\( H_\infty \) FULL-INFORMATION CONTROL AND ESTIMATION

8.1: Differential games

- \( H_2 \) control optimizes average-case performance. Assumes complete and perfect plant knowledge.

- \( H_\infty \) control optimizes worst-case performance (gain). Also assumes perfect and complete plant knowledge.
  - \( H_\infty \) does not guarantee robustness directly.
  - We will see more next chapter re. \( \mu \)-synthesis to guarantee robustness.

- LQR, LQE and LQG are all posed as \( H_2 \) optimization problems. The cost functions may be modified easily to pose them as \( H_\infty \) optimization problems.

- So, for \( H_\infty \) control, we will have a similar set of problems to solve, but different cost functions to optimize.

- The \( H_\infty \) problem seeks to minimize the maximum gain of some system over the set of disturbance inputs.
  - “Disturbance” inputs \( W \) are true disturbance, reference inputs, etc.
  - Performance “outputs” \( Y \) comprise the system state \( x(t) \), input \( u(t) \), etc.
The $H_{\infty}$ problem may be posed as a two-player game: The designer seeks a control to minimize the system gain; Nature seeks a disturbance input to maximize the gain.

Dynamics specified by differential equations ➞ “Differential Game.”

The state dynamics are modeled by

$$\dot{x}(t) = Ax(t) + B_u u(t) + B_w w(t)$$

The objective function is a real function of the state, control, and disturbance: $J\{x(t), u(t), w(t)\}$.

The solution consists of the optimal control trajectory $u^*(t)$ and the worst-case disturbance input $w^*(t)$.

$$J\{x(u^*, w), u^*, w\} \leq J\{x(u^*, w^*), u^*, w^*\} \leq J\{x(u, w^*), u, w^*\}.$$  

The control solution is then of the mini-max problem

$$u^* = \arg\min_u (\max_w J\{x(u, w), u, w\}).$$

Lagrange multipliers may be used to convert a constrained mini-max problem into an unconstrained mini-max problem of higher order

$$J_a(x, u, w, \lambda) = J(x, u, w) + \int_0^{T_f} \lambda^T(t)\{Ax(t) + B_u u(t) + B_w w(t) - \dot{x}(t)\} \, dt.$$  

A necessary condition for a saddle point is that the variation of $J_a$ must be zero.
8.2: Full-information control

- Our first control design will assume “full information:”
  - Plant state known,
  - Input “disturbance” known.

- The plant is modeled as

$$\dot{x}(t) = Ax(t) + B_u u(t) + B_w w(t)$$

$$y(t) = C_y x(t) + D_{yu} u(t),$$

where $D_{yu}^T C_y = 0$ and $D_{yu}^T D_{yu} = I$.

- The output consists of two parts: Linear combinations of the state $x(t)$ and linear combinations of the control $u(t)$.
  - The conditions $D_{yu}^T C_y = 0$ and $D_{yu}^T D_{yu} = I$ enforce separation between these two parts, and normalize the contribution of $u(t)$ to the output.
  - For example, $C_y = \begin{bmatrix} I_n & 0 \end{bmatrix}^T$ and $D_{yu} = \begin{bmatrix} 0 & I_m \end{bmatrix}^T$.

- These restrictions may be relaxed at the expense of more difficult mathematics to follow (we won’t relax them!)

- The $\mathcal{H}_\infty$ full-information control problem is to find a feedback controller, using the state and disturbance, that minimizes the closed-loop $\infty$-norm

$$J = \| G_{yw} \|_{\infty, [0, t_f]} = \sup_{\|w(t)\|_{2, [0, t_f]} \neq 0} \frac{\|y(t)\|_{2, [0, t_f]}}{\|w(t)\|_{2, [0, t_f]}}.$$ 

- The controller is a linear system (not necessarily time-invariant), denoted $\mathcal{K}(\cdot)$. 

This objective function may not be used directly as a differential game since it is a function of only the controller, and not of any particular single disturbance signal.

- The \( \sup(\cdot) \) makes the cost \( J \) independent of any particular disturbance input.
- We can drop the \( \sup(\cdot) \) but do not end up with a tractable optimization problem.

We modify the function in a few steps to get a more useful representation that does remove the \( \sup(\cdot) \).

First, consider that the optimum controller must be better than some suboptimal controller having cost \( \gamma \). So,

\[
\left\| G_{yw} \right\|_{\infty,[0,t_f]} = \sup_{\| w(t) \|_{2,[0,t_f]} \neq 0} \frac{\| y(t) \|_{2,[0,t_f]}}{\| w(t) \|_{2,[0,t_f]}} < \gamma.
\]

Controllers that satisfy this bound are suboptimal solutions to the \( \mathcal{H}_\infty \) full-information control problem (or simply suboptimal controllers).

The same result is obtained by squaring both sides of the inequality

\[
\left\| G_{yw} \right\|_{\infty,[0,t_f]}^2 = \sup_{\| w(t) \|_{2,[0,t_f]} \neq 0} \left\{ \frac{\| y(t) \|_{2,[0,t_f]}^2}{\| w(t) \|_{2,[0,t_f]}^2} \right\} < \gamma^2.
\]

To satisfy this strict inequality, the term in \( \{ \cdot \} \) must be bounded away from \( \gamma^2 \).

\[
\frac{\| y(t) \|_{2,[0,t_f]}^2}{\| w(t) \|_{2,[0,t_f]}^2} \leq \gamma^2 - \varepsilon^2.
\]

Regrouping,

\[
\| y(t) \|_{2,[0,t_f]}^2 - \gamma^2 \| w(t) \|_{2,[0,t_f]}^2 \leq -\varepsilon^2 \| w(t) \|_{2,[0,t_f]}^2.
\]
Satisfying this inequality means that the closed-loop infinity norm is bounded by $\gamma$ for all disturbance inputs and for some $\varepsilon$.

We can then use the expression on the left as an objective function for differential game theory

$$J_\gamma(x, u, w) = \|y(t)\|_{2,[0,t_f]}^2 - \gamma^2 \|w(t)\|_{2,[0,t_f]}^2.$$

**Bisection Algorithm**

- If $\gamma$ is initially set high enough (we will see how to test for this later), then a solution exists for $J_\gamma$ (to be developed in the next sections).
- This solution is a suboptimal solution to the original $H_\infty$ optimization problem.
- The goal is to find the smallest $\gamma$ and approximate the $H_\infty$ optimal solution as closely as possible.
- The following algorithm may be used to search for a good suboptimal controller.

**Bisection search algorithm steps**

1. Select $\gamma_u$, $\gamma_l$ such that $\gamma_l \leq \|G_{yw}\|_{\infty,[0,t_f]} \leq \gamma_u$.
2. Test $(\gamma_u - \gamma_l)/\gamma_l < \text{tol}$.
   - If true, stop. $\|G_{yw}\|_{\infty,[0,t_f]} \approx \frac{1}{2}(\gamma_u + \gamma_l)$.
   - If false, continue.
3. With $\gamma = \frac{1}{2}(\gamma_l + \gamma_u)$, test if $\|G_{yw}\|_{\infty,[0,t_f]} < \gamma$ using a test TBD.
   - If true, set $\gamma_u = \gamma$ (test value too high) and go to step 2.
   - If false, set $\gamma_l = \gamma$ (test value too low) and go to step 2.
8.3: The Hamiltonian equations

- We now solve the suboptimal $\mathcal{H}_\infty$ control problem.
- We form the augmented cost function

$$J_{a,\gamma}(u, w, \lambda) = \int_0^{t_f} y^T(t)y(t) - \gamma^2 w^T(t)w(t)$$

$$+ 2\lambda^T(t)\{Ax(t) + Bu(t) + Bw(t) - \dot{x}(t)\} \, dt.$$  

- Note that, by our assumptions on $C_y$ and $D_{yu}$,

$$y^T(t)y(t) = x^T(t)C_y^T C_y x(t) + x^T(t)\underbrace{C_y^T D_{yu}u(t)}_0$$

$$+ u^T(t)\underbrace{D_{yu}^T C_y x(t)}_0 + u^T(t)\underbrace{D_{yu}^T D_{yu} u(t)}_I.$$  

- Next, we form the increment of the cost function

$$\Delta J_{a,\gamma}(u, w, \lambda, \delta u, \delta w, \delta \lambda)$$

$$= J_{a,\gamma}(u + \delta u, w + \delta w, \lambda + \delta \lambda) - J_{a,\gamma}(u, w, \lambda)$$

$$= \int_0^{t_f} (x + \delta x)^T C_y^T C_y (x + \delta x) + (u + \delta u)^T (u + \delta u)$$

$$- \gamma^2 (w + \delta w)^T (w + \delta w)$$

$$+ 2(\lambda + \delta \lambda)^T\{A(x + \delta x) + Bu(u + \delta u) + Bw(w + \delta w) - (\dot{x} + \delta \dot{x})\} \, dt$$

$$- \int_0^{t_f} x^T C_y^T C_y x + u^T u - \gamma^2 w^T w + 2\lambda^T\{Ax + Bu u + Bw w - \dot{x}\} \, dt.$$  

- Expanding and grouping,
\[
\Delta J_{a,y} = \int_0^{t_f} \delta x^T C_y^T C_y \delta x + \delta u^T \delta u - \gamma^2 \delta w^T \delta w \\
+ 2\delta \lambda^T \{A \delta x + B_u \delta u + B_w \delta w - \dot{x}\} + 2u^T \delta u \\
+ 2x^T C_y^T C_y \delta x - 2\gamma^2 w^T \delta w + 2\delta \lambda^T \{A x + B_u u + B_w w - \dot{x}\} \\
+ 2\lambda^T \{A \delta x + B_u \delta u + B_w \delta w - \dot{x}\} \, dt.
\]

- The variation is (set to zero to find optimum)

\[
\delta J_{a,y} = \int_0^{t_f} 2x^T C_y^T C_y \delta x + 2u^T \delta u - 2\gamma^2 w^T \delta w \\
+ 2\delta \lambda^T \{A x + B_u u + B_w w - \dot{x}\} \\
+ 2\lambda^T \{A \delta x + B_u \delta u + B_w \delta w - \dot{x}\} \, dt = 0.
\]

- Integration by parts yields

\[
\int_0^{t_f} \lambda^T(t) \delta \dot{x}(t) \, dt = \lambda^T(t_f) \delta x(t_f) - \lambda^T(0) \delta x(0) - \int_0^{t_f} \dot{\lambda}^T(t) \delta x(t) \, dt.
\]

- We assume that the initial state is fixed so \(\delta x(0) = 0\). Substituting:

\[
\delta J_{a,y} = -2\lambda^T(t_f) \delta x(t_f) + \int_0^{t_f} (2\dot{\lambda}^T + 2x^T C_y^T C_y + 2\lambda^T A) \delta x \\
+ (2\lambda^T B_w - 2\gamma^2 w^T) \delta w + (2u^T + 2\lambda^T B_u) \delta u \\
+ 2\delta \lambda^T \{A x + B_u u + B_w w - \dot{x}\} \, dt = 0.
\]

- The conditions for optimality is then
\[ \dot{\lambda}(t_f) = 0; \]
\[ \dot{\lambda}(t) = -C_y^T C_y x(t) - A^T \lambda(t); \]
\[ u(t) = -B_u^T \lambda(t); \]
\[ w(t) = \gamma^{-2} B_w^T \lambda(t); \]
\[ \dot{x}(t) = Ax(t) + B_u u(t) + B_w w(t). \]

- Combining, we get a Hamiltonian equation

\[
\begin{bmatrix}
\dot{x}(t) \\
\dot{\lambda}(t)
\end{bmatrix} = \begin{bmatrix}
A & -B_u B_u^T + \gamma^{-2} B_w B_w^T \\
-C_y^T C_y & -A^T
\end{bmatrix} \begin{bmatrix}
x(t) \\
\lambda(t)
\end{bmatrix} = \mathcal{H}_\infty \begin{bmatrix}
x(t) \\
\lambda(t)
\end{bmatrix}
\]

with initial condition \( x(0) = 0 \) and final condition \( \lambda(t_f) = 0 \).

- The solution to this system, at final time, is

\[
\begin{bmatrix}
x(t_f) \\
\lambda(t_f)
\end{bmatrix} = e^{\mathcal{H}_\infty (t_f-t)} \begin{bmatrix}
x(t) \\
\lambda(t)
\end{bmatrix} = \begin{bmatrix}
\Phi_{11}(t_f-t) & \Phi_{12}(t_f-t) \\
\Phi_{21}(t_f-t) & \Phi_{22}(t_f-t)
\end{bmatrix} \begin{bmatrix}
x(t) \\
\lambda(t)
\end{bmatrix}.
\]

- Substitute the final condition \( \lambda(t_f) = 0 \),

\[
\begin{bmatrix}
x(t_f) \\
0
\end{bmatrix} = \begin{bmatrix}
\Phi_{11}(t_f-t) & \Phi_{12}(t_f-t) \\
\Phi_{21}(t_f-t) & \Phi_{22}(t_f-t)
\end{bmatrix} \begin{bmatrix}
x(t) \\
\lambda(t)
\end{bmatrix}
\]

- Solving the lower-triangular block

\[
\lambda(t) = -\{\Phi_{22}(t_f-t)\}^{-1} \Phi_{21}(t_f-t) x(t) = P(t)x(t).
\]

- Concluding,

\[
u(t) = -B_u^T P(t)x(t) = B_u^T \{\Phi_{22}(t_f-t)\}^{-1} \Phi_{21}(t_f-t) x(t) = -K(t)x(t).
\]
EXAMPLE 9.1: We wish to control a plant that is modeled as

\[
\dot{x}(t) = x(t) + u(t) + 2w(t)
\]

\[
y(t) = \begin{bmatrix} 10 \\ 0 \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t)
\]

where the objective function is

\[
J_y\{x(t), u(t), w(t)\} = \|y(t)\|_2^2 - \gamma^2 \|w(t)\|_2^2.
\]

- Note: State and control weightings are incorporated into the “output” equation, which is scaled to yield a unity-weight on the control.

- The Hamiltonian system is:

\[
\begin{bmatrix}
\dot{x}(t) \\
\dot{\lambda}(t)
\end{bmatrix} = \begin{bmatrix} 1 & -1 + 4\gamma^{-2} \\ -100 & -1 \end{bmatrix} \begin{bmatrix} x(t) \\
\lambda(t)
\end{bmatrix},
\]

which has state-transition matrix

\[
e^{\mathcal{L}_\infty t} = \mathcal{L}^{-1}[(sI - \mathcal{L}_\infty)^{-1}]
\]

\[
= \mathcal{L}^{-1} \left[ \frac{1}{s^2 - 101 + 400\gamma^{-2}} \begin{bmatrix} s + 1 & -1 + 4\gamma^{-2} \\ -100 & s - 1 \end{bmatrix} \right]
\]

- An $\mathcal{H}_\infty$ full-information suboptimal controller exists if $\Phi_{22}$ is invertible throughout the desired time interval.

- This element of the state-transition matrix is:

\[
\Phi_{22}(t) = \mathcal{L}^{-1} \left[ \frac{s - 1}{s^2 - 101 + 400\gamma^{-2}} \right]
\]

\[
= \begin{cases} 
\frac{a - 1}{2a} e^{at} + \frac{a + 1}{2a} e^{-at}, & \gamma > \sqrt{\frac{400}{101}} \\
1 - t, & \gamma = \sqrt{\frac{101}{400}} \\
\sqrt{\frac{\omega^2 + 1}{\omega^2}} \sin(\omega t + \theta), & \gamma < \sqrt{\frac{400}{101}}
\end{cases}
\]
where \( a = \sqrt{101 - 400\gamma^{-2}} \), \( \omega = \sqrt{-101 + 400\gamma^{-2}} \), and \( \theta = -\tan^{-1}(\omega) \).

- Note that as \( \gamma \) gets smaller, \( a \) gets smaller, \( \omega \) gets bigger, and \( \theta \) gets more negative.

- When \( \gamma < \sqrt{400/101} \) the eigenvalues of the Hamiltonian matrix are imaginary, and the gain does not exist for long time intervals since \( \Phi_{22}(t) \) periodically crosses zero.

- The lack of solution for feedback gain means there is no solution of the differential game for the given bound.

- For the suboptimal bound \( \gamma = 2.03 \) (\( a = 2 \)) and the final time \( t_f = 3 \), the \( \mathcal{H}_\infty \) suboptimal feedback gain is

\[
K(t) = \frac{100e^{2(3-t)} - 100e^{-2(3-t)}}{e^{2(3-t)} + 3e^{-2(3-t)}}.
\]
8.4: Riccati equations to find state feedback

The Riccati equation

- Solving the Hamiltonian for the time-varying state feedback may be done via a Riccati equation.
- Given that \( \lambda(t) = P(t)x(t) \), differentiate to get
  \[
  \dot{\lambda}(t) = \dot{P}(t)x(t) + P(t)\dot{x}(t).
  \]
- Substitute for \( \dot{\lambda}(t) \) and \( \dot{x}(t) \) from the Hamiltonian system
  \[-C^T_y C_y x(t) - A^T \lambda(t) = \dot{P}(t)x(t) + P(t)\{Ax(t) - (B_u B_u^T - \gamma^{-2} B_w B_w^T)\lambda(t)\}.\]
- Then, substituting for \( \lambda(t) = P(t)x(t) \),
  \[
  \{\dot{P}(t) + P(t)A + A^T P(t) + C^T_y C_y - P(t)(B_u B_u^T - \gamma^{-2} B_w B_w^T) P(t)\}x(t) = 0.
  \]
- Since this is valid for any \( x(t) \)
  \[
  \dot{P}(t) = -P(t)A - A^T P(t) - C^T_y C_y + P(t)(B_u B_u^T - \gamma^{-2} B_w B_w^T) P(t),
  \]
  which is the Riccati equation for the \( \mathcal{H}_\infty \) suboptimal control problem, initialized with \( P(t_f) = 0 \).

- A suboptimal \( \mathcal{H}_\infty \) controller exists for performance bound \( \gamma \) iff there exists a solution \( P \succeq 0 \) to
  \[
  PA + A^T P - P(B_u B_u^T - \gamma^{-2} B_w B_w^T) P + C^T_y C_y = 0,
  \]
  where the matrix \( A - (B_u B_u^T - \gamma^{-2} B_w B_w^T) P \) is stable.

- This may be used as the test in the bisection search algorithm for existence of an \( \mathcal{H}_\infty \) controller for performance bound \( \gamma \).

**EXAMPLE 9.1 CONTINUED:** Here, we use the Riccati equation to solve the same problem as in the prior example.
The Riccati equation is \((A = 1, B_u = 1, B_w = 2, C_y = \begin{bmatrix} 10 & 0 \end{bmatrix}^T)\):

\[
P(t) = -2P(t) + (1 - 4y^{-2})P^2(t) - 100,
\]

with final condition \(P(3) = 0\).

The solution is

\[
P(t) = K(t) = \frac{100e^{2(3-t)} - 100e^{-2(3-t)}}{e^{2(3-t)} + 3e^{-2(3-t)}},
\]

which is the same as we found before.

(The solution can be verified by substituting \(P(t)\) back into the Riccati equation). Note, \(P(t) = K(t)\) because \(B_u = 1\).

**Steady-state result**

- The steady-state result may be found by setting \(\dot{P}(t) = 0\) and solving the A.R.E.

- An eigenvector method may again be used to solve for \(P_{ss}\)

\[
P = \Psi_{21}(\Psi_{11})^{-1},
\]

where \(\Psi\) is the eigensystem of the Hamiltonian system, and \(\Psi_{21}\) and \(\Psi_{11}\) are formed from the eigenvectors associated with the stable eigenvalues.

- A suboptimal \(\mathcal{H}_\infty\) controller exists for performance bound \(\gamma\) iff \(\mathcal{L}_\infty\) has no eigenvalues on the imaginary axis, \(\Psi_{11}\) is invertible, and

\[
P = \Psi_{21}(\Psi_{11})^{-1} \geq 0.
\]

- This may also be used in the test in the bisection search algorithm for existence of an \(\mathcal{H}_\infty\) controller for performance bound \(\gamma\).
The suboptimal controller is then given as

\[ u(t) = -B_u^T \Psi_{21}(\Psi_{11})^{-1} x(t) = -K x(t). \]

**EXAMPLE 9.2:** An antenna is required to remain pointed at a satellite in the presence of disturbance torques caused by gravity, wind, squirrels, etc. The state model is

\[
\begin{align*}
\dot{x}(t) &= \begin{bmatrix} 0 & 1 \\ 0 & -1 \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} w(t) \\
 e(t) &= \begin{bmatrix} 1 & 0 \end{bmatrix} x(t).
\end{align*}
\]

- As usual, \( u(t) \) is the control torque, \( w(t) \) is the disturbance torque, and \( e(t) \) is the tracking error.
- A tracking error of less than 0.1 rad is desired, even in the presence of disturbances up to 2 N-m.
- The control magnitude is also required to be less than 10 N-m.
- These specifications are appended to the plant as weighting functions to yield a model in standard form:

\[
\begin{align*}
\dot{x}(t) &= \begin{bmatrix} 0 & 1 \\ 0 & -1 \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ 10 \end{bmatrix} u_1(t) + \begin{bmatrix} 0 \\ 2 \end{bmatrix} w_1(t) \\
y(t) &= \begin{bmatrix} 10 & 0 \\ 0 & 0 \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u_1(t),
\end{align*}
\]

where \( w_1(t) \) and \( u_1(t) \) are the normalized disturbance and control inputs.
- In this form, \( w_1(t) \) is less than 1 and both elements of \( y(t) \) are required to be less than 1. The original control input can be recovered from the normalized control as \( u(t) = 10u_1(t) \).
A full-information controller is generated for $\gamma = 1$:

$$u_1(t) = -\begin{bmatrix} 10.2 & 1.36 \end{bmatrix} x(t) \quad \text{or} \quad u(t) = -\begin{bmatrix} 102 & 13.6 \end{bmatrix} x(t).$$

![Tracking error graph](image1)

![Control effort graph](image2)

- The left plot shows frequency response of tracking error; the right plot shows frequency response of control effort.

- Maximum gain from sinusoidal disturbance to tracking error is $10^{-40/20} = 0.01$, which is less than spec of 0.05.

- Maximum gain from disturbance input to control is 1.26, which is less than spec of 5.

- A second full-information controller is generated for $\gamma = 0.21$.

$$u_1(t) = -\begin{bmatrix} 32.8 & 7.39 \end{bmatrix} x(t) \quad \text{or} \quad u(t) = -\begin{bmatrix} 328 & 73.9 \end{bmatrix} x(t).$$

- Performance is better, but gains are higher. Disturbance rejection is higher, but control bandwidth is also higher.
8.5: $\mathcal{H}_\infty$ Estimation

- When investigating $\mathcal{H}_2$ control, we split the problem into a regulation part (LQR) and an estimation part (LQE).

- When joined, the result turned out to be optimal as well (LQG).

- One reason that this works is because $\mathcal{H}_2$ optimization of $-K(t)x(t)$ is equal to $-K(t)$ times the $\mathcal{H}_2$ optimization of $x(t)$.
  
  - This same property does not hold for $\mathcal{H}_\infty$ estimation; we must be more careful when generating an $\mathcal{H}_\infty$ output feedback controller.
  
  - $\mathcal{H}_\infty$ estimation is still required, so we study it briefly here.

- The plant model that is assumed looks like

  \[
  \dot{x}(t) = Ax(t) + Bu(t) + Bw(t) \\
  m(t) = Cmx(t) + Dmw(t)
  \]

  with constraints $DmwBw^T = 0$ and $DmwDmw^T = I$.

- Goal is to estimate a linear combination of the state, $y(t) = Cyx(t)$.
  
  - One example would be to estimate the state itself: $Cy = I$.

- $\mathcal{H}_\infty$ estimation is the dual problem to $\mathcal{H}_\infty$ regulation. As you might expect, there is a Hamiltonian system involved

  \[
  \begin{bmatrix}
  \dot{x}_h(t) \\
  \dot{y}_h(t)
  \end{bmatrix} =
  \begin{bmatrix}
  -A^T & C_m^T C_m - \gamma^{-2}C_y^T C_y \\
  BwBw^T & A
  \end{bmatrix}
  \begin{bmatrix}
  x_h(t) \\
  y_h(t)
  \end{bmatrix} = \mathcal{Y}_\infty x_h(t).
  \]

- The $\mathcal{H}_\infty$ suboptimal estimator gain may be found in terms of the state-transition matrix of the Hamiltonian system:
where the $\Phi_{ij}$ are terms of the state-transition matrix:

$$
L(t) = Q(t)C_m^T = \Phi_{21}(t)\{\Phi_{11}(t)\}^{-1}C_m^T
$$

\[e^{\mathcal{Y}_\infty t} = \begin{bmatrix}
\Phi_{11}(t) & \Phi_{12}(t) \\
\Phi_{21}(t) & \Phi_{22}(t)
\end{bmatrix}.\]

- A steady-state solution may be found from the eigensystem of $\mathcal{Y}_\infty$ as

$$
Q = \Psi_{22}(\Psi_{12})^{-1},
$$

where

$$
\begin{bmatrix}
\Psi_{12} \\
........
\Psi_{22}
\end{bmatrix}
$$

is a matrix whose columns are the eigenvectors of the estimator Hamiltonian associated with the unstable eigenvalues.

- A suboptimal $\mathcal{H}_\infty$ estimator exists if there is a solution to $\mathcal{Y}_\infty$ that has no eigenvalues on the imaginary axis, $\Psi_{12}$ is invertible, and $Q = \Psi_{22}(\Psi_{12})^{-1} \geq 0$.

  - This may be used in the bisection search algorithm for existence of an $\mathcal{H}_\infty$ estimator for performance bound $\gamma$.

- These result may also be obtained from a Riccati equation

$$
\dot{Q}(t) = Q(t)A^T + AQ(t) + B_wB_w^T - Q(t)\{C_m^TC_m - \gamma^{-2}C_y^TC_y\}Q(t),
$$

with initial condition $Q(0) = 0$.

- A suboptimal $\mathcal{H}_\infty$ estimator exists if there is a solution to

$$
AQ + Q^TA - Q(C_m^TC_m - \gamma^{-2}C_y^TC_y)Q + B_wB_w^T = 0,
$$

such that $Q \geq 0$ and that the matrix $A - Q(C_m^TC_m - \gamma^{-2}C_y^TC_y)$ is stable.

- This may also be used in the bisection search algorithm for existence of an $H_\infty$ estimator for performance bound $\gamma$.

**Example 9.3:** An $H_\infty$ estimator can be applied to the radar range tracking of an aircraft. A model equations for the range $r(t)$ and radial velocity of an aircraft is (range is measured):

$$
\begin{bmatrix}
\dot{r}(t) \\
\ddot{r}(t)
\end{bmatrix} =
\begin{bmatrix}
0 & 1 \\
0 & 0
\end{bmatrix}
\begin{bmatrix}
r(t) \\
\dot{r}(t)
\end{bmatrix} +
\begin{bmatrix}
0 \\
1
\end{bmatrix} w(t)
$$

$$m(t) =
\begin{bmatrix}
1 & 0
\end{bmatrix}
\begin{bmatrix}
r(t) \\
\dot{r}(t)
\end{bmatrix} + v(t).
$$

- The output to be estimated is the entire state:

$$y(t) =
\begin{bmatrix}
1 & 0 \\
0 & 1
\end{bmatrix}
\begin{bmatrix}
r(t) \\
\dot{r}(t)
\end{bmatrix}
$$

- The plant input and measurement error are assumed to be bounded:

$$
\begin{bmatrix}
|w(t)| \\
|v(t)|
\end{bmatrix} \leq
\begin{bmatrix}
4 \text{ m/sec}^2 \\
20 \text{ m}
\end{bmatrix}
$$

- Model equations using normalized inputs are:

$$
\begin{bmatrix}
\dot{r}(t) \\
\ddot{r}(t)
\end{bmatrix} =
\begin{bmatrix}
0 & 1 \\
0 & 0
\end{bmatrix}
\begin{bmatrix}
r(t) \\
\dot{r}(t)
\end{bmatrix} +
\begin{bmatrix}
0 & 0 \\
4 & 0
\end{bmatrix}
\begin{bmatrix}
w_1(t) \\
v_1(t)
\end{bmatrix}
$$

$$m(t) =
\begin{bmatrix}
1 & 0
\end{bmatrix}
\begin{bmatrix}
r(t) \\
\dot{r}(t)
\end{bmatrix} +
\begin{bmatrix}
0 & 20
\end{bmatrix}
\begin{bmatrix}
w_1(t) \\
v_1(t)
\end{bmatrix}
$$

- Normalizing this equation so that the input-output coupling matrix satisfies $D_{mw}D_{mw}^T = I$ gives the measurement equation
\[ m_1(t) = \begin{bmatrix} 1/20 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} r(t) \\ \dot{r}(t) \end{bmatrix} + \begin{bmatrix} 0 & 1 \\ \end{bmatrix} \begin{bmatrix} w_1(t) \\ v_1(t) \end{bmatrix} \]

where \( m(t) = 20m_1(t) \).

\[ \begin{align*}
\dot{x}(t) &= A\hat{x}(t) + L_1(t) \left[ m(t) - \begin{bmatrix} 1/20 & 0 \end{bmatrix} \hat{x}(t) \right] \\
\hat{y}(t) &= C_y\hat{x}(t).
\end{align*} \]

We can write this in terms of the original measurements,

\[ \begin{align*}
\dot{x}(t) &= A\hat{x}(t) + \frac{1}{20}L_1(t) \left[ 20m_1(t) - 20 \begin{bmatrix} 1/20 & 0 \end{bmatrix} \hat{x}(t) \right] \\
&= A\hat{x}(t) + L(t) \left[ m(t) - \begin{bmatrix} 1 & 0 \end{bmatrix} \hat{x}(t) \right],
\end{align*} \]

where \( L(t) = L_1(t)/20 \).

Baseline results are plotted below.
If meas. error decreases from 20 to 2, we get these results.

Estimation gains increase; estimates converge more quickly.

Finally, if reference output is only range rate (we still estimate range, but don’t care about its accuracy), we get these results.

Gains change (slightly)! The $H_\infty$ estimator depends on the output of interest!
Summary

- $\mathcal{H}_\infty$ controller and estimator design is an iterative procedure.
- Perform the bisection algorithm to find the lowest value of the performance bound $\gamma$ that you can, within some tolerance.
  - You will generally back off of this performance bound somewhat, as designs that are nearly optimal can have numeric instabilities.
- For controller design, the conditions for existence of some suboptimal controller with bound $\gamma$ are:
  - There exists a solution $P \geq 0$ to
    \[ PA + A^T P - P(B_u B_u^T - \gamma^{-2} B_w B_w^T) P + C_y^T C_y = 0, \]
    where the matrix $A - (B_u B_u^T - \gamma^{-2} B_w B_w^T) P$ is stable; or,
    - $\mathcal{H}_\infty$ has no eigenvalues on the imaginary axis, $\Psi_{11}$ is invertible, and $P = \Psi_{21}(\Psi_{11})^{-1} \geq 0$.
- For estimator design, the conditions for existence of a suboptimal controller with bound $\gamma$ are:
  - There exists a solution $Q \geq 0$ to
    \[ AQ + Q^T A - Q(C_m^T C_m - \gamma^{-2} C_y^T C_y) Q + B_w B_w^T = 0, \]
    where the matrix $A - Q(C_m^T C_m - \gamma^{-2} C_y^T C_y)$ is stable; or,
    - $\mathcal{V}_\infty$ has no eigenvalues on the imaginary axis, $\Psi_{12}$ is invertible, and $Q = \Psi_{22}(\Psi_{12})^{-1} \geq 0$.
- For controller implementation, with some $\gamma$, solve the differential Riccati equation.
\[
\dot{P}(t) = -P(t)A - A^T P(t) - C_y^T C_y + P(t)(B_u B_u^T - \gamma^{-2} B_w B_w^T)P(t),
\]
backward in time, initialized with \( P(t_f) = 0 \). Then,
\[
u(t) = -K(t)x(t) = -B_u^T P(t)x(t).
\]

- A steady-state suboptimal controller, with some \( \gamma \), uses the
eigensystem \( \Psi \) of the \( \mathcal{L}_\infty \) matrix to find the steady-state gain matrix:
\[
u(t) = -Kx(t) = -B_u^T \Psi_{21}(\Psi_{11})^{-1} x(t),
\]
where \( \Psi_{21} \) and \( \Psi_{11} \) are formed from the eigenvectors associated with
the stable eigenvalues of \( \mathcal{L}_\infty \).

- For estimator implementation, with some \( \gamma \), solve the differential
Riccati equation
\[
\dot{Q}(t) = Q(t)A^T + AQ(t) + B_w B_w^T - Q(t)\{C_m^T C_m - \gamma^{-2} C_y^T C_y\} Q(t),
\]
forward in time, initialized with \( Q(0) = 0 \). Then,
\[
L(t) = Q(t)C_m^T.
\]

- A steady-state suboptimal estimator, with some \( \gamma \), uses the
eigensystem \( \Psi \) of the \( \mathcal{Y}_\infty \) matrix to find the steady-state gain:
\[
L = QC_m^T = \Psi_{22}(\Psi_{12})^{-1} C_m^T,
\]
where \( \Psi_{22} \) and \( \Psi_{12} \) are formed from the eigenvectors associated with
the unstable eigenvalues of \( \mathcal{Y}_\infty \).