a low-order $G(s)$. Note that $G(s)$ must be minimum phase and have equal-order numerator and denominator since we require a stable causal inverse.
- The optimization routines give us the magnitude response of the transfer function, and knowing that it must be minimum-phase we use the gain-phase relationship

$$\angle\tilde{d}_k(j\omega_l) = \frac{2\omega_l}{\pi}\int_0^\infty \frac{\ln[d_k(\omega)] - \ln[d_k(\omega_l)]}{\omega^2 - \omega_l^2}\,\mathrm{d}\omega$$

where $d_k(\omega)$ is the magnitude of the $\mathscr{D}$-scale, and $\angle\tilde{d}_k(j\omega_l)$ is the reconstructed phase at frequency ω_l.

- The complex $\mathscr{D}$-scale is then $\tilde{d}_k(j\omega) = d_k(\omega)e^{j\angle\tilde{d}_k(j\omega)}$.
- We wish to fit a transfer function of order n_o to this data

$$G(s) = \frac{b_{n_o}s^{n_o} + \cdots + b_1 s + b_0}{s^{n_o} + a_{(n_o-1)}s^{(n_o-1)} + \cdots + a_1 s + a_0}.$$

- We need to approximate the "truth" $\tilde{d}_k(j\omega)$ with $G(j\omega)$ which implies

$$\tilde{d}_k(\omega_l) \approx \frac{b_{n_o}(j\omega_l)^{n_o} + \cdots + b_1(j\omega_l) + b_0}{(j\omega_l)^{n_o} + a_{(n_o-1)}(j\omega_l)^{(n_o-1)} + \cdots + a_1(j\omega_l) + a_0}.$$

- Multiplying through by the denominator and subtracting the resulting RHS from both sides, we compute the fitting error as

$$e_k(\omega_l) = \tilde{d}_k(\omega_l)\{j\omega_l\}^{n_o} + \tilde{d}_k(\omega_l)a_{(n_o-1)}\{j\omega_l\}^{(n_o-1)}$$

$$+\cdots+\tilde{d}_k(\omega_l)a_1\{j\omega_l\}+\tilde{d}_k(\omega_l)a_o$$

$$-b_{n_o}\{j\omega_l\}^{n_o}-\cdots-b_1\{j\omega_l\}-b_0.$$

- The errors from all sample frequencies may be computed

$$E_k=\left[\begin{array}{cccc} e_k(\omega_1) & e_k(\omega_2) & \cdots & e_k(\omega_{n_\omega})\end{array}\right]^T.$$

- For a given transfer-function model, $E_k=Y_k-M_k\theta_k$ where

$$Y_k=\left[\begin{array}{cccc}\tilde{d}_k(\omega_1)\{j\omega_1\}^{n_o}, & \tilde{d}_k(\omega_2)\{j\omega_2\}^{n_o}, & \cdots, & \tilde{d}_k(\omega_{n_\omega})\{j\omega_{n_\omega})^{n_o}\end{array}\right]^T,$$

$$M_k=\left[\begin{array}{cccccc} -\tilde{d}_k(\omega_1)\{j\omega_1\}^{(n_o-1)}, & \cdots & -\tilde{d}_k(\omega_1) & \{j\omega_1\}^{n_o} & \cdots & 1 \\ & & \vdots & \vdots & & \vdots \\ -\tilde{d}_k(\omega_{n_\omega})\{j\omega_{n_\omega}\}^{(n_o-1)} & \cdots & -\tilde{d}_k(\omega_{n_\omega}) & \{j\omega_{n_\omega}\}^{n_o} & \cdots & 1\end{array}\right]$$

$$\theta_k=\left[\begin{array}{cccccc} a_{(n_o-1)} & \cdots & a_o & b_{n_o} & \cdots & b_0\end{array}\right]^T.$$

- We already know the weighted least-squares solution to this optimization problem

$$\hat{\theta}=(M_k^HWM_k)^{-1}M_k^HWY_k,$$

where "H" indicates Hermitian transpose and W is an optional (diagonal) matrix weighting the fit differently at different frequencies.

EXAMPLE 10.5: Consider a single-link robot arm, with linearized plant model

$$\left[\begin{array}{c}\dot{\theta}(t)\\ \ddot{\theta}(t)\end{array}\right]=\left[\begin{array}{cc}0 & 1\\ 0.5 & 0\end{array}\right]\left[\begin{array}{c}\theta(t)\\ \dot{\theta}(t)\end{array}\right]+\left[\begin{array}{c}0\\ 0.1\end{array}\right]u(t)+\left[\begin{array}{cc}0 & 0\\ 0.1 & 0\end{array}\right]\left[\begin{array}{c}w(t)\\ v(t)\end{array}\right].$$

- The disturbance input consists of a torque caused by noise and biases in the torque circuit. The torque is bounded, $|w(t)|\le 0.5$.

- The angular position of the arm is the output of interest, and is corrupted by noise $|v(t)| \le 0.05$.

$$m(t) = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} \theta(t) \\ \dot\theta(t) \end{bmatrix} + \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} w(t) \\ v(t) \end{bmatrix}.$$

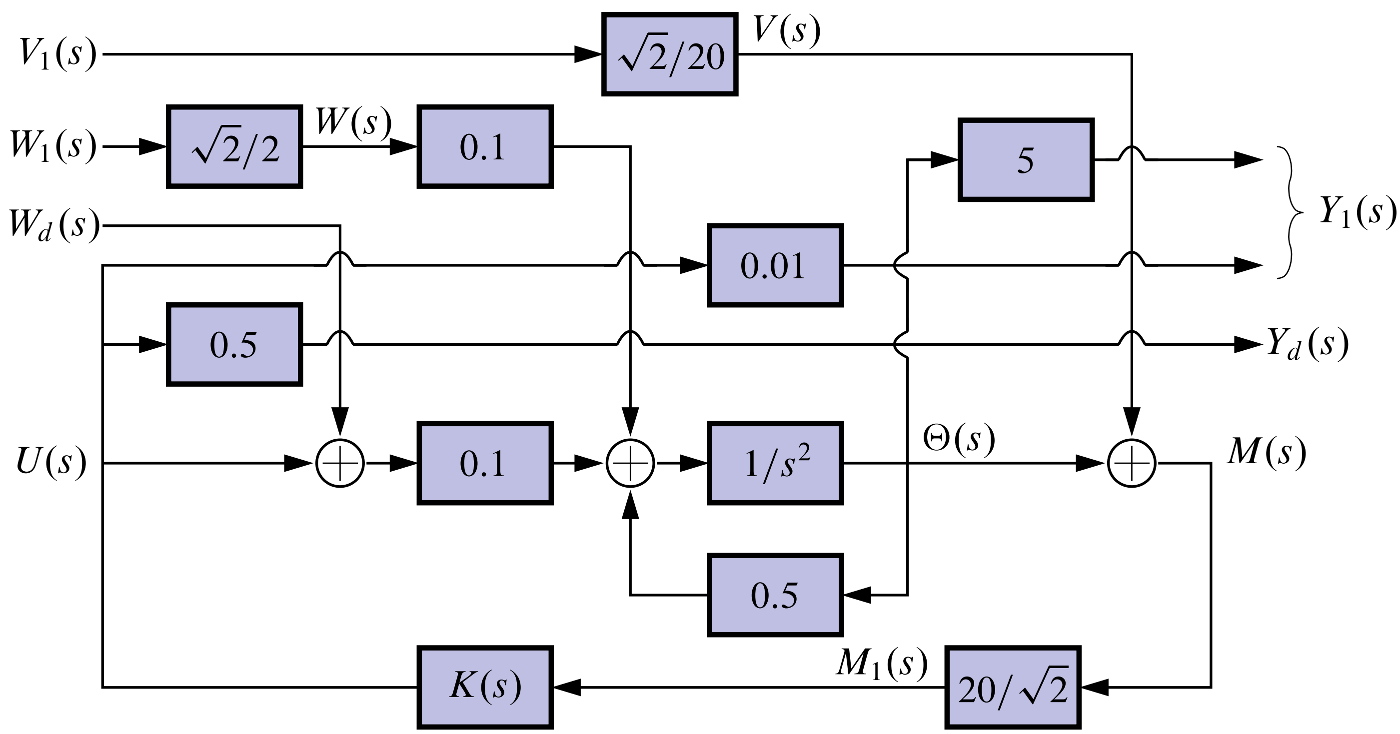
- The desired bounds on the output $\theta(t)$ and control $u(t)$ are 0.2 and 100, respectively.
- In order to obtain good stability margins for this system, an input multiplicative perturbation is added to the plant: $\|\Delta_i\| \le 0.5$.
- Note: Per text equation (10.19) we must normalize the two disturbance inputs $w(t)$ and $v(t)$ so that the overall external input has two-norm less than 1.
  - We must have $|w_1(t)| \le 1/\sqrt{\#\,\text{inputs}} = 1/\sqrt{2}$. We must also have $|v_1(t)| \le 1/\sqrt{2}$.
  - This means that $v_1(t)$ must be scaled by $\sqrt{2}/20$ to achieve the desired $|v(t)| \le 0.05$.
  - This means that $w_1(t)$ must be scaled by $\sqrt{2}/2$ to achieve the desired $|w(t)| \le 0.5$.
- The re-scaled state equation is then

$$\begin{bmatrix} \dot\theta(t) \\ \ddot\theta(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0.5 & 0 \end{bmatrix} \begin{bmatrix} \theta(t) \\ \dot\theta(t) \end{bmatrix} + \begin{bmatrix} 0 \\ 0.1 \end{bmatrix} u(t) + \left[\begin{array}{c:cc} 0 & 0 & 0 \\ 0 & \dfrac{\sqrt{2}}{20} & 0 \end{array}\right] \begin{bmatrix} u(t) \\ \hdashline w_1(t) \\ v_1(t) \end{bmatrix}.$$

- The output equation does not match the desired format, so we must scale $m(t)$ to get $m_1(t)$. Let $m_1(t) = \dfrac{20}{\sqrt{2}} m(t)$. Then,

$$\begin{bmatrix} m_1(t) \\ \hdashline y_1(t) \end{bmatrix} = \begin{bmatrix} \dfrac{20}{\sqrt{2}} & 0 \\ \hdashline 5 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \theta(t) \\ \dot{\theta}(t) \end{bmatrix} + \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0.01 & 0 & 0 \end{bmatrix} \begin{bmatrix} u(t) \\ \hdashline w_1(t) \\ v_1(t) \end{bmatrix}.$$

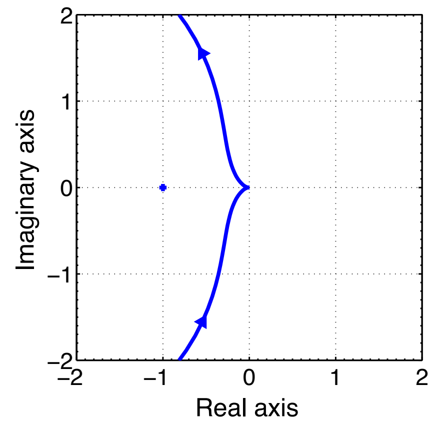
- The overall system diagram for control design is:



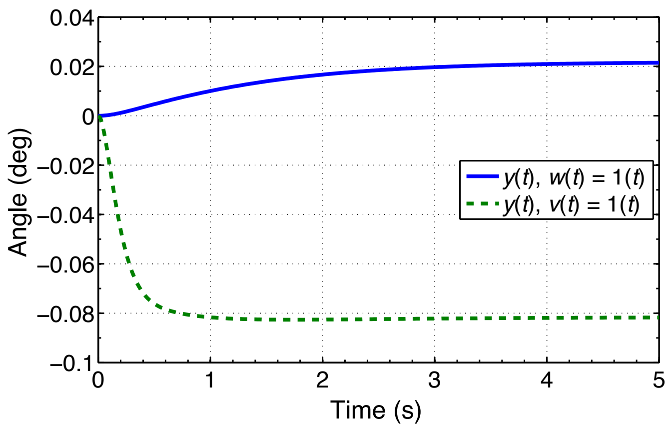
- Several $\mathscr{D}$-$\mathscr{K}$ iterations were performed, starting with the standard $\mathscr{H}_\infty$ controller design of order 2.
- The $\mathscr{D}$-scale transfer functions added an additional 8 poles to the system, for iterations 2...
- Results of the controllers are:

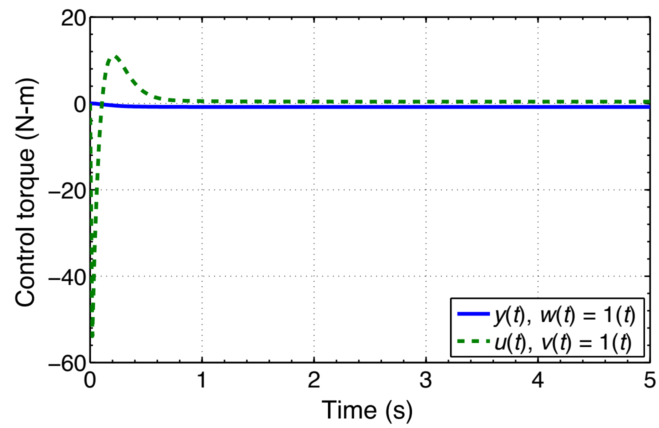
Iteration #	1	2	3	4
Controller order	2	10	10	10
Maximum μ	1.66	0.88	0.79	0.76

- We see that the final controller design guarantees robust performance ($\mu < 1$) and is far better than the initial $\mathscr{H}_\infty$ design.



- The Nyquist plot for this system is plotted. Stability is excellent.
- Step responses to disturbance inputs are plotted. Output is within spec: $|u(t)| \le 100$ and $|y(t)| \le 0.2$.





EXAMPLE 10.6: The book also has an excellent example of LQG versus robust μ-synthesis design. We don't have time to cover it here, but it is worth reading.

9.6: Controller order reduction

- The controller produced by a $\mathscr{D}$-$\mathscr{K}$ iteration is usually very high order because of the dynamics added by the $\mathscr{D}$-scales and the inverse $\mathscr{D}$-scales.
- High-order controllers increase hardware quantity and complexity.
  - For digital control, high-order controllers increase the computational burden and therefore the required processor speed.
  - The sampling rate is also higher, since high-order controllers have faster dynamics than low-order controllers.
- We would like to reduce the controller order in a systematic way, by discarding the parts that have the least effect.

Balanced truncation

- A systematic way to do this is via a technique called "balanced truncation".
- Consider the generic state-space model

$$\dot{x}(t) = Ax(t) + Bu(t)$$

$$y(t) = Cx(t) + Du(t).$$

- For this system, we define the controllability Gramian to be

$$L_c = \int_0^\infty e^{At} B B^T e^{A^T t}\, \mathrm{d}t,$$

which can be computed by solving the Lyapunov equation

$$AL_c + L_c A^T + BB^T = 0.$$

- The controllability Gramian may be interpreted as the steady-state state covariance matrix of $x(t)$, when the system is subject to white noise input with unity spectral density.
- States with larger variances can be thought of as being more strongly coupled to the input, while states with smaller variances are more weakly coupled to the input.
- The observability Gramian is defined as

$$L_o = \int_0^\infty e^{A^T t} C^T C e^{At}\,\mathrm{d}t,$$

which can be computed by solving the following Lyapunov equation:

$$A^T L_o + L_o A + C^T C = 0.$$

- States with larger variances can be though of as more strongly coupled to the output, while states with smaller variances are more weakly coupled to the output.
- If a state is weakly coupled to both the input and the output, it is reasonable that we can excise the state from the model.
- However, it is possible for a state to be weakly coupled to the input, but strongly coupled to the output (for example). Do we delete it? We need a balanced realization in order to be sure.
- A balanced realization is one with $L_c = L_o$, and diagonal. It is then easy to see which states to delete.
- To get the balanced realization, we perform a transformation

$$\tilde{x}(t) = T^{-1}x(t).$$

- This gives a new model with $A \Leftarrow T^{-1}AT$; $B \Leftarrow T^{-1}B$; $C \Leftarrow CT$; $D \Leftarrow D$.

- The effect on Gramians is:

$$L_c \Leftarrow T^{-1}L_cT^{-T}, \qquad \text{and} \qquad L_o \Leftarrow T^TL_oT.$$

- We desire to find a transformation such that

$$L_c = L_o = \Sigma = \begin{bmatrix} \sigma_1 & & & 0 \\ & \sigma_2 & & \\ & & \ddots & \\ 0 & & & \sigma_n \end{bmatrix}.$$

- The diagonal elements are labeled σ since they are the singular values of both Gramians. They are ordered from greatest to least.
- To find T:
  1. Factor the original controllability Gramian: $L_c = R^TR$, where $R > 0$. This may be done via Cholesky factorization.
  2. Generate the singular value decomposition of the matrix RL_oR^T: $RL_oR^T = U\Sigma^2U^T$.
  3. Generate the transformation matrix $T = R^TU\Sigma^{-1/2}$.
- Verify that the resulting $L_c = L_o = \Sigma$. Note that $U^T = U^{-1}$ because U is orthonormal.

$$\begin{aligned} L_c &\Leftarrow T^{-1}L_cT^{-T} \\ &= \Sigma^{1/2}U^{-1}R^{-T}L_cR^{-1}U^{-T}\Sigma^{T/2} \\ &= \Sigma^{1/2}U^{-1}R^{-T}(R^TR)R^{-1}U\Sigma^{T/2} \\ &= \Sigma. \\ L_o &\Leftarrow T^TL_oT \\ &= \Sigma^{-T/2}U^TRL_oR^TU\Sigma^{-1/2} \end{aligned}$$

$$= \Sigma^{-T/2}U^T(U\Sigma^2U^T)U\Sigma^{-1/2}$$
$$= \Sigma.$$

- For the state-space system transformed by T, we get

$$\begin{bmatrix} \dot{\tilde{x}}_1(t) \\ \dot{\tilde{x}}_2(t) \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} \tilde{x}_1(t) \\ \tilde{x}_2(t) \end{bmatrix} + \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} u(t)$$

$$y(t) = \begin{bmatrix} C_1 & C_2 \end{bmatrix} \begin{bmatrix} \tilde{x}_1(t) \\ \tilde{x}_2(t) \end{bmatrix} + Du(t).$$

- In this system, the top states are most coupled to the input and output, and the bottom states are least coupled to input and output. We remove the lower states, resulting in a modified controller:

$$\dot{\tilde{x}}_1(t) = A_{11}\tilde{x}_1(t) + B_1u(t)$$
$$y(t) = C_1\tilde{x}_1(t) + Du(t).$$

- This procedure will always produce a stable controller provided the original controller is stable and minimal. However, the feedback loop may no longer be stable. May need to remove fewer states.