$\mathscr{H}_\infty$ FULL-INFORMATION CONTROL AND ESTIMATION

8.1: Differential games

- $\mathscr{H}_2$ control optimizes average-case performance. Assumes complete and perfect plant knowledge.
- $\mathscr{H}_\infty$ control optimizes worst-case performance (gain). Also assumes perfect and complete plant knowledge.
  - $\mathscr{H}_\infty$ does *not* guarantee robustness directly.
  - We will see more next chapter re. μ-synthesis to guarantee robustness.
- LQR, LQE and LQG are all posed as $\mathscr{H}_2$ optimization problems. The cost functions may be modified easily to pose them as $\mathscr{H}_\infty$ optimization problems.
- So, for $\mathscr{H}_\infty$ control, we will have a similar set of problems to solve, but different cost functions to optimize.
- The $\mathscr{H}_\infty$ problem seeks to minimize the maximum gain of some system over the set of disturbance inputs.
  - "Disturbance" inputs W are true disturbance, reference inputs, etc.
  - Performance "outputs" Y comprise the system state $x(t)$, input $u(t)$, etc.

- The $\mathscr{H}_\infty$ problem may be posed as a two-player game: The designer seeks a control to minimize the system gain; Nature seeks a disturbance input to maximize the gain.
- Dynamics specified by differential equations ➠ "Differential Game."
- The state dynamics are modeled by

$$\dot{x}(t) = Ax(t) + B_u u(t) + B_w w(t)$$

- The objective function is a real function of the state, control, and disturbance: $J\{x(t), u(t), w(t)\}$.
- The solution consists of the optimal control trajectory $u^*(t)$ and the worst-case disturbance input $w^*(t)$.

$$J\{x(u^*, w), u^*, w\} \le J\{x(u^*, w^*), u^*, w^*\} \le J\{x(u, w^*), u, w^*\}.$$

- The control solution is then of the mini-max problem

$$u^* = \arg\min_u(\max_w J\{x(u, w), u, w\}).$$

- Lagrange multipliers may be used to convert a constrained mini-max problem into an unconstrained mini-max problem of higher order

$$J_a(x, u, w, \lambda) = J(x, u, w) + \int_0^{t_f} \lambda^T(t)\{Ax(t) + B_u u(t) + B_w w(t) - \dot{x}(t)\}\,\mathrm{d}t.$$

- A necessary condition for a saddle point is that the variation of J_a must be zero.

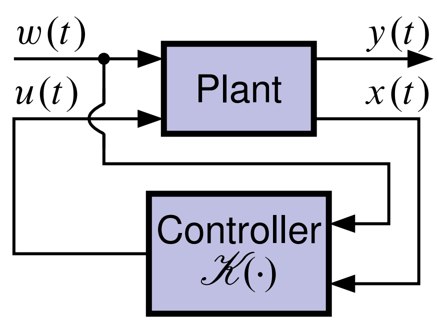
8.2: Full-information control

- Our first control design will assume "full information:"
  - Plant state known,
  - Input "disturbance" known.
- The plant is modeled as

$$\dot{x}(t) = Ax(t) + B_u u(t) + B_w w(t)$$

$$y(t) = C_y x(t) + D_{yu} u(t),$$

  where $D_{yu}^T C_y = 0$ and $D_{yu}^T D_{yu} = I$.
- The output consists of two parts: Linear combinations of the state $x(t)$ and linear combinations of the control $u(t)$.
  - The conditions $D_{yu}^T C_y = 0$ and $D_{yu}^T D_{yu} = I$ enforce separation between these two parts, and normalize the contribution of $u(t)$ to the output.
  - For example, $C_y = \begin{bmatrix} I_n & 0 \end{bmatrix}^T$ and $D_{yu} = \begin{bmatrix} 0 & I_m \end{bmatrix}^T$.
- These restrictions may be relaxed at the expense of more difficult mathematics to follow (we won't relax them!)
- The $\mathscr{H}_\infty$ full-information control problem is to find a feedback controller, using the state and disturbance, that minimizes the closed-loop ∞-norm



$$J = \left\| G_{yw} \right\|_{\infty,[0,t_f]} = \sup_{\|w(t)\|_{2,[0,t_f]} \neq 0} \frac{\|y(t)\|_{2,[0,t_f]}}{\|w(t)\|_{2,[0,t_f]}}.$$

- The controller is a linear system (not necessarily time-invariant), denoted $\mathscr{K}(\cdot)$.

- This objective function may not be used directly as a differential game since it is a function of only the controller, and not of any particular single disturbance signal.
  - The $\sup(\cdot)$ makes the cost J independent of any particular disturbance input.
  - We can drop the $\sup(\cdot)$ but do not end up with a tractable optimization problem.
- We modify the function in a few steps to get a more useful representation that does remove the $\sup(\cdot)$.
- First, consider that the optimum controller must be better than some suboptimal controller having cost γ. So,

$$\left\|G_{yw}\right\|_{\infty,[0,t_f]} = \sup_{\|w(t)\|_{2,[0,t_f]}\neq 0} \frac{\|y(t)\|_{2,[0,t_f]}}{\|w(t)\|_{2,[0,t_f]}} < \gamma.$$

- Controllers that satisfy this bound are suboptimal solutions to the $\mathscr{H}_\infty$ full-information control problem (or simply suboptimal controllers).
- The same result is obtained by squaring both sides of the inequality

$$\left\|G_{yw}\right\|^2_{\infty,[0,t_f]} = \sup_{\|w(t)\|_{2,[0,t_f]}\neq 0} \left\{\frac{\|y(t)\|^2_{2,[0,t_f]}}{\|w(t)\|^2_{2,[0,t_f]}}\right\} < \gamma^2.$$

- To satisfy this strict inequality, the term in $\{\cdot\}$ must be bounded away from γ^2.

$$\frac{\|y(t)\|^2_{2,[0,t_f]}}{\|w(t)\|^2_{2,[0,t_f]}} \le \gamma^2 - \varepsilon^2.$$

- Regrouping,

$$\|y(t)\|^2_{2,[0,t_f]} - \gamma^2 \|w(t)\|^2_{2,[0,t_f]} \le -\varepsilon^2 \|w(t)\|^2_{2,[0,t_f]}.$$

- Satisfying this inequality means that the closed-loop infinity norm is bounded by γ for all disturbance inputs and for some ε.
- We can then use the expression on the left as an objective function for differential game theory

$$J_\gamma(x,u,w) = \|y(t)\|^2_{2,[0,t_f]} - \gamma^2 \|w(t)\|^2_{2,[0,t_f]}.$$

Bisection Algorithm

- If γ is initially set high enough (we will see how to test for this later), then a solution exists for J_γ (to be developed in the next sections).
- This solution is a *sub*optimal solution to the original $\mathscr{H}_\infty$ optimization problem.
- The goal is to find the smallest γ and approximate the $\mathscr{H}_\infty$ optimal solution as closely as possible.
- The following algorithm may be used to search for a good suboptimal controller.

Bisection search algorithm steps

1. Select γ_u, γ_l such that $\gamma_l \le \left\|G_{yw}\right\|_{\infty,[0,t_f]} \le \gamma_u$.
2. Test $(\gamma_u - \gamma_l)/\gamma_l <$ tol.
   - If true, stop. $\left\|G_{yw}\right\|_{\infty,[0,t_f]} \approx \dfrac{1}{2}(\gamma_u + \gamma_l)$.
   - If false, continue.
3. With $\gamma = \dfrac{1}{2}(\gamma_l + \gamma_u)$, test if $\left\|G_{yw}\right\|_{\infty,[0,t_f]} < \gamma$ using a test TBD.
   - If true, set $\gamma_u = \gamma$ (test value too high) and go to step 2.
   - If false, set $\gamma_l = \gamma$ (test value too low) and go to step 2.

8.3: The Hamiltonian equations

- We now solve the suboptimal $\mathscr{H}_\infty$ control problem.
- We form the augmented cost function

$$J_{a,\gamma}(u,w,\lambda) = \int_0^{t_f} y^T(t)y(t) - \gamma^2 w^T(t)w(t) + 2\lambda^T(t)\{Ax(t) + B_u u(t) + B_w w(t) - \dot{x}(t)\}\,\mathrm{d}t.$$

- Note that, by our assumptions on C_y and D_{yu},

$$y^T(t)y(t) = x^T(t)C_y^T C_y x(t) + \underbrace{x^T(t)C_y^T D_{yu}u(t)}_{0} + \underbrace{u^T(t)D_{yu}^T C_y x(t)}_{0} + u^T(t)\underbrace{D_{yu}^T D_{yu}}_{I}u(t).$$

- Next, we form the increment of the cost function

$$\begin{aligned}
&\Delta J_{a,\gamma}(u,w,\lambda,\delta u,\delta w,\delta\lambda)\\
&= J_{a,\gamma}(u+\delta u, w+\delta w, \lambda+\delta\lambda) - J_{a,\gamma}(u,w,\lambda)\\
&= \int_0^{t_f} (x+\delta x)^T C_y^T C_y (x+\delta x) + (u+\delta u)^T(u+\delta u)\\
&\quad -\gamma^2(w+\delta w)^T(w+\delta w)\\
&\quad +2(\lambda+\delta\lambda)^T\{A(x+\delta x) + B_u(u+\delta u) + B_w(w+\delta w) - (\dot{x}+\delta\dot{x})\}\,\mathrm{d}t\\
&\quad -\int_0^{t_f} x^T C_y^T C_y x + u^T u - \gamma^2 w^T w + 2\lambda^T\{Ax + B_u u + B_w w - \dot{x}\}\,\mathrm{d}t.
\end{aligned}$$

- Expanding and grouping,

$$\begin{aligned}\Delta J_{a,\gamma} = \int_0^{t_f} & \delta x^T C_y^T C_y \delta x + \delta u^T \delta u - \gamma^2 \delta w^T \delta w \\ & + 2\delta\lambda^T \{A\delta x + B_u \delta u + B_w \delta w - \delta\dot{x}\} + 2u^T \delta u \\ & + 2x^T C_y^T C_y \delta x - 2\gamma^2 w^T \delta w + 2\delta\lambda^T \{Ax + B_u u + B_w w - \dot{x}\} \\ & + 2\lambda^T \{A\delta x + B_u \delta u + B_w \delta w - \delta\dot{x}\}\, \mathrm{d}t.\end{aligned}$$

- The variation is (set to zero to find optimum)

$$\begin{aligned}\delta J_{a,\gamma} = \int_0^{t_f} & 2x^T C_y^T C_y \delta x + 2u^T \delta u - 2\gamma^2 w^T \delta w \\ & + 2\delta\lambda^T \{Ax + B_u u + B_w w - \dot{x}\} \\ & + 2\lambda^T \{A\delta x + B_u \delta u + B_w \delta w - \delta\dot{x}\}\, \mathrm{d}t = 0.\end{aligned}$$

- Integration by parts yields

$$\int_0^{t_f} \lambda^T(t)\delta\dot{x}(t)\, \mathrm{d}t = \lambda^T(t_f)\delta x(t_f) - \lambda^T(0)\delta x(0) - \int_0^{t_f} \dot{\lambda}^T(t)\delta x(t)\, \mathrm{d}t.$$

- We assume that the initial state is fixed so $\delta x(0) = 0$. Substituting:

$$\begin{aligned}\delta J_{a,\gamma} = -2\lambda^T(t_f)\delta x(t_f) + \int_0^{t_f} & (2\dot{\lambda}^T + 2x^T C_y^T C_y + 2\lambda^T A)\delta x \\ & + (2\lambda^T B_w - 2\gamma^2 w^T)\delta w + (2u^T + 2\lambda^T B_u)\delta u \\ & + 2\delta\lambda^T (Ax + B_u u + B_w w - \dot{x})\, \mathrm{d}t = 0.\end{aligned}$$

- The conditions for optimality is then

$$\begin{aligned}\lambda(t_f) &= 0;\\ \dot{\lambda}(t) &= -C_y^T C_y x(t) - A^T\lambda(t);\\ u(t) &= -B_u^T\lambda(t);\\ w(t) &= \gamma^{-2}B_w^T\lambda(t);\\ \dot{x}(t) &= Ax(t) + B_u u(t) + B_w w(t).\end{aligned}$$

- Combining, we get a Hamiltonian equation

$$\begin{bmatrix}\dot{x}(t)\\ \dot{\lambda}(t)\end{bmatrix} = \underbrace{\begin{bmatrix}A & -B_uB_u^T + \gamma^{-2}B_wB_w^T\\ -C_y^TC_y & -A^T\end{bmatrix}}_{\mathscr{H}_\infty \text{ Hamiltonian Matrix}}\begin{bmatrix}x(t)\\ \lambda(t)\end{bmatrix} = \mathscr{Z}_\infty\begin{bmatrix}x(t)\\ \lambda(t)\end{bmatrix}$$

with initial condition $x(0) = 0$ and final condition $\lambda(t_f) = 0$.

- The solution to this system, at final time, is

$$\begin{bmatrix}x(t_f)\\ \lambda(t_f)\end{bmatrix} = e^{\mathscr{Z}_\infty(t_f-t)}\begin{bmatrix}x(t)\\ \lambda(t)\end{bmatrix} = \begin{bmatrix}\Phi_{11}(t_f-t) & \Phi_{12}(t_f-t)\\ \Phi_{21}(t_f-t) & \Phi_{22}(t_f-t)\end{bmatrix}\begin{bmatrix}x(t)\\ \lambda(t)\end{bmatrix}.$$

- Substitute the final condition $\lambda(t_f) = 0$,

$$\begin{bmatrix}x(t_f)\\ 0\end{bmatrix} = \begin{bmatrix}\Phi_{11}(t_f-t) & \Phi_{12}(t_f-t)\\ \Phi_{21}(t_f-t) & \Phi_{22}(t_f-t)\end{bmatrix}\begin{bmatrix}x(t)\\ \lambda(t)\end{bmatrix}$$

- Solving the lower-triangular block

$$\lambda(t) = -\{\Phi_{22}(t_f-t)\}^{-1}\Phi_{21}(t_f-t)x(t) = P(t)x(t).$$

- Concluding,

$$u(t) = -B_u^TP(t)x(t) = B_u^T\{\Phi_{22}(t_f-t)\}^{-1}\Phi_{21}(t_f-t)x(t) = -K(t)x(t).$$

EXAMPLE 9.1: We wish to control a plant that is modeled as

$$\dot{x}(t) = x(t) + u(t) + 2w(t)$$
$$y(t) = \begin{bmatrix} 10 \\ 0 \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t)$$

where the objective function is

$$J_\gamma\{x(t), u(t), w(t)\} = \|y(t)\|_2^2 - \gamma^2 \|w(t)\|_2^2 .$$

- Note: State and control weightings are incorporated into the "output" equation, which is scaled to yield a unity-weight on the control.
- The Hamiltonian system is:

$$\begin{bmatrix} \dot{x}(t) \\ \dot{\lambda}(t) \end{bmatrix} = \begin{bmatrix} 1 & -1 + 4\gamma^{-2} \\ -100 & -1 \end{bmatrix} \begin{bmatrix} x(t) \\ \lambda(t) \end{bmatrix},$$

which has state-transition matrix

$$\begin{aligned} e^{\mathscr{Z}_\infty t} &= \mathcal{L}^{-1}[(sI - \mathscr{Z}_\infty)^{-1}] \\ &= \mathcal{L}^{-1}\left[\frac{1}{s^2 - 101 + 400\gamma^{-2}} \begin{bmatrix} s+1 & -1 + 4\gamma^{-2} \\ -100 & s-1 \end{bmatrix}\right] \end{aligned}$$

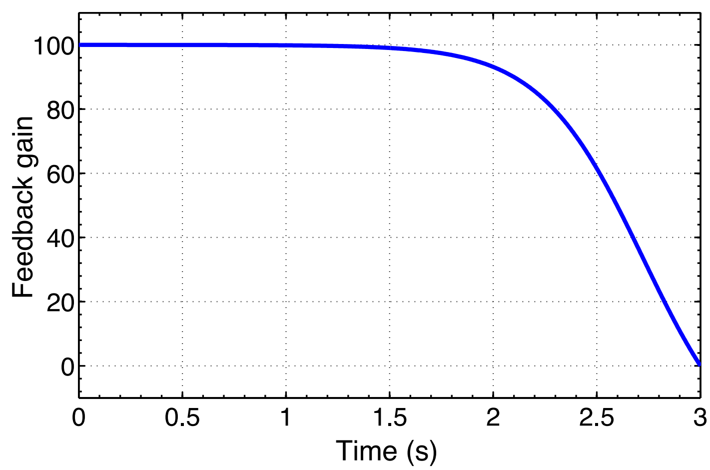
- An $\mathscr{H}_\infty$ full-information suboptimal controller exists if Φ_{22} is invertible throughout the desired time interval.
- This element of the state-transition matrix is:

$$\begin{aligned} \Phi_{22}(t) &= \mathcal{L}^{-1}\left[\frac{s-1}{s^2 - 101 + 400\gamma^{-2}}\right] \\ &= \begin{cases} \dfrac{a-1}{2a} e^{at} + \dfrac{a+1}{2a} e^{-at}, & \gamma > \sqrt{\dfrac{400}{101}} \\ 1 - t, & \gamma = \sqrt{\dfrac{400}{101}} \\ \sqrt{\dfrac{\omega^2 + 1}{\omega^2}} \sin(\omega t + \theta), & \gamma < \sqrt{\dfrac{400}{101}} \end{cases} \end{aligned}$$

where $a = \sqrt{101 - 400\gamma^{-2}}$, $\omega = \sqrt{-101 + 400\gamma^{-2}}$, and $\theta = -\tan^{-1}(\omega)$.

- Note that as γ gets smaller, a gets smaller, ω gets bigger, and θ gets more negative.
- When $\gamma < \sqrt{400/101}$ the eigenvalues of the Hamiltonian matrix are imaginary, and the gain does not exist for long time intervals since $\Phi_{22}(t)$ periodically crosses zero.
- The lack of solution for feedback gain means there is no solution of the differential game for the given bound.
- For the suboptimal bound $\gamma = 2.03$ ($a = 2$) and the final time $t_f = 3$, the $\mathscr{H}_\infty$ suboptimal feedback gain is

$$K(t) = \frac{100e^{2(3-t)} - 100e^{-2(3-t)}}{e^{2(3-t)} + 3e^{-2(3-t)}}.$$



8.4: Riccati equations to find state feedback

The Riccati equation

- Solving the Hamiltonian for the time-varying state feedback may be done via a Riccati equation.
- Given that $\lambda(t) = P(t)x(t)$, differentiate to get

$$\dot{\lambda}(t) = \dot{P}(t)x(t) + P(t)\dot{x}(t).$$

- Substitute for $\dot{\lambda}(t)$ and $\dot{x}(t)$ from the Hamiltonian system

$$-C_y^T C_y x(t) - A^T \lambda(t) = \dot{P}(t)x(t) + P(t)\{Ax(t) - (B_u B_u^T - \gamma^{-2} B_w B_w^T)\lambda(t)\}.$$

- Then, substituting for $\lambda(t) = P(t)x(t)$,

$$\{\dot{P}(t) + P(t)A + A^T P(t) + C_y^T C_y - P(t)(B_u B_u^T - \gamma^{-2} B_w B_w^T)P(t)\}x(t) = 0.$$

- Since this is valid for any $x(t)$

$$\dot{P}(t) = -P(t)A - A^T P(t) - C_y^T C_y + P(t)(B_u B_u^T - \gamma^{-2} B_w B_w^T)P(t),$$

  which is the Riccati equation for the $\mathscr{H}_\infty$ suboptimal control problem, initialized with $P(t_f) = 0$.

- A suboptimal $\mathscr{H}_\infty$ controller exists for performance bound γ iff there exists a solution $P \geq 0$ to

$$PA + A^T P - P(B_u B_u^T - \gamma^{-2} B_w B_w^T)P + C_y^T C_y = 0,$$

  where the matrix $A - (B_u B_u^T - \gamma^{-2} B_w B_w^T)P$ is stable.

  - This may be used as the test in the bisection search algorithm for existence of an $\mathscr{H}_\infty$ controller for performance bound γ.

EXAMPLE 9.1 CONTINUED: Here, we use the Riccati equation to solve the same problem as in the prior example.

- The Riccati equation is ($A = 1$, $B_u = 1$, $B_w = 2$, $C_y = \begin{bmatrix} 10 & 0 \end{bmatrix}^T$):

$$\dot{P}(t) = -2P(t) + (1 - 4\gamma^{-2})P^2(t) - 100,$$

with final condition $P(3) = 0$.

- The solution is

$$P(t) = K(t) = \frac{100e^{2(3-t)} - 100e^{-2(3-t)}}{e^{2(3-t)} + 3e^{-2(3-t)}},$$

which is the same as we found before.

- (The solution can be verified by substituting $P(t)$ back into the Riccati equation). Note, $P(t) = K(t)$ because $B_u = 1$.

Steady-state result

- The steady-state result may be found by setting $\dot{P}(t) = 0$ and solving the A.R.E.
- An eigenvector method may again be used to solve for P_{ss}

$$P = \Psi_{21}(\Psi_{11})^{-1},$$

where Ψ is the eigensystem of the Hamiltonian system, and Ψ_{21} and Ψ_{11} are formed from the eigenvectors associated with the *stable* eigenvalues.

- A suboptimal $\mathscr{H}_\infty$ controller exists for performance bound γ iff $\mathscr{Z}_\infty$ has no eigenvalues on the imaginary axis, Ψ_{11} is invertible, and $P = \Psi_{21}(\Psi_{11})^{-1} \geq 0$.
  - This may also be used in the test in the bisection search algorithm for existence of an $\mathscr{H}_\infty$ controller for performance bound γ.

- The suboptimal controller is then given as

$$u(t) = -B_u^T \Psi_{21}(\Psi_{11})^{-1} x(t) = -Kx(t).$$

EXAMPLE 9.2: An antenna is required to remain pointed at a satellite in the presence of disturbance torques caused by gravity, wind, squirrels, etc. The state model is

$$\dot{x}(t) = \begin{bmatrix} 0 & 1 \\ 0 & -1 \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} w(t)$$

$$e(t) = \begin{bmatrix} 1 & 0 \end{bmatrix} x(t).$$

- As usual, $u(t)$ is the control torque, $w(t)$ is the disturbance torque, and $e(t)$ is the tracking error.
- A tracking error of less than 0.1 rad is desired, even in the presence of disturbances up to 2 N-m.
- The control magnitude is also required to be less than 10 N-m.
- These specifications are appended to the plant as weighting functions to yield a model in standard form:

$$\dot{x}(t) = \begin{bmatrix} 0 & 1 \\ 0 & -1 \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ 10 \end{bmatrix} u_1(t) + \begin{bmatrix} 0 \\ 2 \end{bmatrix} w_1(t)$$

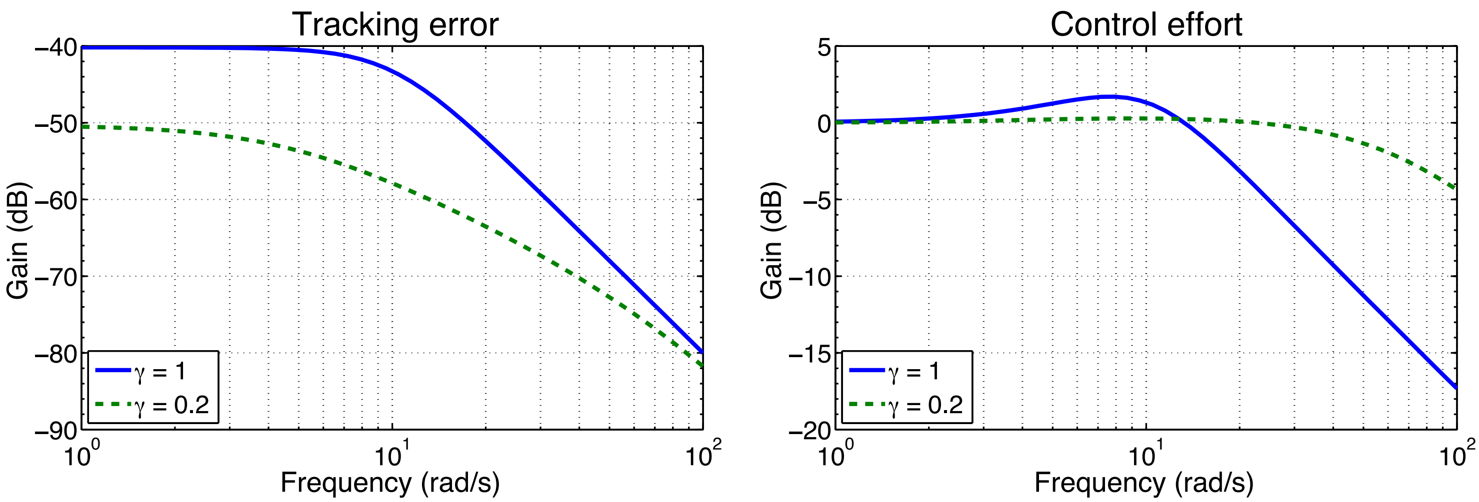
$$y(t) = \begin{bmatrix} 10 & 0 \\ 0 & 0 \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u_1(t),$$

where $w_1(t)$ and $u_1(t)$ are the normalized disturbance and control inputs.

- In this form, $w_1(t)$ is less than 1 and both elements of $y(t)$ are required to be less than 1. The original control input can be recovered from the normalized control as $u(t) = 10u_1(t)$.

- A full-information controller is generated for $\gamma = 1$:

$$u_1(t) = -\left[\begin{array}{cc} 10.2 & 1.36 \end{array}\right] x(t) \quad \text{or} \quad u(t) = -\left[\begin{array}{cc} 102 & 13.6 \end{array}\right] x(t).$$



- The left plot shows frequency response of tracking error; the right plot shows frequency response of control effort.
- Maximum gain from sinusoidal disturbance to tracking error is $10^{-40/20} = 0.01$, which is less than spec of 0.05.
- Maximum gain from disturbance input to control is 1.26, which is less than spec of 5.
- A second full-information controller is generated for $\gamma = 0.21$.

$$u_1(t) = -\left[\begin{array}{cc} 32.8 & 7.39 \end{array}\right] x(t) \quad \text{or} \quad u(t) = -\left[\begin{array}{cc} 328 & 73.9 \end{array}\right] x(t).$$

- Performance is better, but gains are higher. Disturbance rejection is higher, but control bandwidth is also higher.

8.5: $\mathscr{H}_\infty$ Estimation

- When investigating $\mathscr{H}_2$ control, we split the problem into a regulation part (LQR) and an estimation part (LQE).
- When joined, the result turned out to be optimal as well (LQG).
- One reason that this works is because $\mathscr{H}_2$ optimization of $-K(t)x(t)$ is equal to $-K(t)$ times the $\mathscr{H}_2$ optimization of $x(t)$.
  - This same property does not hold for $\mathscr{H}_\infty$ estimation; we must be more careful when generating an $\mathscr{H}_\infty$ output feedback controller.
  - $\mathscr{H}_\infty$ estimation is still required, so we study it briefly here.
- The plant model that is assumed looks like

$$\dot{x}(t) = Ax(t) + B_u u(t) + B_w w(t)$$

$$m(t) = C_m x(t) + D_{mw} w(t)$$

with constraints $D_{mw} B_w^T = 0$ and $D_{mw} D_{mw}^T = I$.

- Goal is to estimate a linear combination of the state, $y(t) = C_y x(t)$.
  - One example would be to estimate the state itself: $C_y = I$.
- $\mathscr{H}_\infty$ estimation is the dual problem to $\mathscr{H}_\infty$ regulation. As you might expect, there is a Hamiltonian system involved

$$\begin{bmatrix} \dot{x}_{\hat{h}}(t) \end{bmatrix} = \left[\begin{array}{c|c} -A^T & C_m^T C_m - \gamma^{-2} C_y^T C_y \\ \hline B_w B_w^T & A \end{array}\right] \begin{bmatrix} x_{\hat{h}}(t) \end{bmatrix} = \mathscr{Y}_\infty x_{\hat{h}}(t).$$

- The $\mathscr{H}_\infty$ suboptimal estimator gain may be found in terms of the state-transition matrix of the Hamiltonian system:

$$L(t) = Q(t)C_m^T = \Phi_{21}(t)\{\Phi_{11}(t)\}^{-1}C_m^T$$

where the Φ_{ij} are terms of the state-transition matrix:

$$e^{\mathscr{Y}_\infty t} = \left[\begin{array}{c:c} \Phi_{11}(t) & \Phi_{12}(t) \\ \hdashline \Phi_{21}(t) & \Phi_{22}(t) \end{array}\right].$$

- A steady-state solution may be found from the eigensystem of $\mathscr{Y}_\infty$ as

$$Q = \Psi_{22}(\Psi_{12})^{-1},$$

where

$$\left[\begin{array}{c} \Psi_{12} \\ \hdashline \Psi_{22} \end{array}\right]$$

is a matrix whose columns are the eigenvectors of the estimator Hamiltonian associated with the unstable eigenvalues.

- A suboptimal $\mathscr{H}_\infty$ estimator exists if there is a solution to $\mathscr{Y}_\infty$ that has no eigenvalues on the imaginary axis, Ψ_{12} is invertible, and $Q = \Psi_{22}(\Psi_{12})^{-1} \geq 0$.
  - This may be used in the bisection search algorithm for existence of an $\mathscr{H}_\infty$ estimator for performance bound γ.
- These result may also be obtained from a Riccati equation

$$\dot{Q}(t) = Q(t)A^T + AQ(t) + B_w B_w^T - Q(t)\{C_m^T C_m - \gamma^{-2}C_y^T C_y\}Q(t),$$

with initial condition $Q(0) = 0$.

- A suboptimal $\mathscr{H}_\infty$ estimator exists if there is a solution to

$$AQ + Q^T A - Q(C_m^T C_m - \gamma^{-2}C_y^T C_y)Q + B_w B_w^T = 0,$$

such that $Q \geq 0$ and that the matrix $A - Q(C_m^T C_m - \gamma^{-2} C_y^T C_y)$ is stable.

- This may also be used in the bisection search algorithm for existence of an $\mathscr{H}_\infty$ estimator for performance bound γ.

EXAMPLE 9.3: An $\mathscr{H}_\infty$ estimator can be applied to the radar range tracking of an aircraft. A model equations for the range $r(t)$ and radial velocity of an aircraft is (range is measured):

$$\begin{bmatrix} \dot{r}(t) \\ \ddot{r}(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} r(t) \\ \dot{r}(t) \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} w(t)$$

$$m(t) = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} r(t) \\ \dot{r}(t) \end{bmatrix} + v(t).$$

- The output to be estimated is the entire state:

$$y(t) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} r(t) \\ \dot{r}(t) \end{bmatrix}$$

- The plant input and measurement error are assumed to be bounded:

$$\begin{bmatrix} |w(t)| \\ |v(t)| \end{bmatrix} \leq \begin{bmatrix} 4\,\text{m/sec}^2 \\ 20\,\text{m} \end{bmatrix}$$

- Model equations using normalized inputs are:

$$\begin{bmatrix} \dot{r}(t) \\ \ddot{r}(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} r(t) \\ \dot{r}(t) \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 4 & 0 \end{bmatrix} \begin{bmatrix} w_1(t) \\ v_1(t) \end{bmatrix}$$

$$m(t) = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} r(t) \\ \dot{r}(t) \end{bmatrix} + \begin{bmatrix} 0 & 20 \end{bmatrix} \begin{bmatrix} w_1(t) \\ v_1(t) \end{bmatrix}$$

- Normalizing this equation so that the input-output coupling matrix satisfies $D_{mw} D_{mw}^T = I$ gives the measurement equation

$$m_1(t) = \begin{bmatrix} \frac{1}{20} & 0 \end{bmatrix} \begin{bmatrix} r(t) \\ \dot{r}(t) \end{bmatrix} + \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} w_1(t) \\ v_1(t) \end{bmatrix}$$

where $m(t) = 20m_1(t)$.

- The resulting $\mathscr{H}_\infty$ estimator is given as

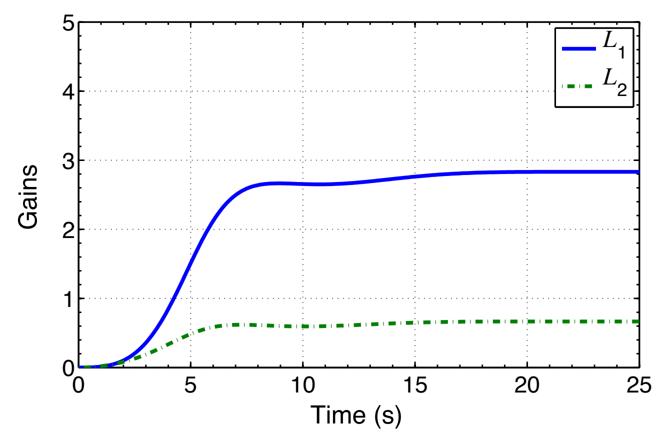
$$\dot{\hat{x}}(t) = A\hat{x}(t) + L_1(t)\left[m_1(t) - \begin{bmatrix} \frac{1}{20} & 0 \end{bmatrix} \hat{x}(t)\right]$$
$$\hat{y}(t) = C_y\hat{x}(t).$$

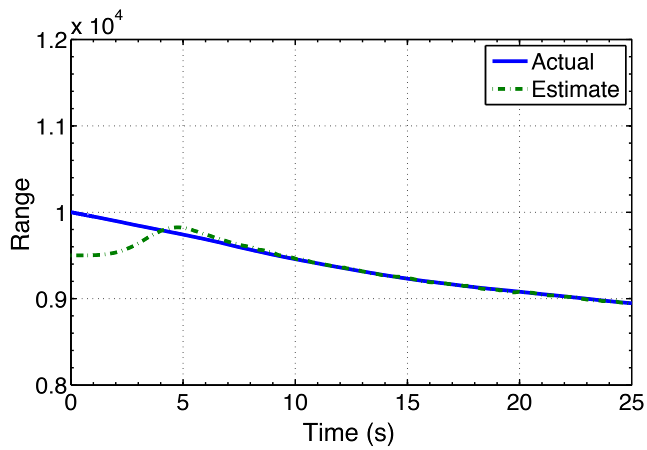
- We can write this in terms of the original measurements,

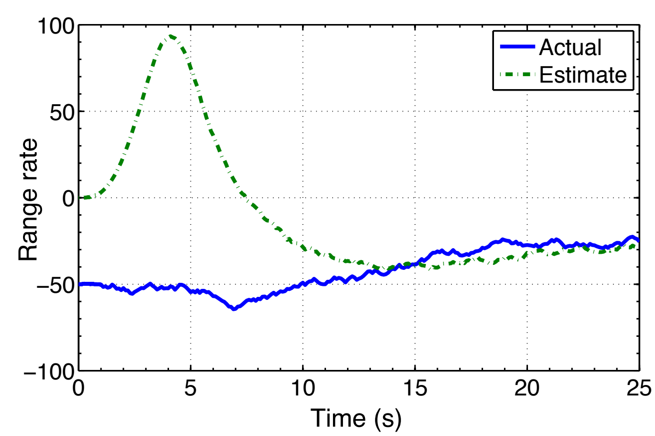
$$\begin{aligned} \dot{\hat{x}}(t) &= A\hat{x}(t) + \frac{1}{20}L_1(t)\left[20m_1(t) - 20\begin{bmatrix} \frac{1}{20} & 0 \end{bmatrix} \hat{x}(t)\right] \\ &= A\hat{x}(t) + L(t)\left[m(t) - \begin{bmatrix} 1 & 0 \end{bmatrix} \hat{x}(t)\right], \end{aligned}$$

where $L(t) = L_1(t)/20$.

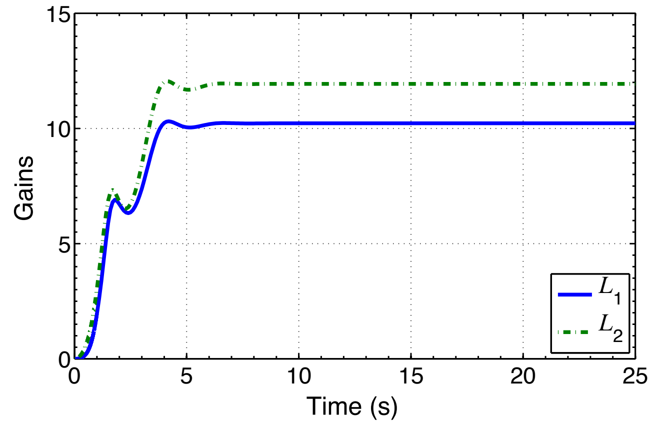
- Baseline results are plotted below.

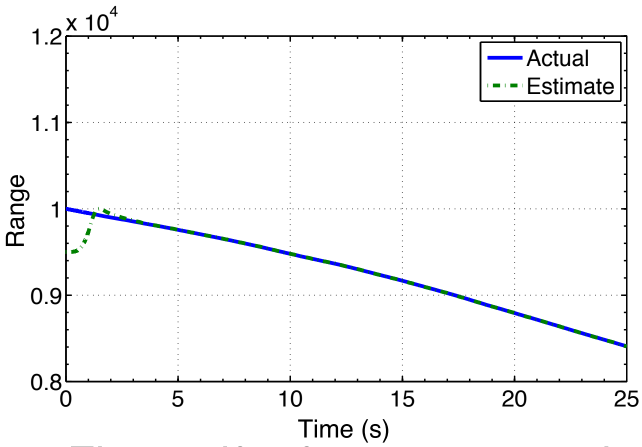


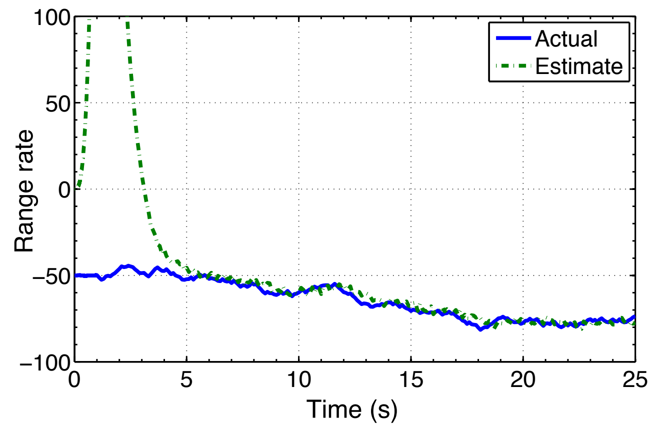




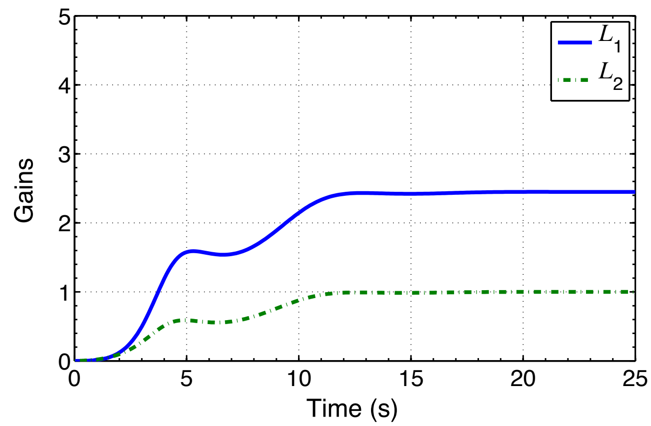
- If meas. error decreases from 20 to 2, we get the these results.
- Estimation gains increase; estimates converge more quickly.

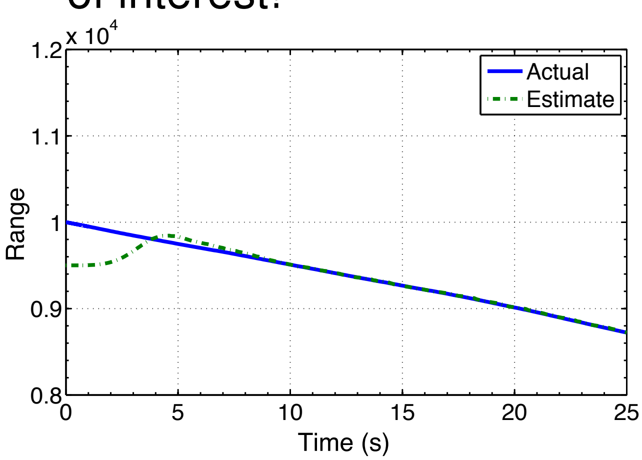


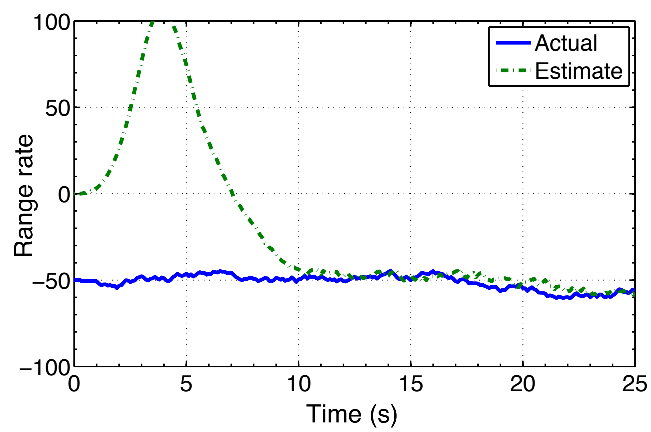




- Finally, if reference output is only range rate (we still estimate range, but don't care about its accuracy), we get these results.
- Gains change (slightly)! The $\mathscr{H}_\infty$ estimator depends on the output of interest!







Summary

- $\mathscr{H}_\infty$ controller and estimator design is an iterative procedure.
- Perform the bisection algorithm to find the lowest value of the performance bound γ that you can, within some tolerance.
  - You will generally back off of this performance bound somewhat, as designs that are nearly optimal can have numeric instabilities.
- For **controller design**, the conditions for existence of some suboptimal controller with bound γ are:
  - There exists a solution $P \geq 0$ to

$$PA + A^T P - P(B_u B_u^T - \gamma^{-2} B_w B_w^T)P + C_y^T C_y = 0,$$

    where the matrix $A - (B_u B_u^T - \gamma^{-2} B_w B_w^T)P$ is stable; *or,*
  - $\mathscr{Z}_\infty$ has no eigenvalues on the imaginary axis, Ψ_{11} is invertible, and $P = \Psi_{21}(\Psi_{11})^{-1} \geq 0$.
- For **estimator design**, the conditions for existence of a suboptimal controller with bound γ are:
  - There exists a solution $Q \geq 0$ to

$$AQ + Q^T A - Q(C_m^T C_m - \gamma^{-2} C_y^T C_y)Q + B_w B_w^T = 0,$$

    where the matrix $A - Q(C_m^T C_m - \gamma^{-2} C_y^T C_y)$ is stable; *or,*
  - $\mathscr{Y}_\infty$ has no eigenvalues on the imaginary axis, Ψ_{12} is invertible, and $Q = \Psi_{22}(\Psi_{12})^{-1} \geq 0$.
- For **controller implementation**, with some γ, solve the differential Riccati equation

$$\dot{P}(t) = -P(t)A - A^T P(t) - C_y^T C_y + P(t)(B_u B_u^T - \gamma^{-2} B_w B_w^T)P(t),$$

backward in time, initialized with $P(t_f) = 0$. Then,

$$u(t) = -K(t)x(t) = -B_u^T P(t)x(t).$$

- A steady-state suboptimal controller, with some γ, uses the eigensystem Ψ of the $\mathscr{Z}_\infty$ matrix to find the steady-state gain matrix:

$$u(t) = -Kx(t) = -B_u^T \Psi_{21}(\Psi_{11})^{-1} x(t),$$

where Ψ_{21} and Ψ_{11} are formed from the eigenvectors associated with the *stable* eigenvalues of $\mathscr{Z}_\infty$.

- For **estimator implementation**, with some γ, solve the differential Riccati equation

$$\dot{Q}(t) = Q(t)A^T + AQ(t) + B_w B_w^T - Q(t)\{C_m^T C_m - \gamma^{-2} C_y^T C_y\}Q(t),$$

forward in time, initialized with $Q(0) = 0$. Then,

$$L(t) = Q(t)C_m^T.$$

- A steady-state suboptimal estimator, with some γ, uses the eigensystem Ψ of the $\mathscr{Y}_\infty$ matrix to find the steady-state gain:

$$L = QC_m^T = \Psi_{22}(\Psi_{12})^{-1} C_m^T,$$

where Ψ_{22} and Ψ_{12} are formed from the eigenvectors associated with the *unstable* eigenvalues of $\mathscr{Y}_\infty$.