LINEAR QUADRATIC GAUSSIAN

6.1: Deriving LQG via separation principle

- We will now start to look at the design of controllers for systems

$$\dot{x}(t) = A(t)x(t) + B_u(t)u(t) + B_w(t)w(t)$$

$$y(t) = C_y(t)x(t) + v(t).$$

where $w(t)$, $v(t)$ are white, Gaussian, zero-mean, independent.

$$v(t) \sim \mathcal{N}(0, S_v), \qquad w(t) \sim \mathcal{N}(0, S_w), \qquad x(0) \sim \mathcal{N}(x_0, \Xi_{x,0}).$$

- Time-varying plant dynamics can be handled quite easily, but we will concentrate on time-invariant case.

OBJECTIVE: Design a compensator $U = G_c(s)Y$ so that the closed-loop system is stable, and system achieves "good performance."

- Our goal will be to minimize the cost function

$$J = \mathbb{E}\left[\frac{1}{2}x^T(t_f)Hx(t_f) + \frac{1}{2}\int_0^{t_f} x^T(t)Qx(t) + u^T(t)Ru(t)\,\mathrm{d}t\right],$$

where $H, Q, R > 0$ and *we measure only the system output.*

- We have seen two versions of this problem already:

1. Noise-free: $u(t) = -K(t)x(t)$.
2. Noisy, but with complete state observations: $u(t) = -K(t)x(t)$. Same $K(t)$ but cost is increased.
3. Now, noisy output observations. We will show that $u(t) = -K(t)\hat{x}(t)$ where $\hat{x}(t)$ is the output of a Kalman filter.

- This is called the separation principle—using estimate $\hat{x}(t)$ as if it were the true state.

KEY POINT: This is exactly the same separation operation that we performed before when we designed state-space controllers via pole-placement techniques.

➠ Validation of that procedure was based only on the *stability* of the closed-loop system.

- We are now saying that the separation into two independent optimal sub-problems is *actually what optimizes the original objective.*

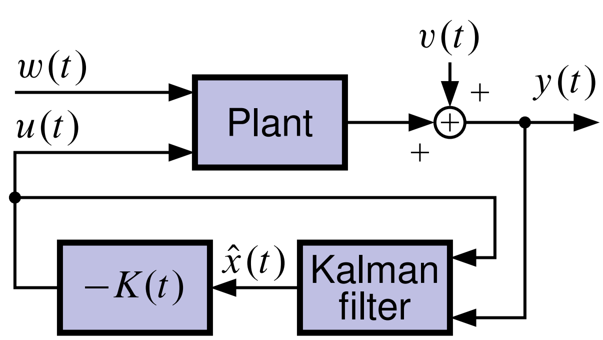
➠ On one hand, the separation is intuitively appealing: $\hat{x}(t)$ is our best guess of what is going on inside the system. So, in some sense, it is clear we should use it in the regulator.

➠ But to extend this intuition further and postulate that it is actually *the optimal* procedure to follow is *not obvious*. (*e.g.*, Noise $\Rightarrow$ Uncertainty $\Rightarrow$ Reduced control gains??)

Separation Principle for Optimal Performance

- Our approach to solving the optimal control problem with finite and noisy measurements will be

1. Rewrite cost and system in terms of the estimator states and dynamics ➠ Can access these.
2. Design a stochastic LQR controller for this revised system ➠ Full state feedback of $\hat{x}(t)$.
3. Connection $u(t) = -K(t)\hat{x}(t)$.



- Start with the cost

$$\begin{aligned}\mathbb{E}[x^T Q x] &= \mathbb{E}[\{x-\hat{x}+\hat{x}\}^T Q\{x-\hat{x}+\hat{x}\}]\\ &= \mathbb{E}[\{x-\hat{x}\}^T Q\{x-\hat{x}\}] + 2\mathbb{E}[\{x-\hat{x}\}^T Q\hat{x}] + \mathbb{E}[\hat{x}^T Q\hat{x}]\\ &= \mathbb{E}[\tilde{x}^T Q\tilde{x}] + 2\mathbb{E}[\tilde{x}^T Q\hat{x}] + \mathbb{E}[\hat{x}^T Q\hat{x}].\end{aligned}$$

- Each term is a scalar, and unchanged by the trace operator,

$$\begin{aligned}\mathbb{E}[x^T Q x] &= \operatorname{trace}(\mathbb{E}[\tilde{x}^T Q\tilde{x}]) + \operatorname{trace}(\mathbb{E}[\tilde{x}^T Q\hat{x}]) + \mathbb{E}[\hat{x}^T Q\hat{x}]\\ &= \operatorname{trace}(\mathbb{E}[\tilde{x}\tilde{x}^T Q]) + \operatorname{trace}(\mathbb{E}[\hat{x}\tilde{x}^T Q]) + \mathbb{E}[\hat{x}^T Q\hat{x}]\\ &= \operatorname{trace}(\Xi_x Q) + \mathbb{E}[\hat{x}^T Q\hat{x}]\end{aligned}$$

  because $\hat{x}$ and $\tilde{x}$ are orthogonal if a Kalman filter is used to estimate x (Proved in text).

- We can now write the LQG cost as

$$\begin{aligned}J &= \mathbb{E}\left[\frac{1}{2}\hat{x}^T(t_f)H\hat{x}(t_f) + \frac{1}{2}\int_0^{t_f}\hat{x}^T(t)Q\hat{x}(t) + u^T(t)Ru(t)\,\mathrm{d}t\right]\\ &\quad+\frac{1}{2}\operatorname{trace}\left\{\Xi_x(t_f)H + \int_0^{t_f}\Xi_x(t)Q\,\mathrm{d}t\right\}.\end{aligned}$$

- The terms within the $\operatorname{trace}\{\}$ operator are independent of the control input $u(t)$ so J is minimized whenever

$$J = \mathbb{E}\left[\frac{1}{2}\hat{x}^T(t_f)H\hat{x}(t_f) + \frac{1}{2}\int_0^{t_f}\hat{x}^T(t)Q\hat{x}(t) + u^T(t)Ru(t)\,\mathrm{d}t\right]$$

  is minimized.

- This looks like a stochastic LQR problem in the (measurable) state $\hat{x}$. But, note that the dynamics of $\hat{x}$ are different from the dynamics of x. Need to show that the assumptions of LQR are met.

- Estimator dynamics:

$$\dot{\hat{x}}(t) = A\hat{x}(t) + B_u u(t) + L(t)\left[y(t) - C_y\hat{x}(t)\right].$$

KEY FACT: The innovations process $r(t) = y(t) - C_y\hat{x}(t)$ is white, Gaussian $\sim \mathcal{N}(0, S_v)$. (Shown in section A12 of book appendix).

- So, LQG problem is to minimize

$$J = \mathbb{E}\left[\frac{1}{2}\hat{x}^T(t_f)H\hat{x}(t_f) + \frac{1}{2}\int_0^{t_f} \hat{x}^T(t)Q\hat{x}(t) + u^T(t)Ru(t)\,\mathrm{d}t\right] + \text{const}$$

subject to the dynamics

$$\dot{\hat{x}}(t) = A\hat{x}(t) + B_u u(t) + L(t)r(t)$$

$$L(t) \quad \text{known,}$$

$$r(t) \sim \mathcal{N}(0, S_v).$$

- This is a (strange looking) stochastic LQR problem.
  - ➠ Solution independent of the white driving noise.
  - ➠ Optimal solution a linear feedback law $u(t) = -K(t)\hat{x}(t)$.
  - ➠ $K(t)$ is from solution of LQR with data (A, B, Q, R) … same.
  - ➠ Therefore, LQG = combination of LQR/LQE is

PERFORMANCE OPTIMAL.

6.2: Satellite-tracking example

EXAMPLE: A satellite tracking antenna, subject to random wind torques, can be modeled as

$$\begin{bmatrix} \dot\theta(t) \\ \ddot\theta(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & -0.1 \end{bmatrix} \begin{bmatrix} \theta(t) \\ \dot\theta(t) \end{bmatrix} + \begin{bmatrix} 0 \\ 0.001 \end{bmatrix} u(t) + \begin{bmatrix} 0 \\ 0.001 \end{bmatrix} w(t)$$

$$y(t) = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} \theta(t) \\ \dot\theta(t) \end{bmatrix} + v(t),$$

where

- $\theta(t)$ is the pointing error of the antenna in degrees,
- $u(t)$ is the control torque in $\mathrm{N\,m}$, and
- $w(t)$ is the wind torque (disturbance input) in $\mathrm{N\,m}$.

- The wind torque is assumed to be white noise with a spectral density of $S_w = 5000\,\mathrm{N}^2\,\mathrm{m}^2\,\mathrm{Hz}^{-1}$.
- The measurement noise is white with a spectral density $S_v = 1\,\mathrm{deg}^2\,\mathrm{Hz}^{-1}$.
- The cost function to be minimized is

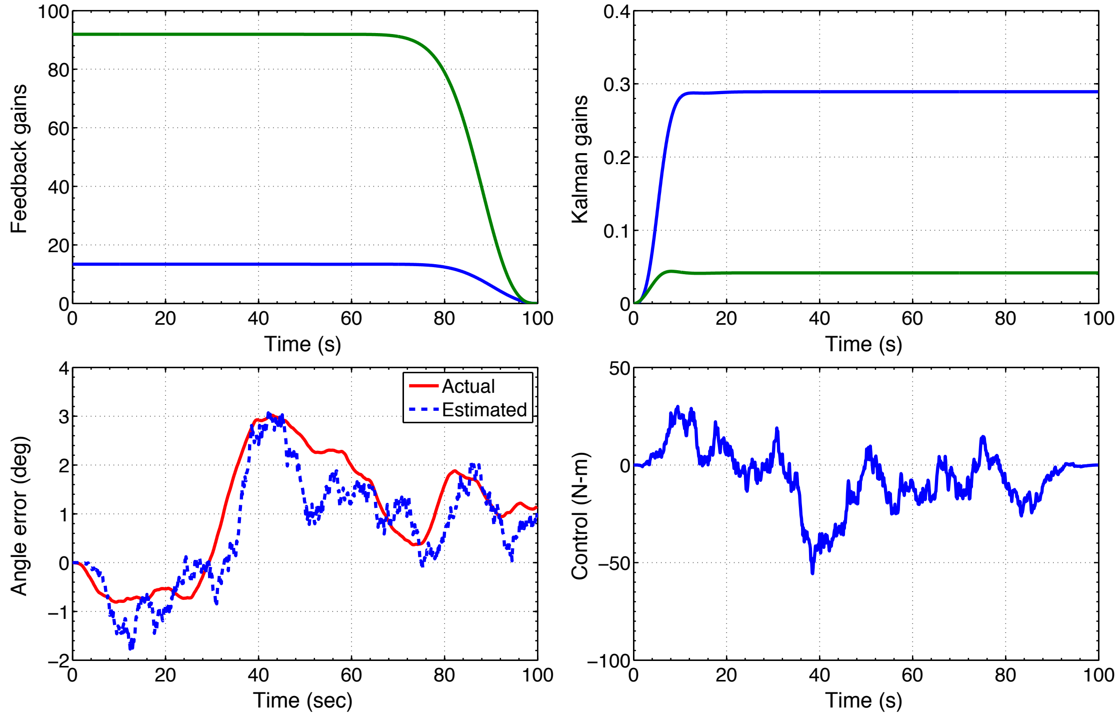
$$J = \mathbb{E}\left[\int_0^{100} q\theta^2(t) + ru^2(t)\,\mathrm{d}t\right].$$

- The simulation is initialized with

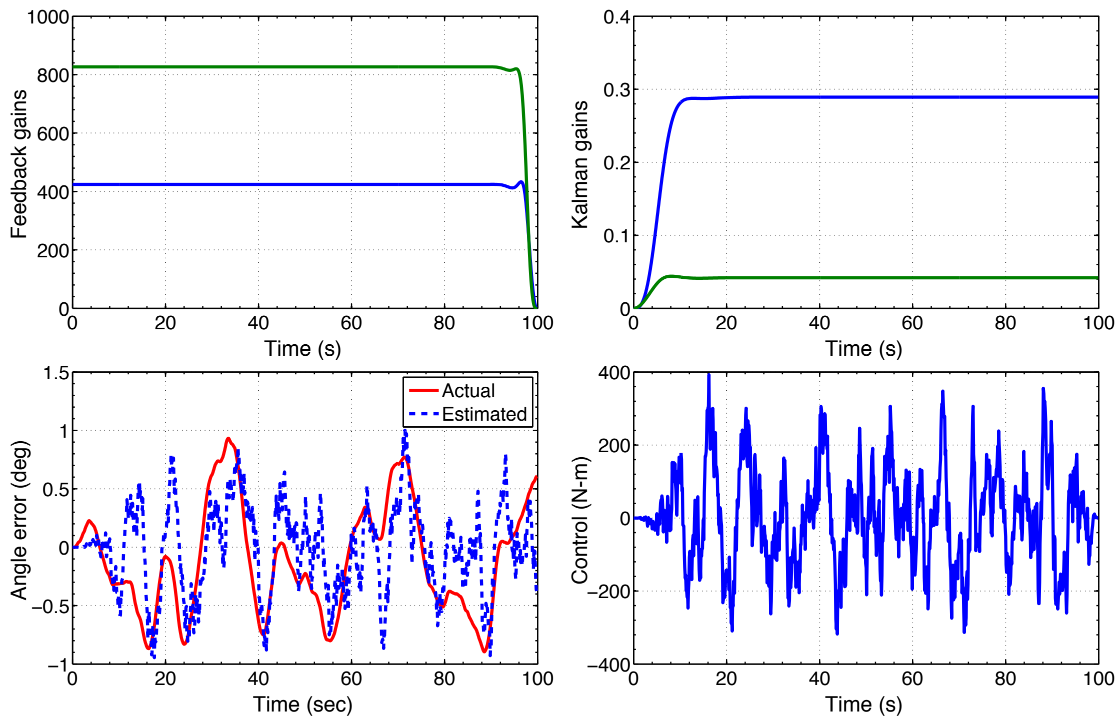
$$\hat{x}(0) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \quad \Xi_x(0) = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

- The MATLAB code is from Burl. Results follow.

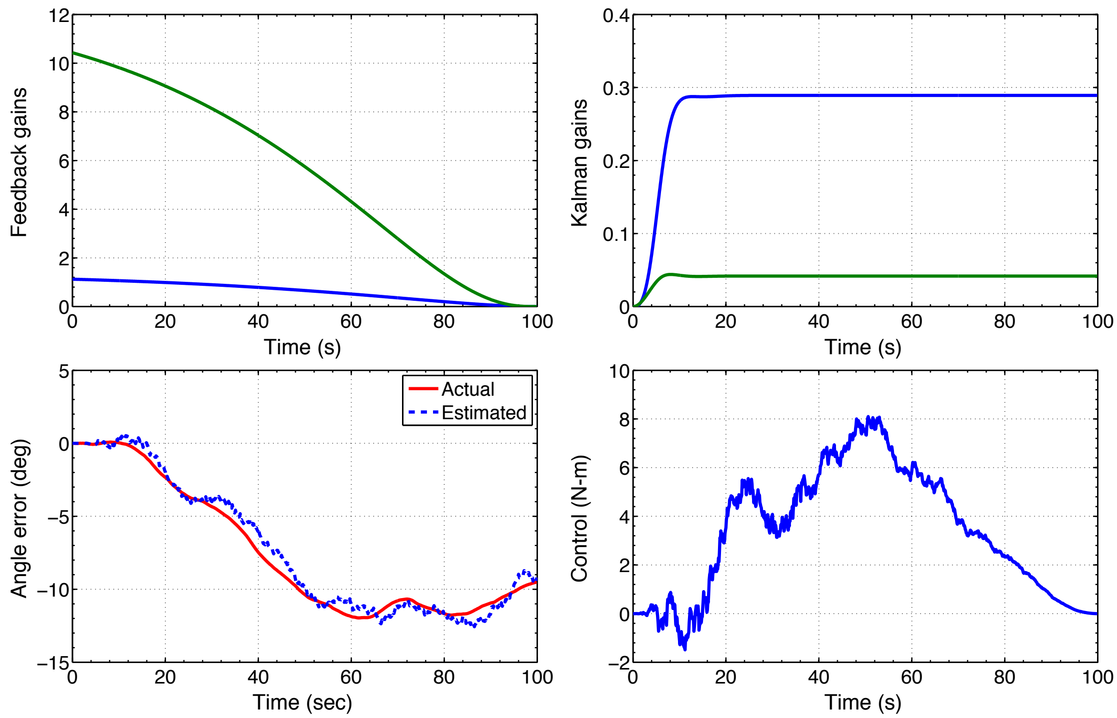
- First, the baseline example with: $q = 180$; $r = 1$; $S_w = 5000$; $S_v = 1$.



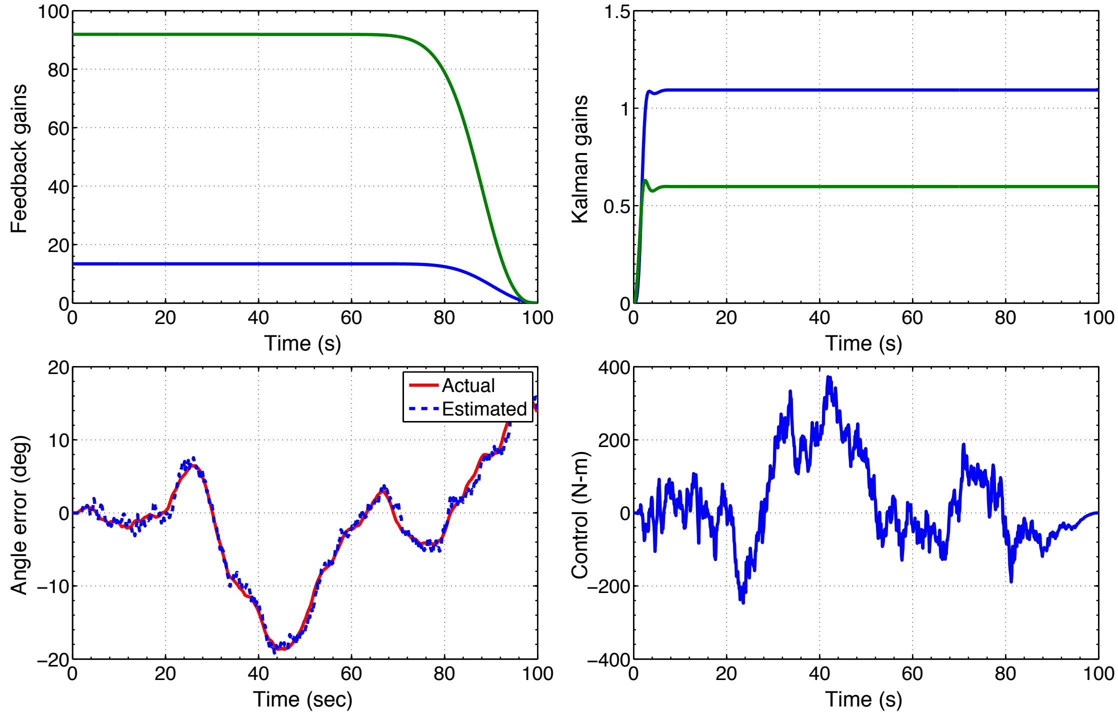
- Increased regulation requirement: $q = 18000$; $r = 1$; $S_w = 5000$; $S_v = 1$. State feedback gains are larger and approach steady state more rapidly, but Kalman gains remain the same. Larger control inputs and better control performance.



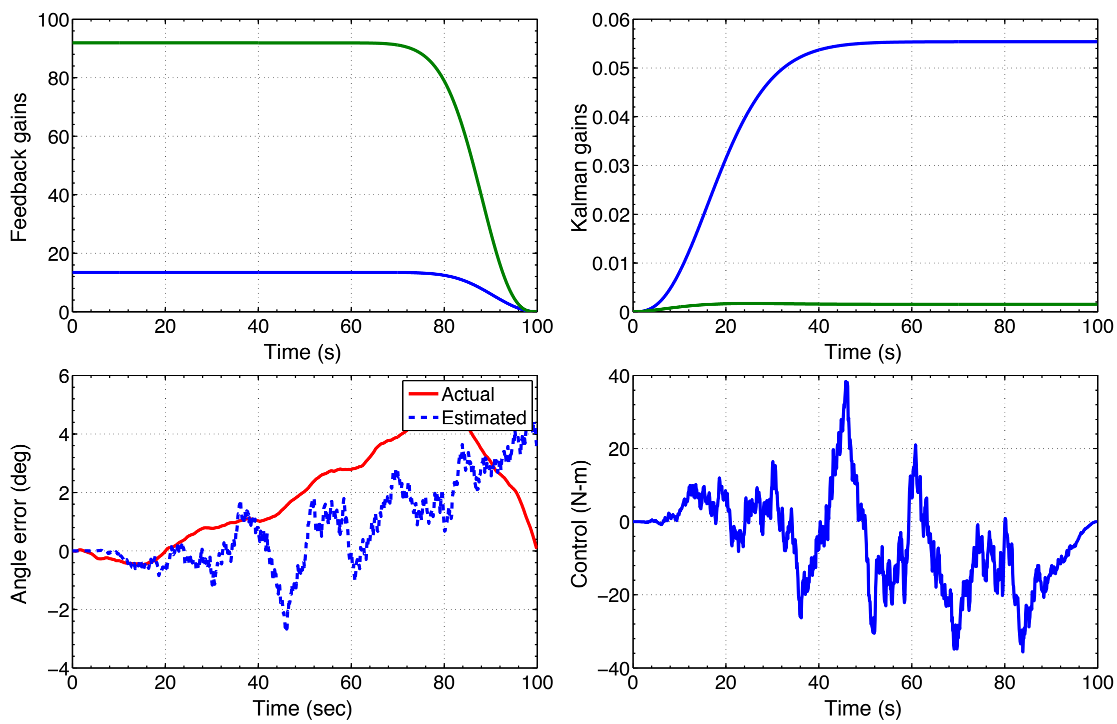
- Increased control-effort penalty: $q = 180$; $r = 100$; $S_w = 5000$; $S_v = 1$. State feedback gains smaller, converge more slowly. Kalman gains remain the same. Smaller control inputs, increased angle errors.



- High disturbance: $q = 180$; $r = 1$; $S_w = 500000$; $S_v = 1$. Kalman gains larger, converge much more rapidly. State feedback gains remain the same. Large gains make estimator faster but larger plant noise means that estimate is less accurate. Control input also increased due to larger gains and increased plant noise. Angle error increased by roughly 10, which is square-root of increase in spectral density.



- Large sensor noise: $q = 180$; $r = 1$; $S_w = 5000$; $S_v = 100$. Kalman gains smaller, converge more slowly. State feedback gains remain same. Smaller Kalman gains make estimator slower and larger measurement noise makes estimates less accurate. Control is poorer due to increased measurement noise.



6.3: Steady-state LQG control and the compensator

- Assume time-invariant plant dynamics.
- Use steady-state cost. Analysis (Stengel) shows (where $z(t) = C_z x(t)$)

$$\begin{aligned} J_{LQG} &= \lim_{t\to\infty} \mathbb{E}[z^T(t)Q_z z(t) + u^T(t)Ru(t)] \\ &= \text{trace}\left\{ \underline{\underline{\Sigma_{x,\text{ss}}}}\, \underline{L_{\text{ss}} S_v L_{\text{ss}}^T} + \underline{\Xi_{x,\text{ss}}}\, \underline{\underline{C_z^T Q_z C_z}} \right\} \\ &= \text{trace}\left\{ \underline{\underline{\Sigma_{x,\text{ss}}}}\, \underline{B_w S_w B_w^T} + \underline{\Xi_{x,\text{ss}}}\, \underline{\underline{K_{\text{ss}}^T R K_{\text{ss}}}} \right\} \end{aligned}$$

where terms are labeled $\underline{\underline{\text{regulator}}}$ or $\underline{\text{estimator}}$ and

$$A^T\Sigma_{x,\text{ss}} + \Sigma_{x,\text{ss}}A + C_z^T Q_z C_z - \Sigma_{x,\text{ss}} B_u R^{-1} B_u^T \Sigma_{x,\text{ss}} = 0$$

$$A\Xi_{x,\text{ss}} + \Xi_{x,\text{ss}}A^T + B_w S_w B_w^T - \Xi_{x,\text{ss}} C_y^T S_v^{-1} C_y \Xi_{x,\text{ss}} = 0$$

and $L_{\text{ss}} = \Xi_{x,\text{ss}} C_y^T S_v^{-1}$; $K_{\text{ss}} = R^{-1} B_u^T \Sigma_{x,\text{ss}}$.

- Can evaluate steady-state cost from the solution of two Riccati equations. More complicated than the stochastic LQR result

$$J_{LQR} = \text{trace}\left\{ (Q + K_{\text{ss}}^T R K_{\text{ss}})\Sigma_{x,\text{ss}}] \right\}.$$

- Reason: J_{LQG} must also account for cost increase because of estimation error.
- In LQG, $\hat{x} \neq x$ in general. Concepts of (1) Regulation error ($x \neq 0$) and (2) Estimation error ($\hat{x} \neq x$). Both contribute to the cost.

Further interpretations

FAST REGULATOR: To see that both errors contribute, consider the result of a very fast regulator ➠ Control effort cheap, $R \to 0$.

- Precise analysis requires we find impact of $R \to 0$ on $\Sigma_{x,\text{ss}}$ but

$$J_{LQG} = \text{trace}\Big\{\Sigma_{x,\text{ss}} L_{\text{ss}} S_v L_{\text{ss}}^T + \underbrace{\Xi_{x,\text{ss}} C_z^T Q_z C_z}_{\text{not fn of } R}\Big\}$$

$$\geq \text{trace}\left\{\Xi_{x,\text{ss}} C_z^T Q_z C_z\right\} \text{ for all } R$$

In particular $\lim\limits_{R\to 0} J_{LQG} \geq \text{trace}\left\{\Xi_{x,\text{ss}} C_z^T Q_z C_z\right\}$.

- Even in the limit of no control penalty, the cost is lower-bounded by a term associated with the estimation error in the state $\Xi_{x,\text{ss}}$.

FAST ESTIMATOR: Low sensor noise and fast estimation, $S_v \to 0$.

- Use second identity to show $\lim\limits_{S_v\to 0} J_{LQG} \geq \text{trace}\left\{\Sigma_{x,\text{ss}} B_w S_w B_w^T\right\}$.
- Regulation error remains: lower bound on performance with fast estimator.
- Both cases illustrate that it is futile to make either the estimator or the regulator much faster than the other.
- Ultimate performance limited and quickly reach point of diminishing returns.

RULE OF THUMB: Select R, S_v so that estimator and regulator poles are roughly the same (order of magnitude) distance to origin ➠ Similar levels of authority. Since S_v is generally physical, this places a design constraint on R.

The compensator

- Classical control system design techniques create a compensator $D(s)$ to control a plant $G(s)$.
- State-space methods use state feedback and estimation.

- Can combine the state-feedback law and estimation law to derive a compensator. Usually called $K(s)$.

REGULATOR: $u(t) = -K(t)\hat{x}(t)$ where

$$K(t) = R^{-1}B_u^T P(t)$$

$$-\dot{P}(t) = A^T P(t) + P(t)A + C_z^T Q_z C_z - P(t)B_u R^{-1}B_u^T P(t); \quad P(t_f) = H.$$

ESTIMATOR: Dynamics

$$\dot{\hat{x}}(t) = A\hat{x}(t) + B_u u(t) + L(t)\left[y(t) - C_y\hat{x}(t)\right], \qquad \hat{x}(0) = \mathbb{E}[x_0]$$

$$L(t) = \Xi_x(t)C_y^T S_v^{-1}$$

$$\dot{\Xi}_x(t) = A\Xi_x(t) + \Xi_x(t)A^T + B_w S_w B_w^T - \Xi_x(t)C_y^T S_v^{-1}C_y\Xi_x(t); \; \Xi_x(0) = \Xi_{x,0}.$$

COMPENSATOR: Combine the above to get (where $x_c = \hat{x}$)

$$\dot{x}_c(t) = (A - B_u K(t) - L(t)C_y)x_c(t) + L(t)y(t)$$

$$u(t) = -K(t)x_c(t).$$

- In steady-state, $L(t) = L_{\text{ss}}$ and $K(t) = K_{\text{ss}}$ and we get

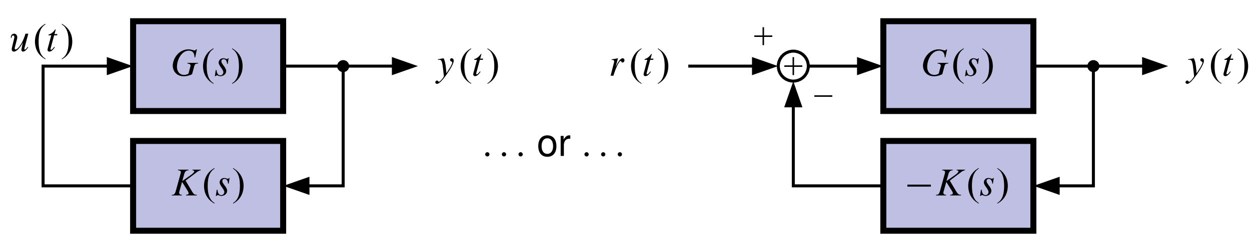
$$\dot{x}_c(t) = (A - B_u K_{\text{ss}} - L_{\text{ss}}C_y)x_c(t) + L_{\text{ss}}y(t)$$

$$u(t) = -K_{\text{ss}}x_c(t).$$

- We can then generate a transfer function for the compensator

$$K(s) = \bar{C}(sI - \bar{A})^{-1}\bar{B}.$$

(what are $\bar{A}$, $\bar{B}$, and $\bar{C}$?)



Closed-loop poles (steady-state)

- The regulator has poles at $\det(sI - A + B_u K_{ss}) = 0$;
- The estimator has poles at $\det(sI - A + L_{ss} C_y) = 0$;
- The closed-loop poles are the combination of regulator and estimator poles (via the separation principle).
- The compensator poles are at $\det(sI - A + B_u K_{ss} + L_{ss} C_y) = 0$ which are *neither* the estimator nor the regulator poles. In particular, the compensator might be unstable!

6.4: Reference tracking

- So far, we have discussed only regulator design—forcing the system state to zero as quickly as possible and keeping it there.
- This is the same as "tracking" the reference input 0.
- What if we want to track a more complicated reference signal?

Tracking constant reference inputs

TRACKING VIA COORDINATE TRANSLATION: A constant plant output can be generated if the plant state is constant.

- Generate the desired constant plant output with a specific state x_d and regulate around that state.

$$u(t) = -K[x(t) - x_d(t)].$$

- The desired state is chosen so that $C_y x_d = r$.
- Often, many different x_d will work in this equation.
- There is usually steady-state error when using this reference-tracking method. Form $\bar{x}(t) = x(t) - x_d(t)$.
- Then,

$$\dot{\bar{x}}(t) = \dot{x}(t) = A[\bar{x}(t) + x_d] + B_u[-K\bar{x}(t)] = (A - B_u K)\bar{x}(t) + Ax_d.$$

- Notice that this differential equation is not homogeneous—it has input Ax_d. So, in general, $\bar{x}(t) \not\to 0$.
- We have an additional constraint $Ax_d = 0$. This can be met only if x_d is in the null-space of A.

- In order for A to have a null-space, it must be singular—have an eigenvalue of 0. An integrator! (More on this later if A does not have an integrator).
- In general, A must have as many 0 eigenvalues as there are reference inputs.
- If $C_y x_d = r$ and $A x_d = 0$ then the plant model becomes

$$\dot{\bar{x}}(t) = A\bar{x}(t) + B_u u(t)$$

and standard LQR theory can be applied to generate $u(t)$.

FEEDFORWARD CONTROL: An alternate approach, which works even when A does not contain integrators, is

$$u(t) = -Kx(t) + K_r r$$

where $K_r r$ is feedforward control (like the $\bar{N}$ approach in ECE5520). The closed-loop system is

$$\begin{aligned}\dot{x}(t) &= (A - B_u K)x(t) + B_u K_r r\\ y(t) &= C_y x(t).\end{aligned}$$

- Assuming stability, using the final value theorem,

$$y(\infty) = -C_y(A - B_u K)^{-1} B_u K_r r.$$

The inverse exists because the closed-loop A_{cl} matrix is stable, so has no eigenvalues at 0.

- Set $r = y(\infty)$ and solve for K_r. For a square system,

$$K_r = -[C_y(A - B_u K)^{-1} B_u]^{-1}.$$

- K_r is the inverse of the dc-gain of the system.

- Not a robust method. Feed-forward control is more sensitive to plant modeling errors than feedback control.

EXAMPLE: Consider the state equation

$$\dot{x}(t) = \begin{bmatrix} 0 & 1 \\ -10 & -1 \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t)$$

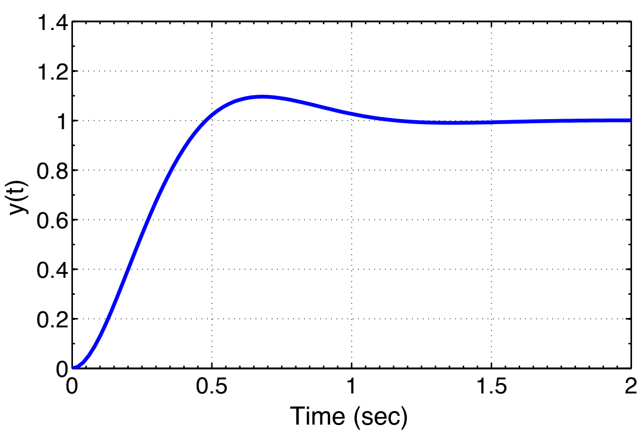
$$y(t) = \begin{bmatrix} 1 & 0 \end{bmatrix} x(t)$$

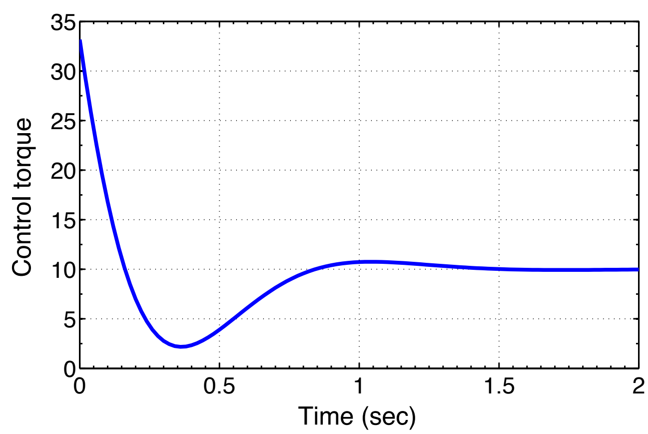
and the LQR cost function

$$J = \int_0^\infty y^2(t) + 0.001u^2(t)\,\mathrm{d}t.$$

- Using feedforward control,

$$u(t) = -\begin{bmatrix} 23 & 5.9 \end{bmatrix} x(t) + 33r.$$

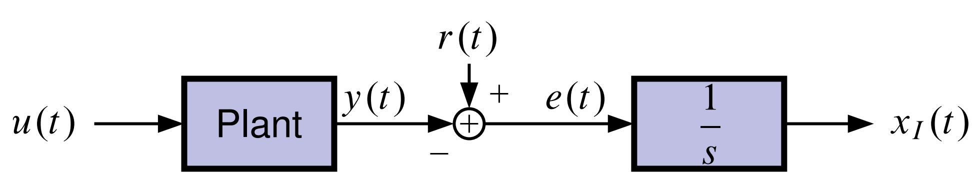




- Zero steady-state error has been achieved since our model is "perfect".

INTEGRAL CONTROL: We saw in the first method that constant reference tracking could be achieved if the plant contained integrator(s).

- If the plant does not contain integrators, we can add them:



- The integrator error equation is

$$\begin{aligned}\dot{x}_I(t) = e(t) &= r(t) - y(t)\\ &= r(t) - C_y x(t).\end{aligned}$$

  Note that there are as many integrators as there are reference inputs.

- We can include the integral state into our normal state-space form by augmenting the system dynamics

$$\left[\begin{array}{c}\dot{x}(t)\\ \hline \dot{x}_I(t)\end{array}\right] = \left[\begin{array}{c|c} A & 0 \\ \hline -C_y & 0\end{array}\right]\left[\begin{array}{c}x(t)\\ \hline x_I(t)\end{array}\right] + \left[\begin{array}{c}B_u\\ \hline 0\end{array}\right]u(t) + \left[\begin{array}{c}0\\ \hline I\end{array}\right]r(t).$$

- Note that the new "A" matrix has an open-loop eigenvalue at the origin. This corresponds to increasing the system type, and integrates out steady-state error.

- LQR may be used to generate a state feedback for the augmented plant. Use a cost function that penalizes the integral of the error

$$\begin{aligned}J &= \int_0^{t_f} x_I^T(t)x_I(t) + u^T(t)Ru(t)\,\mathrm{d}t\\ &= \int_0^{t_f} \left[\begin{array}{c|c} x^T(t) & x_I^T(t)\end{array}\right]\left[\begin{array}{c|c} 0 & 0\\ \hline 0 & I\end{array}\right]\left[\begin{array}{c}x(t)\\ \hline x_I(t)\end{array}\right] + u^T(t)Ru(t)\,\mathrm{d}t.\end{aligned}$$

- The control weighting matrix R is generated by trial-and-error since the physical meaning of the cost function is nebulous.

- The control law is,

$$u(t) = -K(t)\left[\begin{array}{c}x(t)\\ \hline x_I(t)\end{array}\right] = -\left[\begin{array}{c|c} K_x & K_I\end{array}\right]\left[\begin{array}{c}x(t)\\ \hline x_I(t)\end{array}\right].$$

- Generally, integral control is more robust and precise than the other two methods.

EXAMPLE: For the same system considered in the example for feedforward control, use integral control instead.

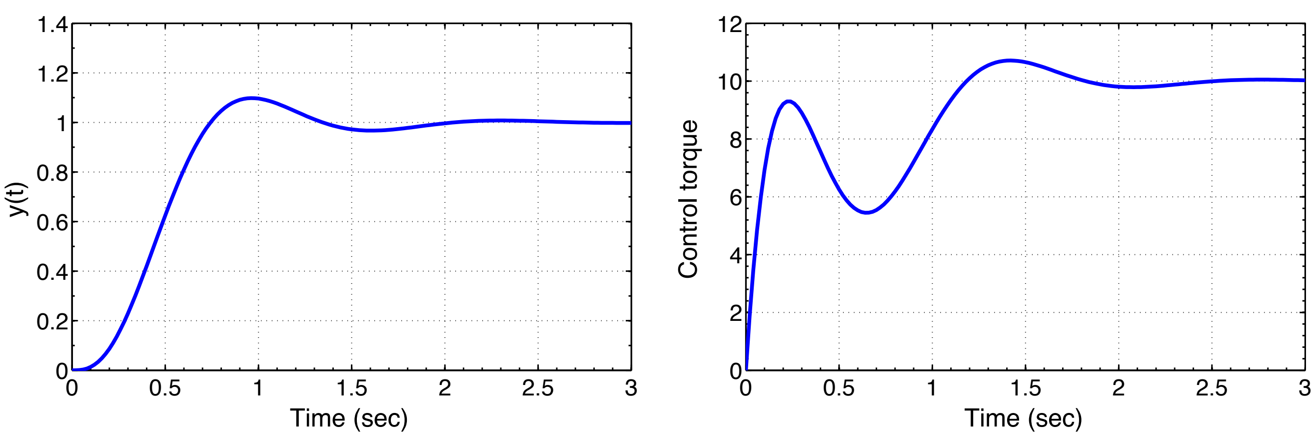
- The augmented system becomes

$$\begin{bmatrix} \dot{x}(t) \\ \hdashline \dot{x}_I(t) \end{bmatrix} = \left[\begin{array}{cc:c} 0 & 1 & 0 \\ -10 & -1 & 0 \\ \hdashline -1 & 0 & 0 \end{array}\right] \begin{bmatrix} x(t) \\ \hdashline x_I(t) \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \\ \hdashline 0 \end{bmatrix} u(t) + \begin{bmatrix} 0 \\ 0 \\ \hdashline 1 \end{bmatrix} r(t).$$

- Ignoring the reference input and applying LQR gives

$$u(t) = -\begin{bmatrix} 31 & 7 \end{bmatrix} x(t) + 100 \int_0^t \{r - y(t)\}\,\mathrm{d}t.$$

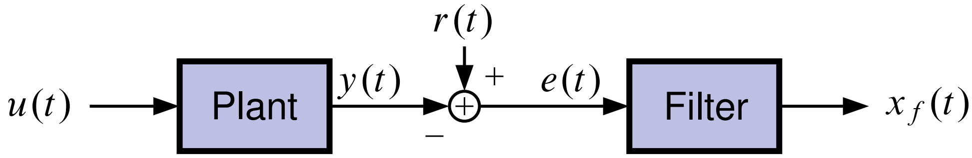
The control weighting $R = 0.0001$ was chosen by trial and error to yield desirable pole locations.



Tracking time-varying reference inputs

- The integration method of eliminating steady-state error may be generalized in order to track more complicated reference-inputs.
- Integration works with dc signals since it puts an open-loop pole at dc.

- Generally, increasing the open-loop gain in a certain band of frequencies will improve closed-loop tracking of those frequencies.
- The system dynamics are augmented by a filter:



- Zero steady-state tracking error is obtained when the poles in the Laplace transform of the reference input are also poles of the loop transfer function.
- This is called the *internal (signal) model principle.*

6.5: Designing for disturbance rejection

- LQG is designed to reject white-noise disturbances, but may be suboptimal when applied to nonwhite disturbances.
- Can modify LQG to reject nonwhite disturbance.

Feedforward disturbance cancellation with full information

- Assume that the state and disturbance are measured exactly.

$$\dot{x}(t) = Ax(t) + B_u u(t) + B_{w_o} w_o$$

$$y(t) = C_y x(t)$$

where w_o is a constant disturbance input.

- Use LQR to generate a state-feedback K matrix, ignoring disturbance input.
- Use

$$u(t) = -Kx(t) + K_w w_0.$$

- The closed-loop state equation is

$$\dot{x}(t) = (A - B_u K)x(t) + B_u K_w w_0 + B_{w_0} w_0.$$

- The steady-state output due to disturbance input is

$$y_w(\infty) = -C_y(A - B_u K)^{-1} B_{w_0} w_0$$

and the steady-state output due to the feedforward control is

$$y_u(\infty) = -C_y(A - B_u K)^{-1} B_u K_w w_0.$$

- These two terms should cancel to eliminate steady-state error due to disturbance

$$-C_y(A-B_uK)^{-1}B_{w_0}w_0 - C_y(A-B_uK)^{-1}B_uK_ww_0 = 0$$

which gives

$$K_w = -\{C_y(A-B_uK)^{-1}B_u\}^{-1}C_y(A-B_uK)^{-1}B_{w_0}.$$

- As with reference tracking, feedforward disturbance canceling is not very robust.

EXAMPLE: The angular velocity of a field-controlled dc motor is required to equal a set point regardless of the load torque. The state equation of the motor is

$$\begin{bmatrix} \dot{\omega}(t) \\ \dot{\tau}_M(t) \end{bmatrix} = \begin{bmatrix} -1 & 1 \\ 0 & -0.1 \end{bmatrix} \begin{bmatrix} \omega(t) \\ \tau_M(t) \end{bmatrix} + \begin{bmatrix} 0 \\ 0.1 \end{bmatrix} u(t) + \begin{bmatrix} 1 \\ 0 \end{bmatrix} \tau_L(t)$$

$$y(t) = \begin{bmatrix} 1 & 0 \end{bmatrix} x(t),$$

where ω is the angular velocity; τ_M is the motor torque and τ_L is the load torque.

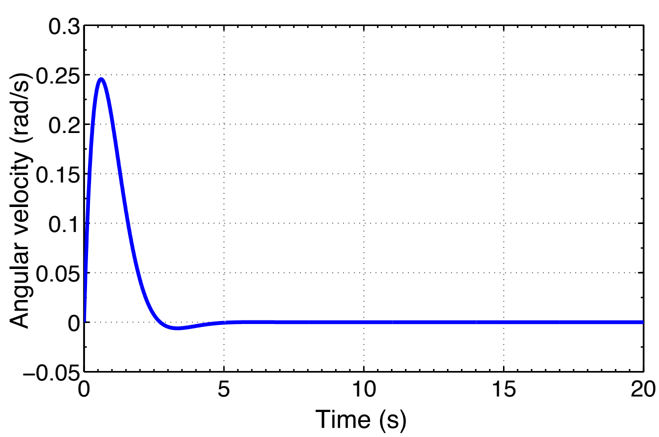
- The LQR cost function is

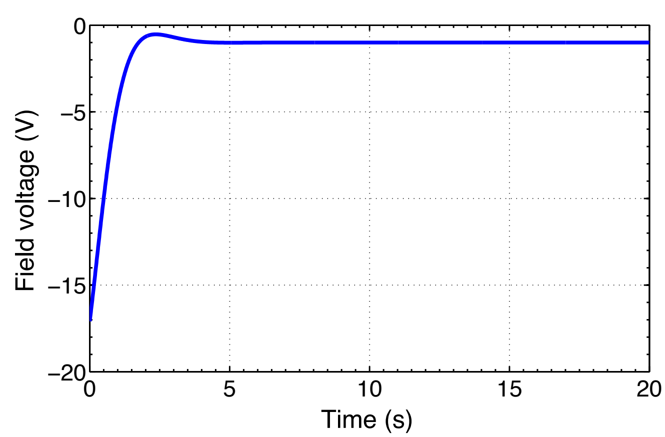
$$J = \int_0^\infty \omega^2(t) + 0.01u^2(t)\,\mathrm{d}t.$$

- We get

$$u(t) = -\begin{bmatrix} 14.6 & 16.1 \end{bmatrix} x(t) - 17.1\tau_L(t).$$

- Simulation of step load torque with set-point of zero:

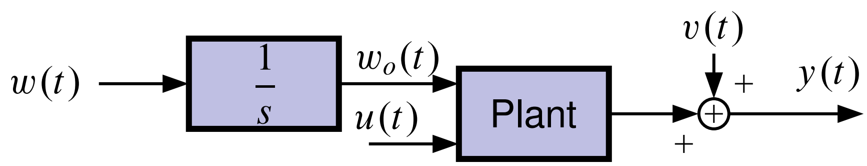




- Simulation output "perfect" due to exact model used.

Feedforward disturbance cancellation with partial information

- Usually we have only noisy partial state measurements.
- Estimate both the plant state and disturbance using a Kalman filter.
- To estimate disturbance, the plant model must be augmented with a filter that generates a nearly-constant disturbance.



- The integral has an artificial noise input that allows the disturbance input to change slowly.
- The plant state model for designing the Kalman filter is then

$$\begin{bmatrix} \dot{x}(t) \\ \dot{w}_0(t) \end{bmatrix} = \begin{bmatrix} A & B_{w_0} \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x(t) \\ w_0(t) \end{bmatrix} + \begin{bmatrix} B_u \\ 0 \end{bmatrix} u(t) + \begin{bmatrix} 0 \\ I \end{bmatrix} w(t)$$

$$y(t) = \begin{bmatrix} C_y & 0 \end{bmatrix} \begin{bmatrix} x(t) \\ w_0(t) \end{bmatrix} + v(t)$$

- The disturbance-canceling controller that uses the noisy partial measurements is then

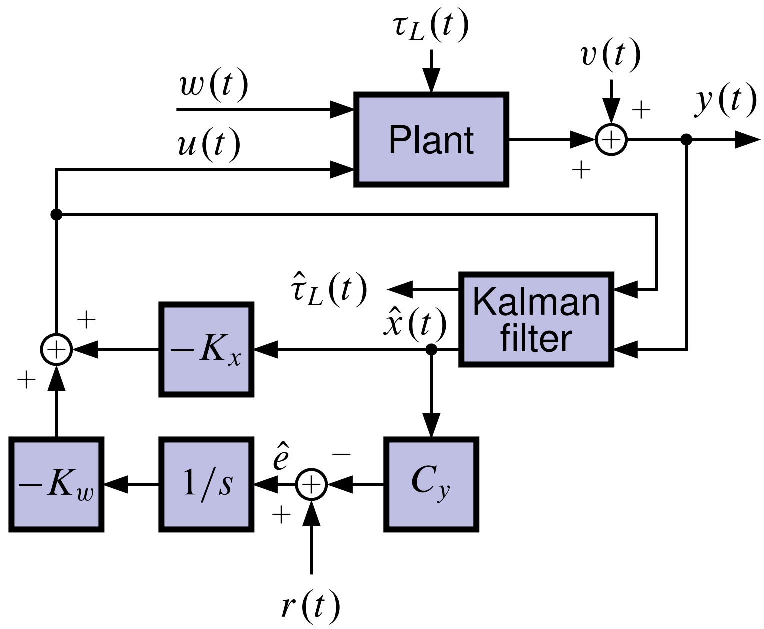
$$u(t) = -K\hat{x}(t) - K_w \hat{w}_0(t),$$

where K is calculated via LQR and K_w is calculated as before.

Integral control for disturbance canceling

- Integral control may also be used to eliminate steady-state errors due to step-like disturbances.

- The Kalman filter must estimate both the plant state and the disturbance input; if it estimates only the plant state it will be biased by the disturbance (the assumptions made when deriving the Kalman filter are violated by non white-noise disturbance).



EXAMPLE: The disturbance canceling example from before.

$$\begin{bmatrix} \dot{\omega}(t) \\ \dot{\tau}_M(t) \end{bmatrix} = \begin{bmatrix} -1 & 1 \\ 0 & -0.1 \end{bmatrix} \begin{bmatrix} \omega(t) \\ \tau_M(t) \end{bmatrix} + \begin{bmatrix} 0 \\ 0.1 \end{bmatrix} u(t)$$

$$+ \begin{bmatrix} 1 \\ 0 \end{bmatrix} \tau_L(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} w(t)$$

$$y(t) = \begin{bmatrix} 1 & 0 \end{bmatrix} x(t) + v(t),$$

- The augmented state for the purpose of designing a Kalman filter is

$$\begin{bmatrix} \dot{\omega}(t) \\ \dot{\tau}_M(t) \\ \dot{\tau}_L(t) \end{bmatrix} = \begin{bmatrix} -1 & 1 & 1 \\ 0 & -0.1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \omega(t) \\ \tau_M(t) \\ \tau_L(t) \end{bmatrix} + \begin{bmatrix} 0 \\ 0.1 \\ 0 \end{bmatrix} u(t)$$

$$+ \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} w(t) \\ w_T(t) \end{bmatrix}$$

$$y(t) = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} x(t) + v(t),$$

where $w_T(t)$ is the white noise that drives changes in the load torque. The noise spectral densities are

$$S_{\begin{bmatrix} w \\ w_T \end{bmatrix}} = \begin{bmatrix} 0.001 & 0 \\ 0 & 0.01 \end{bmatrix}; \quad S_v = 0.001,$$

where the spectral density for w_T was selected by trial and error.

- The resulting steady-state Kalman gain matrix is

$$L = \begin{bmatrix} 1.8 & 0.1 & 3.2 \end{bmatrix}^T.$$

- Incorporating this information, we have so far the dynamics of the estimator:

$$\begin{aligned} \dot{\hat{x}}(t) &= A\hat{x}(t) + Bu(t) + L(y(t) - \hat{y}(t)) \\ &= (A - LC)\hat{x}(t) + Bu(t) + Ly(t) \end{aligned}$$

$$\begin{bmatrix} \dot{\hat{\omega}}(t) \\ \dot{\hat{\tau}}_M(t) \\ \dot{\hat{\tau}}_L(t) \end{bmatrix} = \begin{bmatrix} -2.8 & 1 & 1 \\ -0.1 & -0.1 & 0 \\ -3.2 & 0 & 0 \end{bmatrix} \begin{bmatrix} \hat{\omega}(t) \\ \hat{\tau}_M(t) \\ \hat{\tau}_L(t) \end{bmatrix} + \begin{bmatrix} 0 \\ 0.1 \\ 0 \end{bmatrix} u(t) + \begin{bmatrix} 1.8 \\ 0.1 \\ 3.2 \end{bmatrix} y(t).$$

- For LQR integral state feedback, the augmented state equation is

$$\begin{bmatrix} \dot{\omega}(t) \\ \dot{\tau}_M(t) \\ \dot{x}_I(t) \end{bmatrix} = \begin{bmatrix} -1 & 1 & 0 \\ 0 & -0.1 & 0 \\ -1 & 0 & 0 \end{bmatrix} \begin{bmatrix} \omega(t) \\ \tau_M(t) \\ x_I(t) \end{bmatrix} + \begin{bmatrix} 0 \\ 0.1 \\ 0 \end{bmatrix} u(t) + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} r(t),$$

and the cost function is

$$J = \int_0^\infty x_I^2(t) + 0.0001u^2(t)\,\mathrm{d}t.$$

- For this cost function, we find that

$$K = \begin{bmatrix} 60 & 34 & -100 \end{bmatrix}.$$

- Note that the integrator state implements $\dot{\hat{x}}_I(t) = r(t) - \hat{\omega}(t)$ so we can augment the estimator dynamics:

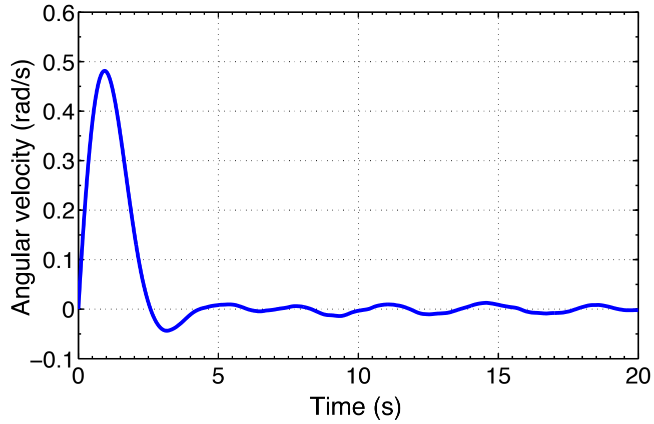
$$\begin{bmatrix} \dot{\hat{\omega}}(t) \\ \dot{\hat{\tau}}_M(t) \\ \dot{\hat{\tau}}_L(t) \\ \dot{\hat{x}}_I(t) \end{bmatrix} = \begin{bmatrix} -2.8 & 1 & 1 & 0 \\ -0.1 & -0.1 & 0 & 0 \\ -3.2 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \hat{\omega}(t) \\ \hat{\tau}_M(t) \\ \hat{\tau}_L(t) \\ \hat{x}_I(t) \end{bmatrix} + \begin{bmatrix} 1.8 \\ 0.1 \\ 3.2 \\ 0 \end{bmatrix} y(t) + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} r(t)$$

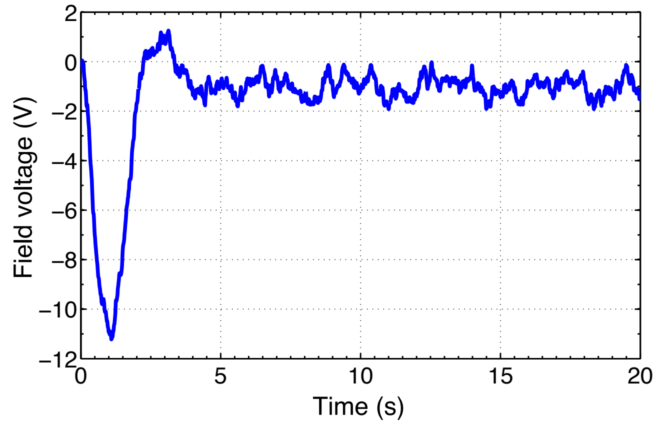
$$+ \begin{bmatrix} 0 \\ 0.1 \\ 0 \\ 0 \end{bmatrix} \begin{bmatrix} -60 & -34 & 0 & 100 \end{bmatrix} \begin{bmatrix} \hat{\omega}(t) \\ \hat{\tau}_M(t) \\ \hat{\tau}_L(t) \\ \hat{x}_I(t) \end{bmatrix}$$

$$\begin{bmatrix} \dot{\hat{\omega}}(t) \\ \dot{\hat{\tau}}_M(t) \\ \dot{\hat{\tau}}_L(t) \\ \dot{\hat{x}}_I(t) \end{bmatrix} = \begin{bmatrix} -2.8 & 1 & 1 & 0 \\ -6.1 & -3.5 & 0 & 10 \\ -3.2 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \hat{\omega}(t) \\ \hat{\tau}_M(t) \\ \hat{\tau}_L(t) \\ \hat{x}_I(t) \end{bmatrix} + \begin{bmatrix} 1.8 \\ 0.1 \\ 3.2 \\ 0 \end{bmatrix} y(t) + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} r(t)$$

$$u(t) = \begin{bmatrix} -60 & -34 & 0 & 100 \end{bmatrix} \begin{bmatrix} \hat{\omega}(t) \\ \hat{\tau}_M(t) \\ \hat{\tau}_L(t) \\ \hat{x}_I(t) \end{bmatrix}.$$

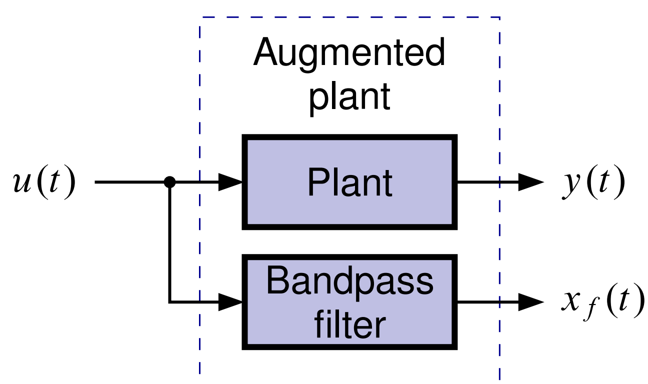
- Response to step load torque.





Frequency-shaped LQG

- One final "trick" we look at is used to shape the frequency response of the LQG system.
- For example, a system may have a resonance frequency we do not wish to excite. Standard control design would place a notch filter at this frequency.
- To perform frequency-shaped LQG design, we place a bandpass filter in parallel with the plant dynamics.



- This adds states that we wish to penalize.
- When designing a Kalman filter, we need only to estimate the true plant states as we have access to the filter states.

Appendix: Steady-state LQG cost

- We earlier stated (where $z(t) = C_z x(t)$)

$$\begin{aligned} J &= \lim_{t\to\infty} \mathbb{E}[z^T(t)Q_z z(t) + u^T(t)Ru(t)] \\ &= \operatorname{trace}\left\{\underline{\underline{\Sigma_{x,\text{ss}}}}\,\underline{L_{\text{ss}}S_v L_{\text{ss}}^T} + \underline{\Xi_{x,\text{ss}}}\,\underline{\underline{C_z^T Q_z C_z}}\right\} \\ &= \operatorname{trace}\left\{\underline{\underline{\Sigma_{x,\text{ss}}}}\,\underline{B_w S_w B_w^T} + \underline{\Xi_{x,\text{ss}}}\,\underline{\underline{K_{\text{ss}}^T R K_{\text{ss}}}}\right\}. \end{aligned}$$

- Here, we show that the second and third lines are the same (not yet sure how to show that the first and second are the same).
- Start with the second line:

$$\begin{aligned} &\operatorname{trace}\left\{\Sigma_{x,\text{ss}}L_{\text{ss}}S_v L_{\text{ss}}^T + \Xi_{x,\text{ss}}C_z^T Q_z C_z\right\} \\ &= \operatorname{trace}\left\{\Sigma_x\left(\underbrace{\Xi_x C_y^T S_v^{-1} C_y \Xi_x}_{A\Xi_x+\Xi_x A^T+B_w S_w B_w^T}\right) + \Xi_x C_z^T Q_z C_z\right\} \end{aligned}$$

- But,

$$C_z^T Q_z C_z = \underbrace{\Sigma_x B_u R^{-1} B_u^T \Sigma_x}_{K^T RK} - A^T\Sigma_x - \Sigma_x A$$

so

$$\begin{aligned} &\operatorname{trace}\left\{\Sigma_{x,\text{ss}}L_{\text{ss}}S_v L_{\text{ss}}^T + \Xi_{x,\text{ss}}C_z^T Q_z C_z\right\} \\ &= \operatorname{trace}\left\{\Sigma_x B_w S_w B_w^T + \Sigma_x(A\Xi_x + \Xi_x A^T) + \Xi_x\left(K^T RK - A^T\Sigma_x - \Sigma_x A\right)\right\} \\ &= \operatorname{trace}\left\{\Sigma_x B_w S_w B_w^T + \Xi_x K^T RK\right\} \\ &\qquad + \underbrace{\operatorname{trace}\left\{\Sigma_x(A\Xi_x + \Xi_x A^T)\right\} - \operatorname{trace}\left\{\Xi_x(A^T\Sigma_x + \Sigma_x A)\right\}}_{\text{equal}} \\ &= \operatorname{trace}\left\{\Sigma_x B_w S_w B_w^T + \Xi_x K^T RK\right\} \end{aligned}$$

where we used the knowledge that

$$A^T \Sigma_{x,\mathrm{ss}} + \Sigma_{x,\mathrm{ss}} A + C_z^T Q_z C_z - \Sigma_{x,\mathrm{ss}} B_u R^{-1} B_u^T \Sigma_{x,\mathrm{ss}} = 0$$

$$A \Xi_{x,\mathrm{ss}} + \Xi_{x,\mathrm{ss}} A^T + B_w S_w B_w^T - \Xi_{x,\mathrm{ss}} C_y^T S_v^{-1} C_y \Xi_{x,\mathrm{ss}} = 0$$

and $L_{\mathrm{ss}} = \Xi_{x,\mathrm{ss}} C_y^T S_v^{-1}$; $K_{\mathrm{ss}} = R^{-1} B_u^T \Sigma_{x,\mathrm{ss}}$.