OUTPUT-FEEDBACK CONTROL

7.1: Open-loop and closed-loop estimators

Open-loop estimators

- State feedback is impractical since we don't know the state!
- But, what if we can estimate the state?

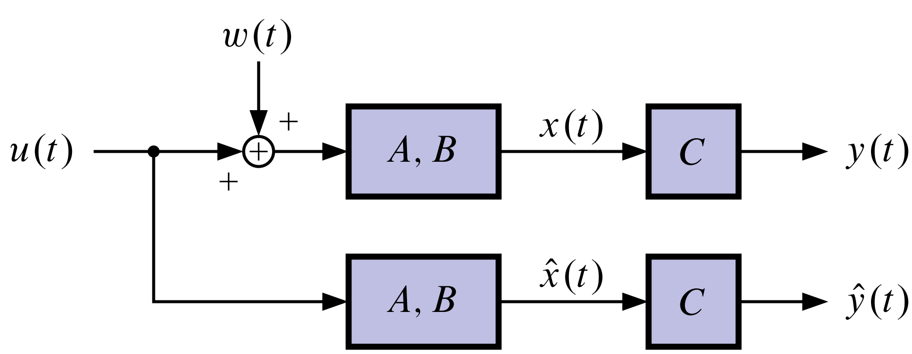
IDEA: Since we know system dynamics, simulate system in real-time.

- If $x(t)$ is the true state, $\hat{x}(t)$ is called the state estimate.
- We want $\hat{x}(t) = x(t)$, or at least $\hat{x}(t) \to x(t)$. How do we build this?
- To start, use our knowledge of the plant

$$\dot{x}(t) = Ax(t) + Bu(t).$$

- Let our state estimate be

$$\dot{\hat{x}}(t) = A\hat{x}(t) + Bu(t).$$



- This is called an "open-loop estimator."
- Some troubling issues:

- We need our model to be very accurate!
- What do we use for $\hat{x}(0)$?
- What does disturbance do?

■ Let's analyze our open-loop estimator by examining the state-estimate error

$$\tilde{x}(t) = x(t) - \hat{x}(t).$$

■ We want $\tilde{x}(t) = 0$. For our estimator,

$$\begin{aligned}\dot{\tilde{x}}(t) &= \dot{x}(t) - \dot{\hat{x}}(t)\\ &= Ax(t) + Bu(t) - A\hat{x}(t) - Bu(t)\\ &= A\tilde{x}(t).\end{aligned}$$

■ So,

$$\tilde{x}(t) = e^{At}\tilde{x}(0).$$

■ Hence, $\hat{x}(t) \to x(t)$ if A is stable!

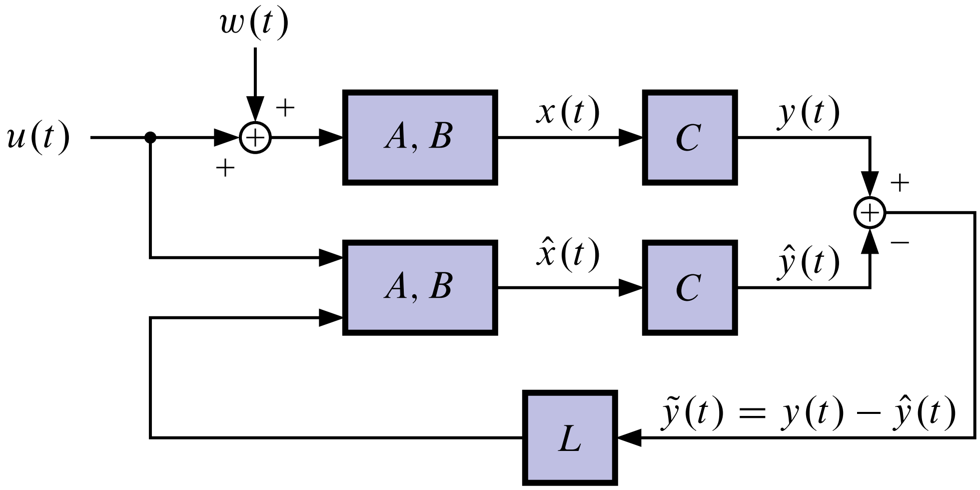
- (This is not too impressive though since $\hat{x}(t) \to x(t)$ because both $\hat{x}(t)$ and $x(t)$ go to zero).

■ We need to improve our estimator:

- Speed up convergence.
- Reduce sensitivity to model uncertainties.
- Counteract disturbances.
- Have convergence even when A is unstable.

■ Key Point: Use feedback of measured output.

Closed-loop estimator



- This is called a "closed-loop estimator" (assuming $\hat{y} = C\hat{x}$).
- Note: If $L = 0$ we have an open-loop estimator.

$$\dot{\hat{x}}(t) = A\hat{x}(t) + Bu(t) + L\left(y(t) - \hat{y}(t)\right).$$

- Let's look at the error.

$$\begin{aligned}
\dot{\tilde{x}}(t) &= \dot{x}(t) - \dot{\hat{x}}(t) \\
&= Ax(t) + Bu(t) - A\hat{x}(t) - Bu(t) - L\left(y(t) - C\hat{x}(t)\right) \\
&= A\tilde{x}(t) - L\left(Cx(t) - C\hat{x}(t)\right) \\
&= (A - LC)\,\tilde{x}(t); \\
\tilde{x}(t) &= e^{(A-LC)t}\tilde{x}(0),
\end{aligned}$$

or, $\hat{x}(t) \to x(t)$ if $A - LC$ is stable, for any value of $\hat{x}(0)$ and any $u(t)$, whether or not A is stable.

- In fact, we can look at the dynamics of the state estimate error to quantitatively evaluate how $\hat{x}(t) \to x(t)$.

$$\dot{\tilde{x}}(t) = (A - LC)\,\tilde{x}(t)$$

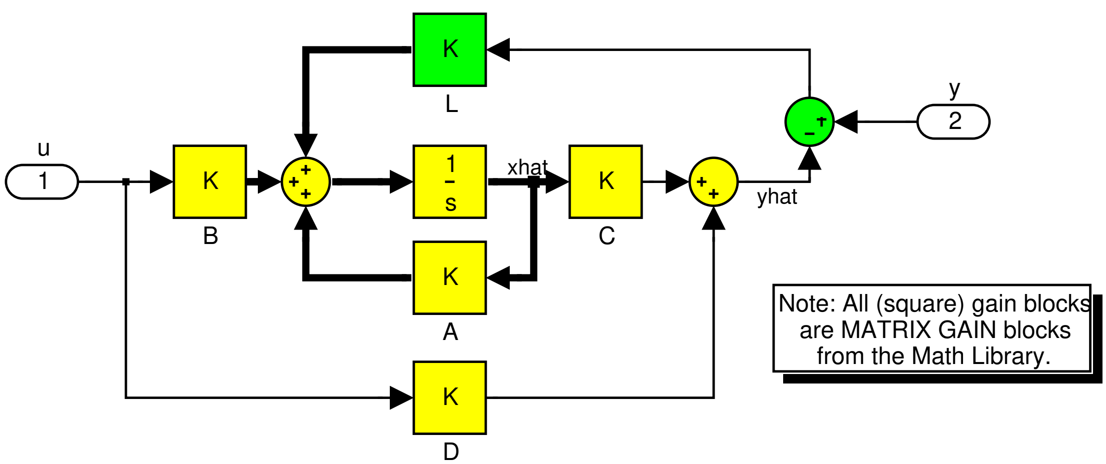
has dynamics related to the roots of the characteristic equation

$$\chi_{ob}(s) = \det(sI - A + LC) = 0.$$

- So, for our estimator, we specify the convergence rate of $\hat{x}(t) \to x(t)$ by choosing desired pole locations: Choose L such that

$$\chi_{ob,des}(s) = \det(sI - A + LC).$$

- This is called the "observer gain problem".
- In Simulink, the following diagram implements a closed-loop estimator. The output is `xhat`.



Integration dynamics

- It can be helpful to add integration dynamics to the observer to ensure that steady-state errors go to zero.
- Instead of comparing $\hat{y}(t)$ to $y(t)$, we compare the integral of $\hat{y}(t)$ to the integral of $y(t)$.
- To do so, we define a new "state" that integrates the output

$$x_i(t) = \int_0^t y(\tau)\,d\tau \qquad \text{or} \qquad \dot{x}_i(t) = y(t) = Cx(t) + Du(t).$$

- Then, we can combine the true system dynamics with the integration dynamics

$$\underbrace{\begin{bmatrix} \dot{x}_i(t) \\ \dot{x}(t) \end{bmatrix}}_{\dot{x}_a(t)} = \underbrace{\begin{bmatrix} 0 & C \\ 0 & A \end{bmatrix}}_{A_a} \underbrace{\begin{bmatrix} x_i(t) \\ x(t) \end{bmatrix}}_{x_a(t)} + \underbrace{\begin{bmatrix} D \\ B \end{bmatrix}}_{B_a} u(t)$$

$$y(t) = \underbrace{\begin{bmatrix} I & 0 \end{bmatrix}}_{C_a} \underbrace{\begin{bmatrix} x_i(t) \\ x(t) \end{bmatrix}}_{x_a(t)}.$$

- The estimator then is designed such that

$$\begin{aligned} \dot{\hat{x}}_a(t) &= A_a\hat{x}_a(t) + B_a u(t) + L_a(x_i(t) - \hat{x}_i(t)) \\ &= A_a\hat{x}_a(t) + B_a u(t) + L_a C_a(x_a(t) - \hat{x}_a(t)). \end{aligned}$$

- The estimation error has dynamics

$$\begin{aligned} \dot{\tilde{x}}(t) &= (A_a x_a(t) + B_a u(t)) - (A_a\hat{x}_a(t) + B_a u(t) + L_a C_a\tilde{x}_a(t)) \\ &= (A_a - L_a C_a)\tilde{x}_a(t). \end{aligned}$$

- Therefore, we need to design an L_a matrix such that $(A_a - L_a C_a)$ has nice properties.
- As a check, is the augmented system even observable?
- Notice that the new observability matrix is

$$\mathcal{O}_a = \begin{bmatrix} C_a \\ C_a A_a \\ C_a A_a^2 \\ \vdots \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & C \\ 0 & CA \\ \vdots & \vdots \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & \mathcal{O} \end{bmatrix}.$$

- So, if the original system is observable, the integrator-augmented system is also observable.

7.2: The observer gain design problem

- We would like a method for computing the observer gain vector L given a set of desired closed-loop observer gains $\chi_{ob,des}(s)$.

Bass–Gura inspired method

- We want a specific characteristic equation for $A - LC$.
- Suppose $\{C, A\}$ is observable. Put $\{A, B, C, D\}$ in *observer canonical form.*
- That is, find T such that

$$T^{-1}AT = A_o = \begin{bmatrix} -a_1 & 1 & & 0 \\ -a_2 & & \ddots & \vdots \\ \vdots & & & 1 \\ -a_n & 0 & \cdots & 0 \end{bmatrix} \quad \text{and} \quad CT = C_o = \begin{bmatrix} 1 & 0 & \cdots & 0 \end{bmatrix}$$

where $\det(sI - A) = s^n + a_1 s^{n-1} + \cdots + a_n$.

- Apply feedback L_o to the observer realization to end up with $A_o - L_o C_o$.
- Note, $L_o = \begin{bmatrix} l_1 & \cdots & l_n \end{bmatrix}^T$, so

$$L_o C_o = \begin{bmatrix} l_1 & 0 & \cdots & 0 \\ l_2 & & & \\ \vdots & & & \\ l_n & 0 & \cdots & 0 \end{bmatrix}.$$

- Useful because characteristic equation obvious.

$$A_o - L_oC_o = \begin{bmatrix} -(a_1+l_1) & 1 & & 0 \\ -(a_2+l_2) & & \ddots & \vdots \\ \vdots & & & 1 \\ -(a_n+l_n) & 0 & \cdots & 0 \end{bmatrix},$$

still in observer form!

- After feedback with L_o the characteristic equation is

$$\det(sI - A_o + L_oC_o) = s^n + (a_1+l_1)s^{n-1} + \cdots + (a_n+l_n).$$

- If we set $l_1 = \alpha_1 - a_1, \ldots, l_n = \alpha_n - a_n$ then we get the desired characteristic polynomial.

- Now, we transform back to the original realization

$$\begin{aligned} \det(sI - A_o + L_oC_o) &= \det(sI - TA_oT^{-1} + TL_oC_oT^{-1}) \\ &= \det(sI - A + TL_oC). \end{aligned}$$

- So, if we use feedback

$$\begin{aligned} L &= TL_o \\ &= T\begin{bmatrix} (\alpha_1 - a_1) & \cdots & (\alpha_n - a_n) \end{bmatrix}^T \end{aligned}$$

we will have the desired characteristic polynomial.

- One remaining question: What is T? We know $T = \mathcal{O}^{-1}\mathcal{O}_o$ and

$$\mathcal{O}_o = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ a_1 & 1 & & \vdots \\ \vdots & & \ddots & 0 \\ a_{n-1} & \cdots & a_1 & 1 \end{bmatrix}^{-1}$$

- So,

$$L = \mathcal{O}^{-1}\begin{bmatrix} 1 & 0 & \cdots & 0 \\ a_1 & 1 & & \vdots \\ \vdots & & \ddots & 0 \\ a_{n-1} & \cdots & a_1 & 1 \end{bmatrix}^{-1}\begin{bmatrix} (\alpha_1 - a_1) \\ (\alpha_2 - a_2) \\ \vdots \\ (\alpha_n - a_n) \end{bmatrix}.$$

- This is the *Bass–Gura formula* for L.

Duality!

- This is another example of duality. Notice that if you substitute $A \leftarrow A^T$, $B \leftarrow C^T$, $L \leftarrow K^T$ in the Bass–Gura controller design procedure, you will get exactly this equation. Why?
- Notice that we wish to design certain pole locations:

$$\text{eig}(A - LC) = \text{eig}(A^T - C^T L^T)$$

which has the same form as the controller design problem

$$\text{eig}(A - BK)$$

if we make the above substitutions.

The Ackermann-inspired method

- The observer-gain problem is "dual" to the controller-gain problem. Replace a controller-design procedure's inputs $A \leftarrow A^T$, $B \leftarrow C^T$, $K \leftarrow L^T$. Design the controller. Then, L will be the observer gains.
- Recall, $K = \begin{bmatrix} 0 & \cdots & 1 \end{bmatrix}\mathcal{C}(A,B)^{-1}\chi_d(A)$.
- By duality,

$$L = \Big[\begin{bmatrix} 0 & \cdots & 1 \end{bmatrix}\mathcal{C}(A,B)^{-1}\chi_d(A)\Big]^T_{\substack{A\leftarrow A^T \\ B\leftarrow C^T}}.$$

$$= \chi_d^T(A^T)\begin{bmatrix} C^T & A^TC^T & \cdots & A^{n-1^T}C^T \end{bmatrix}^{-T}\begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}$$

$$= \chi_d(A)\begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix}^{-1}\begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}$$

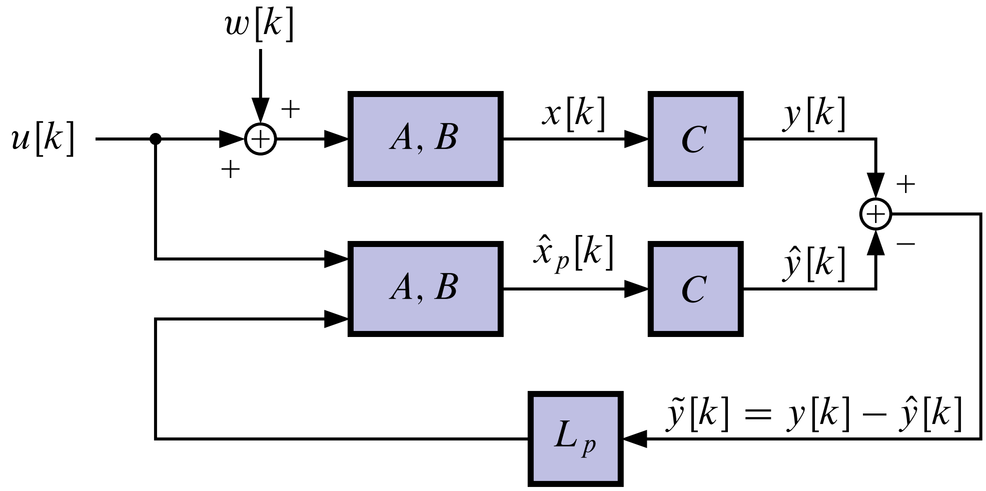
$$= \chi_d(A)\mathcal{O}(C,A)^{-1}\begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}.$$

- In MATLAB,

```
L=acker(A',C',poles)';
L=place(A',C',poles)';
```

7.3: Discrete-time prediction estimator

- In discrete-time, we can do the same thing. The picture looks like



- The update equation for the closed-loop (prediction) estimator is

$$\hat{x}_p[k+1] = A\hat{x}_p[k] + Bu[k] + L_p\left(y[k] - C\hat{x}_p[k]\right).$$

- The prediction estimation error can likewise be written as

$$\tilde{x}[k+1] = \left(A - L_pC\right)\tilde{x}[k],$$

which has dynamics related to the roots of the characteristic equation

$$\chi_{ob}(z) = \det\left(zI - A + L_pC\right) = 0.$$

- We specify the convergence rate of $\hat{x}_p[k] \to x[k]$ by choosing desired pole locations: Choose L_p such that

$$\chi_{ob,des}(z) = \det\left(zI - A + L_pC\right).$$

EXAMPLE: Let $G(s) = \dfrac{1}{s^2}$ and measure $y[k]$. Let

$$x[k] = \begin{bmatrix} y[k] \\ \dot{y}[k] \end{bmatrix},$$

then

$$A = \begin{bmatrix} 1 & T \\ 0 & 1 \end{bmatrix}, \qquad C = \begin{bmatrix} 1 & 0 \end{bmatrix}.$$

- We desire $\tilde{x}[k]$ to decay with poles $z_p = 0.8 \pm 0.2j$, or

$$\chi_{ob,des}(z) = z^2 - 1.6z + 0.68.$$

$$\begin{aligned}\det(zI - A + L_pC) &= \det\begin{bmatrix} z - 1 + l_1 & -T \\ l_2 & z - 1 \end{bmatrix} \\ &= z^2 + z(l_1 - 2) + l_2T - l_1 + 1.\end{aligned}$$

- So,

$$\begin{aligned} l_1 - 2 &= -1.6 \\ l_2T - l_1 + 1 &= 0.68 \end{aligned}$$

or

$$L_p = \begin{bmatrix} 0.4 \\ 0.08/T \end{bmatrix}.$$

- The estimator is

$$\begin{aligned}\hat{x}_p[k+1] &= A\hat{x}_p[k] + Bu[k] + L_p\left(y[k] - C\hat{x}_p[k]\right) \\ &= \left(A - L_pC\right)\hat{x}_p[k] + Bu[k] + L_py[k],\end{aligned}$$

or

$$\begin{bmatrix} \hat{y}_p[k+1] \\ \hat{\dot{y}}_p[k+1] \end{bmatrix} = \begin{bmatrix} 0.6 & T \\ \dfrac{-0.08}{T} & 1 \end{bmatrix}\begin{bmatrix} \hat{y}_p[k] \\ \hat{\dot{y}}_p[k] \end{bmatrix} + \begin{bmatrix} \dfrac{T^2}{2} \\ T \end{bmatrix}u[k] + \begin{bmatrix} 0.4 \\ \dfrac{0.08}{T} \end{bmatrix}y[k].$$

- In general, we can arbitrarily select the prediction estimator poles iff $\{C, A\}$ is observable.

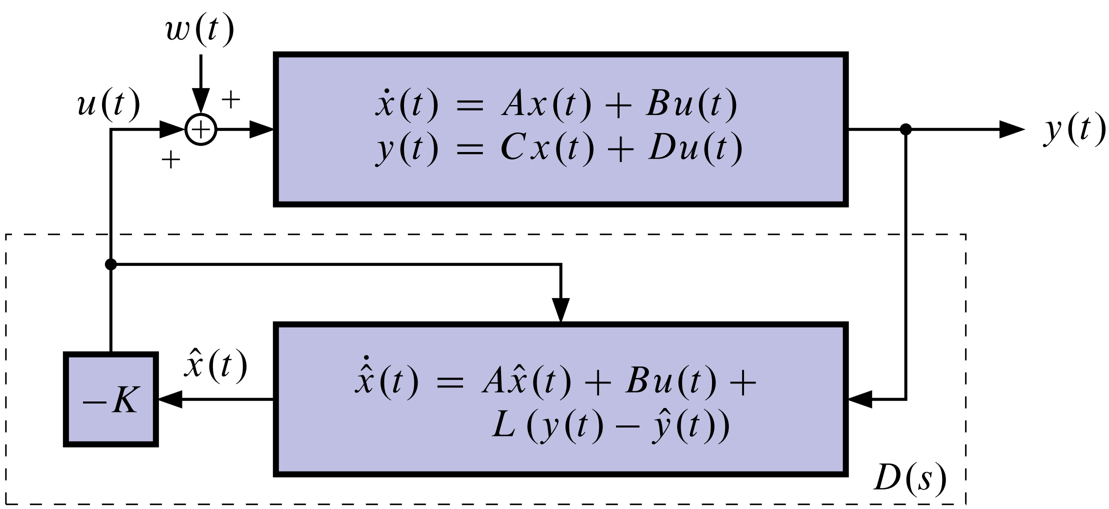
- The observer-gain problem is "dual" to the controller-gain problem. Replace a controller-design procedure's inputs $A \leftarrow A^T$, $B \leftarrow C^T$, $K \leftarrow L_p^T$. Design the controller. Then, L_p will be the observer gains.

7.4: Compensator design: Separation principle

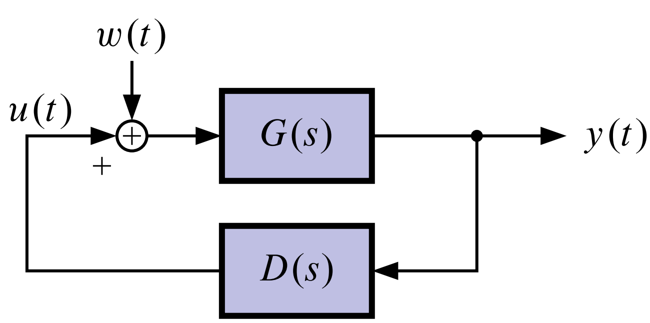
- Now that we have a structure to estimate the state $x(t)$, let's feed back $\hat{x}(t)$ to control the plant. That is,

$$u(t) = r(t) - K\hat{x}(t),$$

where K was designed assuming that $u(t) = r(t) - Kx(t)$. Is this going to work? How risky is it to interconnect two well-behaved, stable systems? (Assume $r(t) = 0$ for now).



- What is inside the dotted line is equivalent to our classical design compensator:



where $G(s)$ is described by

$$\dot{x}(t) = Ax(t) + Bu(t)$$

$$y(t) = Cx(t) + Du(t),$$

and $D(s)$ is described by

$$\begin{aligned}\dot{\hat{x}}(t) &= A\hat{x}(t) + Bu(t) + L\left(y(t) - \hat{y}(t)\right)\\ u(t) &= -K\hat{x}(t),\end{aligned}$$

so that we have

$$\begin{aligned}\dot{x}(t) &= Ax(t) - BK\hat{x}(t)\\ \dot{\hat{x}}(t) &= (A - BK)\,\hat{x}(t) + L\left(Cx(t) + Du(t) - C\hat{x}(t) - Du(t)\right)\\ &= (A - BK - LC)\,\hat{x}(t) + LCx(t).\end{aligned}$$

- Our combined closed-loop-system state equations are

$$\begin{bmatrix} \dot{x}(t) \\ \hline \dot{\hat{x}}(t) \end{bmatrix} = \left[\begin{array}{c|c} A & -BK \\ \hline LC & A - BK - LC \end{array}\right] \begin{bmatrix} x(t) \\ \hline \hat{x}(t) \end{bmatrix},$$

or, in terms of $\tilde{x}(t)$,

$$\begin{aligned}\dot{\tilde{x}}(t) &= \dot{x}(t) - \dot{\hat{x}}(t)\\ &= Ax(t) - BK\hat{x}(t) - LCx(t) - (A - BK - LC)\,\hat{x}(t)\\ &= (A - LC)\,\tilde{x}(t)\\ \dot{x}(t) &= Ax(t) - BK\left(x(t) - \tilde{x}(t)\right),\end{aligned}$$

or,

$$\begin{bmatrix} \dot{x}(t) \\ \hline \dot{\tilde{x}}(t) \end{bmatrix} = \left[\begin{array}{c|c} A - BK & BK \\ \hline 0 & A - LC \end{array}\right] \begin{bmatrix} x(t) \\ \hline \tilde{x}(t) \end{bmatrix}.$$

- Note, we have simply changed coordinates:

$$\begin{bmatrix} x(t) \\ \hline \hat{x}(t) \end{bmatrix} = \underbrace{\left[\begin{array}{c|c} I & 0 \\ \hline I & -I \end{array}\right]}_{T} \begin{bmatrix} x(t) \\ \hline \tilde{x}(t) \end{bmatrix}.$$

- With this change of coordinates, we may easily find the closed-loop poles of the combined state-feedback/estimator system: The eigenvalues of a block-upper-triangular 2×2 matrix are the collection of the eigenvalues of the two diagonal blocks.
- Therefore, the $2n$ poles are the n eigenvalues of $A - BK$ combined with the n eigenvalues of $A - LC$.
- *But*, we *designed* the n eigenvalues of $A - BK$ to give good (stable) state-feedback performance, and we *designed* the n eigenvalues of $A - LC$ to give good (stable) estimator performance. Therefore, the closed-loop system is also stable.
- This is such an astounding conclusion that it has been given a special name: The “Separation Principle”.
- It implies that we can design a compensator in two steps:

  1. Design state-feedback assuming the state $x(t)$ is available.
  2. Design the estimator to estimate the state as $\hat{x}(t)$.

  and, that $u(t) = -K\hat{x}(t)$ works!

7.5: The compensator: Continuous- and discrete-time

The continuous-time compensator

- What is $D(s)$? We know the closed-loop poles, which are the roots of $1 + D(s)G(s) = 0$, and we know the plant's open-loop poles. What are the dynamics of $D(s)$ itself?
- Start with a state-space representation of $D(s)$

$$\dot{\hat{x}}(t) = (A - BK - LC)\hat{x}(t) + Ly(t) - LDu(t)$$
$$u(t) = -K\hat{x}(t)$$

so that

$$D(s) = \frac{U(s)}{Y(s)} = -K(sI - A + BK + LC - LDK)^{-1}L.$$

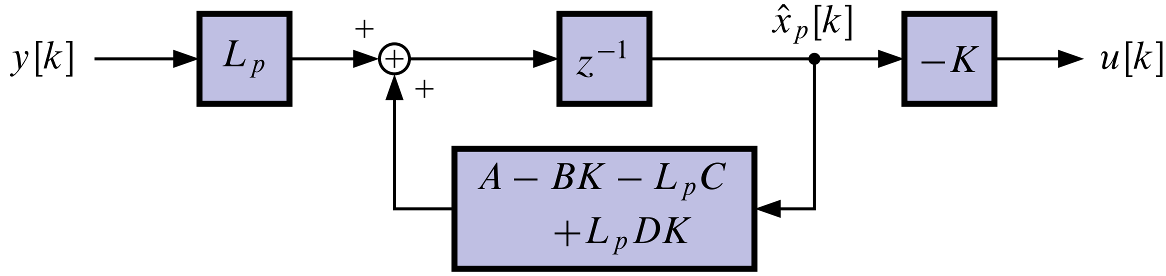
- The poles of $D(s)$ are the roots of $\det(sI - A + BK + LC - LDK) = 0.$
- These are neither the controller poles *nor* the estimator poles.
- $D(s)$ may be unstable even if the plant is stable and the closed-loop system is stable.

The discrete-time compensator

- The results in discrete-time are essentially the same.

$$D(z) = \frac{U(z)}{Y(z)} = -K\left(zI - A + BK + L_pC - L_pDK\right)^{-1} L_p.$$

- The poles of $D(z)$ are the roots of $\det\left(zI - A + BK + L_pC - L_pDK\right) = 0.$
- The compensator has the block-diagram:



- Note that the control signal $u[k]$ only contains information about $y[k-1]$, $y[k-2]$, ..., not about $y[k]$.
- So, our compensator is not taking advantage of the most current measurement. (More on this later).

```
for loop:
  u=-K*xhatp;
  % now, wait until sample time...
  A2D(y);
  D2A(u);  % u depends on past y, not current.
  xhatp=(A-B*K-Lp*C+Lp*D*K)*xhatp+Lp*y;
end loop.
```

EXAMPLE: $G(s) = \dfrac{1}{s^2}$; $T = 1.4$ seconds.

- Design a compensator such that response is dominated by the poles $z_p = 0.8 \pm 0.25j$.
- System description

$$A = \begin{bmatrix} 1 & 1.4 \\ 0 & 1 \end{bmatrix}, \qquad B = \begin{bmatrix} 1 \\ 1.4 \end{bmatrix}, \qquad C = \begin{bmatrix} 1 & 0 \end{bmatrix}, \qquad D = [0],$$

- Control design: Find K such that $\det(zI - A + BK) = z^2 - 1.6z + 0.7$. This leads to $K = \begin{bmatrix} 0.05 & 0.25 \end{bmatrix}$.

- Estimator design: Choose poles to be faster than control roots. Let's radially project $z_p = 0.8 \pm 0.25j$ toward the origin, or $z_{pe} = 0.8 z_p$.
- Find L_p such that $\det\left(zI - A + L_p C\right) = z^2 - 1.28z + 0.45$. This leads to $L_p = \begin{bmatrix} 0.72 \\ 0.12 \end{bmatrix}$.
- So, our compensator is

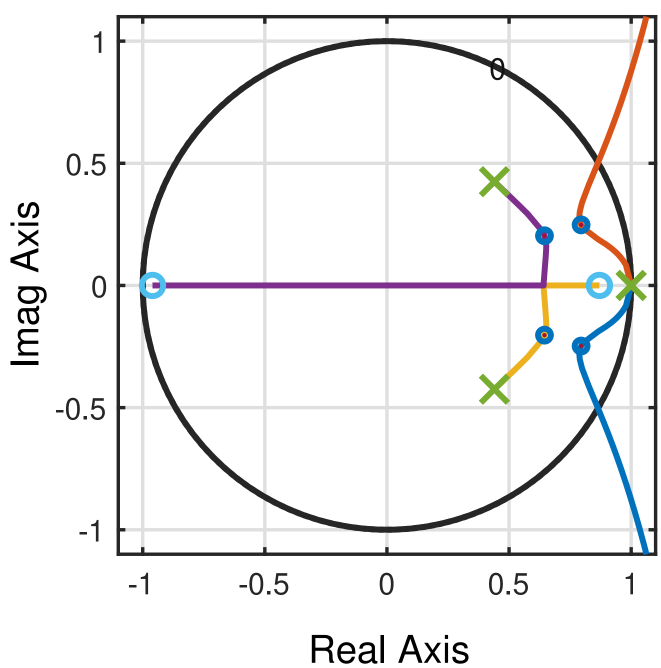
$$\hat{x}_p[k+1] = \begin{bmatrix} 0.23 & 1.16 \\ -0.19 & 0.65 \end{bmatrix} \hat{x}_p[k] + \begin{bmatrix} 0.72 \\ 0.12 \end{bmatrix} y[k]$$

$$u[k] = \begin{bmatrix} -0.05 & -0.25 \end{bmatrix} \hat{x}_p[k],$$

or,

$$\begin{aligned} D(z) &= -0.68 \frac{z - 0.87}{z - 0.44 \pm 0.423j} \\ &= -0.68 \frac{z - 0.87}{z^2 - 0.88z + 0.374}. \end{aligned}$$

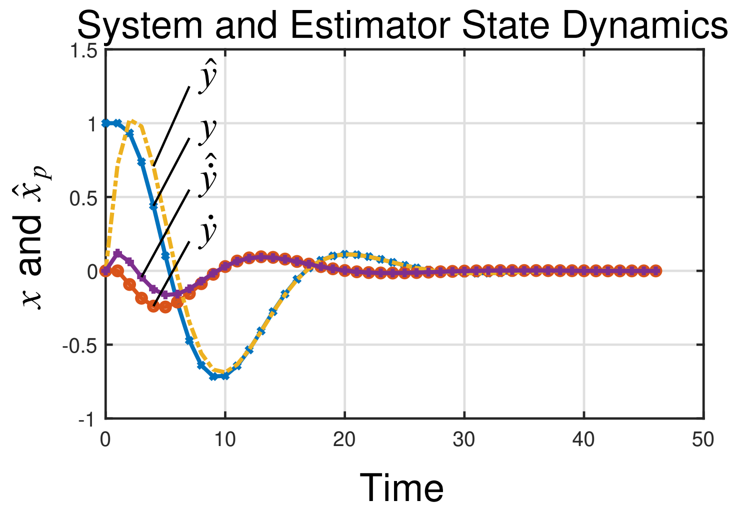
- Let's see the root locus of $D(z)G(z)$. (Note, since $D(z)$ has a negative sign, must use $0°$ locus).

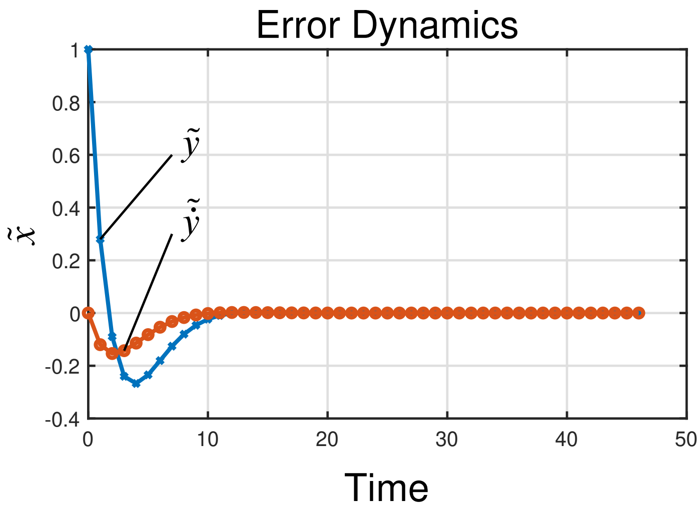


- Using the state representation of the plant and our compensator, we can simulate the closed-loop system to find $x[k]$, $\hat{x}_p[k]$, $\tilde{x}[k]$, and $u[k]$.

EXAMPLE: In MATLAB,

```
Q=[A  -B*K; L*C  A-B*K-L*C];
% get resp. of u, x, xhat to init. condition
[u,X]=dinitial(Q,zeros([4,1]),[0 0 -K],0,x0);
% get the estimate error.
xtilde=X(:,1:2)-X(:,3:4);
```





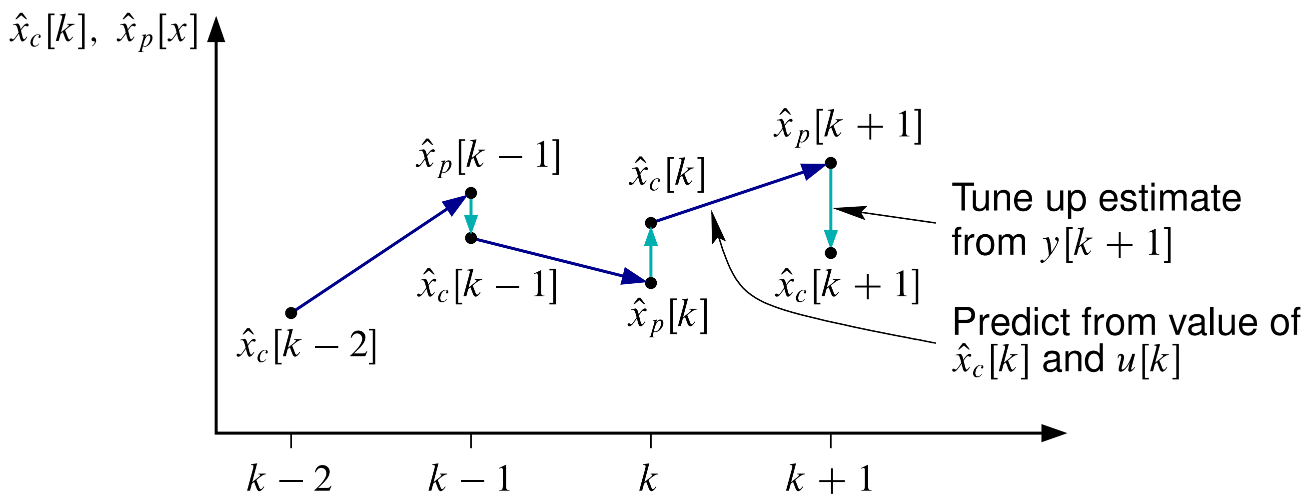
7.6: Current estimator/ compensator

- Using the prediction estimator to build our compensator, we found that the control effort $u[k]$ did not utilize the most current measurement $y[k]$, only past values of y: $y[k-1]$, $y[k-2]\ldots$

IMPLICATION: Cannot implement $D(z) = \dfrac{z-a}{z-b}$ (either lead or lag) to control a system since

$$G(z) = \frac{U(z)}{Y(z)} = 1 + \frac{b-a}{z-b}.$$

- That is, $u[k] = f(y[k], \ldots)$. To develop the current estimate $\hat{x}_c[k]$, consider "tuning-up" our prediction estimate $\hat{x}_p[k]$ at time k with $y[k]$.



$\hat{x}_p[k]$: Estimate just before measurement at k

$\hat{x}_c[k]$: Estimate just after measurement at k.

IMPLEMENTATION:

- "Time update": Predict new state from old estimate, system dynamics

$$\hat{x}_p[k] = A\hat{x}_c[k-1] + Bu[k-1].$$

- "Measurement update": Measure output, use it to update the estimate

$$\hat{x}_c[k] = \hat{x}_p[k] + L_c\left(y[k] - C\hat{x}_p[k]\right).$$

- L_c is called the "current estimator gain."

- Questions: How are $\hat{x}_c[k]$ and $\hat{x}_p[k]$ and how are L_c and L_p related?
- What is the $\hat{x}_c[k]$ recursion relation?

$$\begin{aligned}
\hat{x}_c[k+1] &= \hat{x}_p[k+1] + L_c \underbrace{\left(y[k+1] - C\hat{x}_p[k+1]\right)}_{\text{error}} \\
&= (I - L_cC)\,\hat{x}_p[k+1] + L_c y[k+1] \\
&= (I - L_cC)\,(A\hat{x}_c[k] + Bu[k]) + L_c y[k+1] \\
&= (A - L_cCA)\,\hat{x}_c[k] + (B - L_cCB)\,u[k] + L_c y[k+1].
\end{aligned}$$

- So,

$$\hat{x}_c[k] = f(\hat{x}_c[k-1], u[k-1], y[k]).$$

- What about the estimate error?

$$\begin{aligned}
\tilde{x}[k+1] &= x[k+1] - \hat{x}_c[k+1] \\
&= x[k+1] - \left(\hat{x}_p[k+1] + L_cCx[k+1] - L_cC\hat{x}_p[k+1]\right) \\
&= (I - L_cC)\left(x[k+1] - \hat{x}_p[k+1]\right) \\
&= (I - L_cC)\,(Ax[k] + Bu[k] - A\hat{x}_c[k] - Bu[k]) \\
&= (A - L_cC\underbrace{A}_{\text{new!}})\tilde{x}[k].
\end{aligned}$$

- So, the current estimator error has dynamics related to the roots of

$$\det(zI - A + L_cCA) = 0.$$

- What about the $\hat{x}_p$ recursion relation?

$$\hat{x}_p[k+1] = A\hat{x}_c[k] + Bu[k]$$

$$\begin{aligned}
&= A\left(\hat{x}_p[k] + L_c\left(y[k] - C\hat{x}_p[k]\right)\right) + Bu[k] \\
&= A\hat{x}_p[k] + Bu[k] + AL_c\left(y[k] - C\hat{x}_p[k]\right).
\end{aligned}$$

- Compare this equation with the prediction estimate recursive equation. You will notice that is the same except that

$$L_p = AL_c.$$

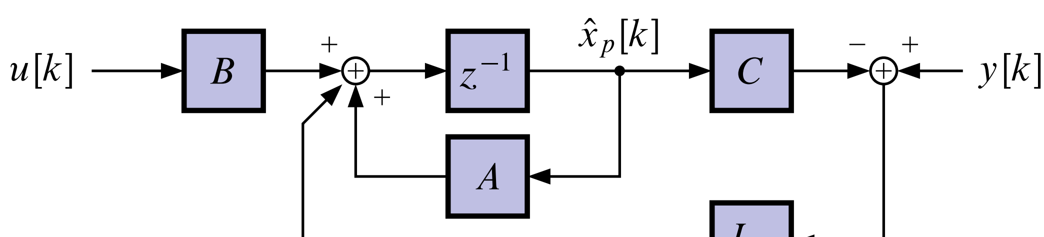
- So, $\hat{x}_p$ in the current-estimator equations is the same quantity $\hat{x}_p$ in the prediction-estimator equations.
- This implies that if we define $\tilde{x} = x - \hat{x}_p$ (not $x - \hat{x}_c$), then

$$\tilde{x}[k+1] = \left(A - L_pC\right)\tilde{x}[k] = (A - AL_cC)\,\tilde{x}[k].$$

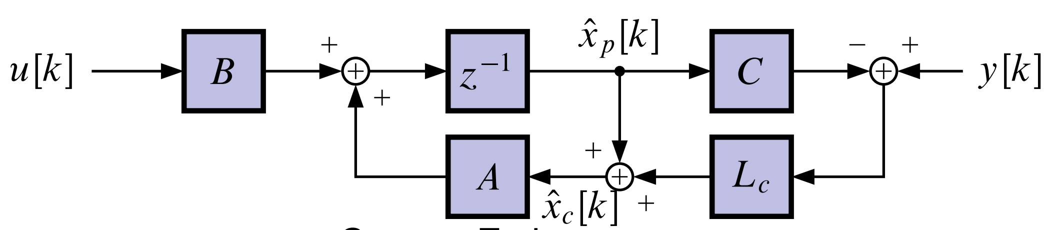
- So, in summary

$$\begin{aligned}
\tilde{x} &= x - \hat{x}_p \quad \Rightarrow \quad \tilde{x}[k+1] = (A - L_cCA)\,\tilde{x}[k] \\
\tilde{x} &= x - \hat{x}_c \quad \Rightarrow \quad \tilde{x}[k+1] = (A - AL_cC)\,\tilde{x}[k].
\end{aligned}$$

- These estimate errors have the same poles. They represent the dynamics of the block diagrams:



Prediction Estimator



Current Estimator

Design of L_c

1. Relate coefficients of

$$\det(zI - A + L_cCA) = \chi_{ob,des}(z).$$

2. Ackermann's formula for L_c

$$L_p = \chi_{ob,des}(z)\mathcal{O}^{-1}\{C, A\}\begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix}$$

replace $C \leftarrow CA$ to find L_c

$$L_c = \chi_{ob,des}(A)\mathcal{O}^{-1}\{CA, A\}\begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix} = \chi_{ob,des}(A)\underbrace{\begin{bmatrix} CA \\ CA^2 \\ \vdots \\ CA^n \end{bmatrix}^{-1}}_{[\mathcal{O}A]^{-1}}\begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix}.$$

- In MATLAB,

```
Lc=acker(A',(C*A)',poles)';
Lc=place(A',(C*A)',poles)';
```

3. Find L_p and then $L_c = A^{-1}L_p$.

7.7: Compensator design using the current estimator

- Plant equations

$$x[k+1] = Ax[k] + Bu[k]$$
$$y[k] = Cx[k].$$

- Estimator equations

$$\hat{x}_p[k+1] = A\hat{x}_c[k] + Bu[k]$$
$$\hat{x}_c[k] = \hat{x}_p[k] + L_c\left(y[k] - C\hat{x}_p[k]\right).$$

- Control

$$u[k] = -K\hat{x}_c[k].$$

- Therefore, we have

$$\begin{aligned} x[k+1] &= Ax[k] - BK\hat{x}_c[k] \\ \hat{x}_c[k+1] &= (I - L_cC)\,A\hat{x}_c[k] + (I - L_cC)\,Bu[k] + L_cy[k+1] \\ &= (I - L_cC)(A - BK)\,\hat{x}_c[k] + L_cy[k+1]. \end{aligned}$$

- With $y[k+1] = CAx[k] + CBu[k]$, then

$$\hat{x}_c[k+1] = (A - BK - L_cCA)\,\hat{x}_c[k] + L_cCAx[k].$$

- Our $2n$-order, closed-loop system is

$$\begin{bmatrix} x[k+1] \\ \hat{x}_c[k+1] \end{bmatrix} = \left[\begin{array}{c|c} A & -BK \\ \hline L_cCA & A - BK - L_cCA \end{array}\right] \begin{bmatrix} x[k] \\ \hat{x}_c[k] \end{bmatrix}.$$

- Compare this to the prediction estimator feedback case:

$$\begin{aligned} L_c &\leftarrow L_p \\ CA &\leftarrow C \end{aligned}$$

- In terms of $\tilde{x}[k+1] = x[k+1] - \hat{x}_c[k+1]$

$$\left[\begin{array}{c} x[k+1] \\ \hline \tilde{x}[k+1] \end{array}\right] = \left[\begin{array}{c|c} A - BK & BK \\ \hline 0 & A - L_cCA \end{array}\right] \left[\begin{array}{c} x[k] \\ \hline \tilde{x}[k] \end{array}\right].$$

- As in the prediction estimator case, the closed-loop poles of our compensated system are the eigenvalues of

$$\text{poles} = \text{eig}\left[\begin{array}{c|c} A - BK & BK \\ \hline 0 & A - L_cCA \end{array}\right]$$

or

$$\begin{aligned} \chi_{cl}(z) &= \chi_{des}(z) \cdot \chi_{ob,des}(z) \\ &= \det(zI - A + BK)\det(zI - A + L_cCA). \end{aligned}$$

- Therefore

$$u[k] = -K\hat{x}_c[k]$$

also works!

- What is $D(z)$ for the current estimator?
- Consider a recurrence relation given earlier

$$\begin{aligned} \hat{x}_c[k+1] &= (I - L_cC)(A - BK)\hat{x}_c[k] + L_c y[k+1] \\ u[k] &= -K\hat{x}_c[k]. \end{aligned}$$

- Taking the z-transform ($\hat{x}_c[0] = 0$)

$$\begin{aligned} z\hat{X}_c(z) &= (I - L_cC)(A - BK)\hat{X}_c(z) + zL_cY(z) \\ U(z) &= -K\hat{X}_c(z) \end{aligned}$$

so

$$D(z)=\frac{U(z)}{Y(z)}=-K\left(zI-(I-L_cC)(A-BK)\right)^{-1}L_cz$$

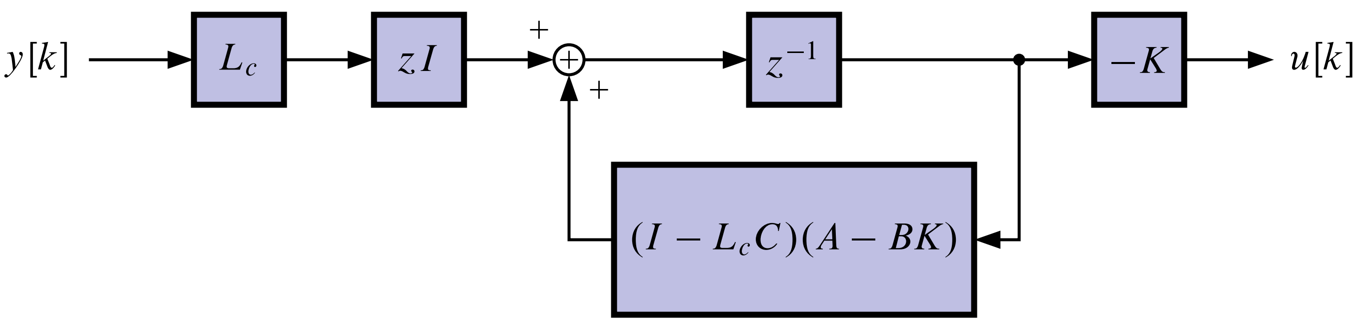
or

$$D(z)=-K\left(zI-A+BK+L_cCA-L_cCBK\right)^{-1}L_cz.$$

- Extra term in $()^{-1}$ and there is always a zero at $z=0$!
- So we always end up with a compensator zero at the origin. The current compensator poles satisfy

$$\det\left(zI-A+BK+L_cCA-L_cCBK\right)=0.$$

- Block diagram:



- We cannot implement the zI block. To see a more useful block diagram, let's write the compensator recurrence relations in standard form, *i.e.,* let's find

$$x_{cc}[k+1]=Ax_{cc}[k]+By[k]$$
$$u[k]=Cx_{cc}[k]+Dy[k].$$

- Start with the control equation

$$\begin{aligned}u[k]&=-K\hat{x}_c[k]\\&=-K\left(\hat{x}_p[k]+L_c\left(y[k]-C\hat{x}_p[k]\right)\right)\\&=-K\left(I-L_cC\right)\hat{x}_p[k]-KL_cy[k].\end{aligned}$$

- Recursion for $\hat{x}_p[k]$?

$$\begin{aligned}\hat{x}_p[k+1] &= A\hat{x}_p[k] + Bu[k] + AL_c\big(y[k] - C\hat{x}_p[k]\big)\\ &= (A - AL_cC)\,\hat{x}_p[k] - BK\hat{x}_c[k] + AL_c y[k]\\ &= (A - AL_cC - BK + BKL_cC)\,\hat{x}_p[k] + (AL_c - BKL_c)\,y[k].\end{aligned}$$

- Therefore

$$\begin{aligned}\hat{x}_p[k+1] &= \bar{A}\hat{x}_p[k] + \bar{B}y[k]\\ u[k] &= \bar{C}\hat{x}_p[k] + \bar{D}y[k]\end{aligned}$$

where

$$\begin{aligned}\bar{A} &= (A - BK)(I - L_cC) & \bar{C} &= -K(I - L_cC)\\ \bar{B} &= (A - BK)L_c & \bar{D} &= -KL_c\end{aligned}$$

- Our modified transfer function for $D(z)$ is

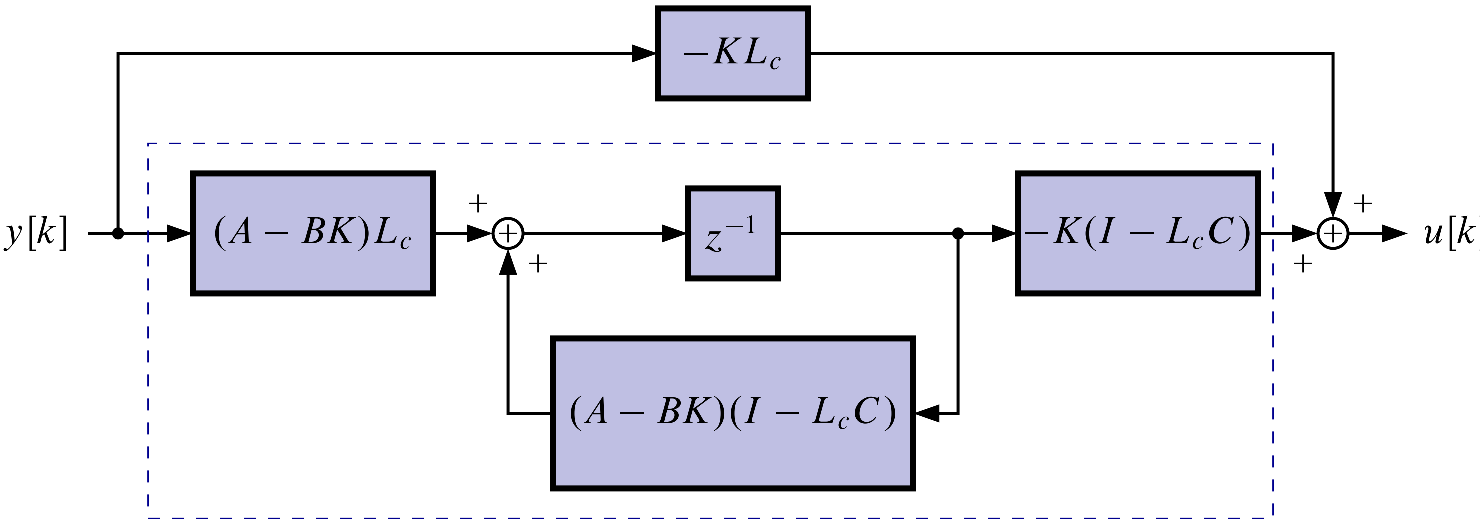
$$D(z) = \bar{D} + \bar{C}(zI - \bar{A})^{-1}\bar{B}$$

or

$$D(z) = -KL_c - K(I - L_cC)\,(zI - (A - BK)(I - L_cC))^{-1}\,(A - BK)L_c.$$

which you can verify is equivalent to the previous expression.

- New block diagram



- Because of delay, everything inside the dotted box can be computed *before* we sample $y[k]$.
- Note the feedthrough term $-KL_c$. So, the compensator responds quickly to plant variations. That is,

$$u[k] = f(y[k], y[k-1], \ldots).$$

IMPLEMENTATION METHOD 1: (not good)

```
xhatp=xhatpnew
A2D(y)
xhatc=xhatp+Lc*(y-C*xhatp)
u=-K*xhatc
D2A(u)
xhatpnew=A*xhatc+B*u
```

IMPLEMENTATION METHOD 2: (good)

```
xhatp=xhatpnew
upartial=-K*(I-Lc*C)*xhatp
A2D(y)
u=upartial-K*Lc*y
```

```
D2A(u)
xhatpnew=A*xhatp+B*u+A*Lc*(y-C*xhatp)
```

EXAMPLE: $G(s) = \dfrac{1}{s^2}$; $T = 1.4$ seconds.

- System description

$$A = \begin{bmatrix} 1 & 1.4 \\ 0 & 1 \end{bmatrix}, \qquad B = \begin{bmatrix} 1 \\ 1.4 \end{bmatrix}, \qquad C = \begin{bmatrix} 1 & 0 \end{bmatrix},$$

- Pick closed-loop poles as we did for prediction estimator

$$\chi_{des}(z) = z^2 - 1.6z + 0.7 \qquad \text{and} \qquad \chi_{ob,des}(z) = z^2 - 1.28z + 0.45.$$

- Control design $K = \begin{bmatrix} 0.05 & 0.25 \end{bmatrix}$.
- Estimator design

$$L_c = A^{-1}L_p = A^{-1}\begin{bmatrix} 0.72 \\ 0.12 \end{bmatrix} = \begin{bmatrix} 0.55 \\ 0.12 \end{bmatrix}.$$

- Our compensator is described by

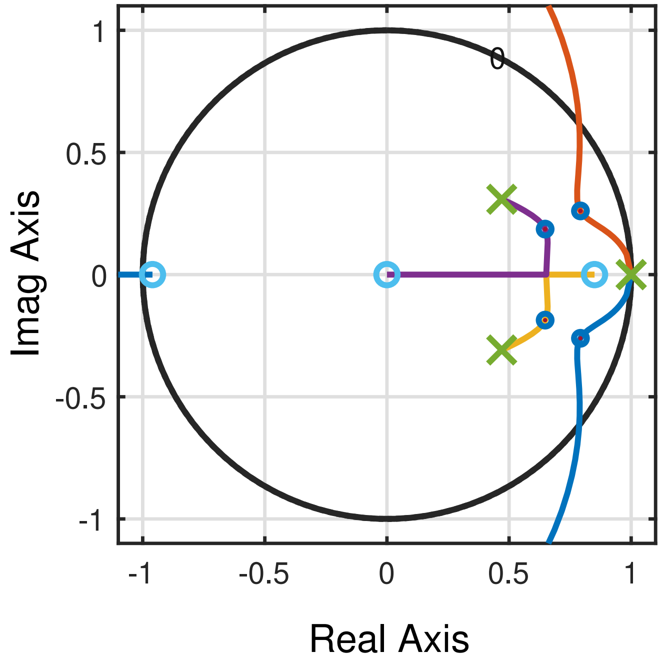
$$\hat{x}_p[k+1] = \begin{bmatrix} 0.29 & 1.16 \\ -0.11 & 0.65 \end{bmatrix}\hat{x}_p[k] + \begin{bmatrix} 0.66 \\ 0.04 \end{bmatrix} y[k]$$

$$u[k] = \begin{bmatrix} 0.0067 & -0.25 \end{bmatrix}\hat{x}_p[k] - 0.06y[k]$$

or

$$\begin{aligned} D(z) &= -0.06\frac{z(z-0.85)}{z - 0.47 \pm 0.31j} \\ &= -0.06\frac{z(z-0.85)}{z^2 - 0.94z + 0.316}. \end{aligned}$$

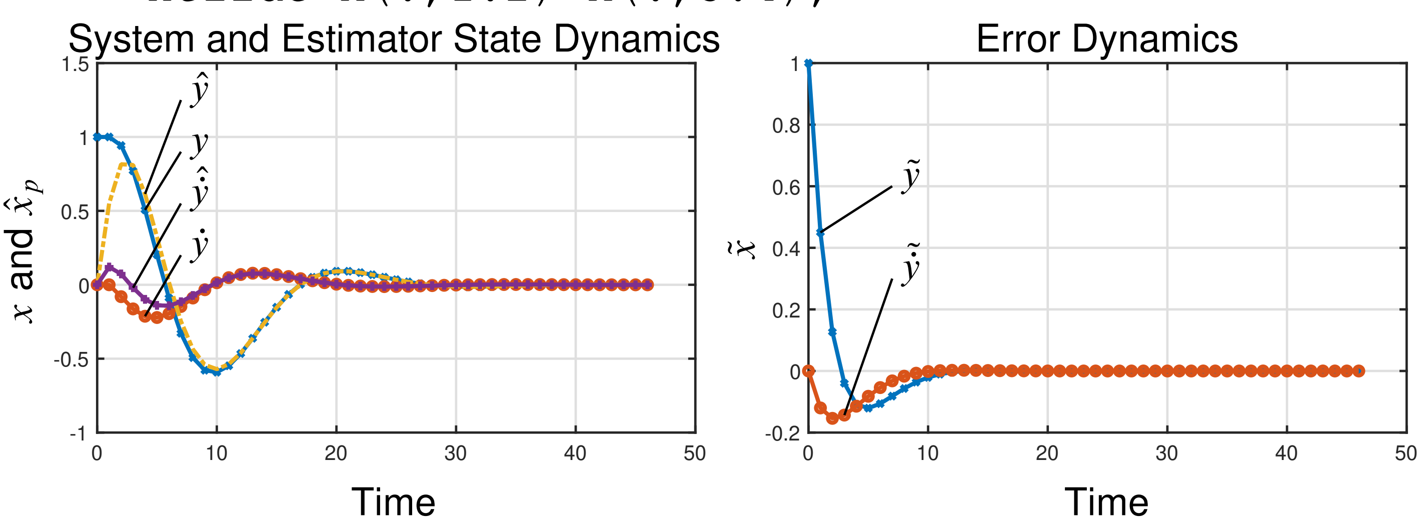
- The root locus



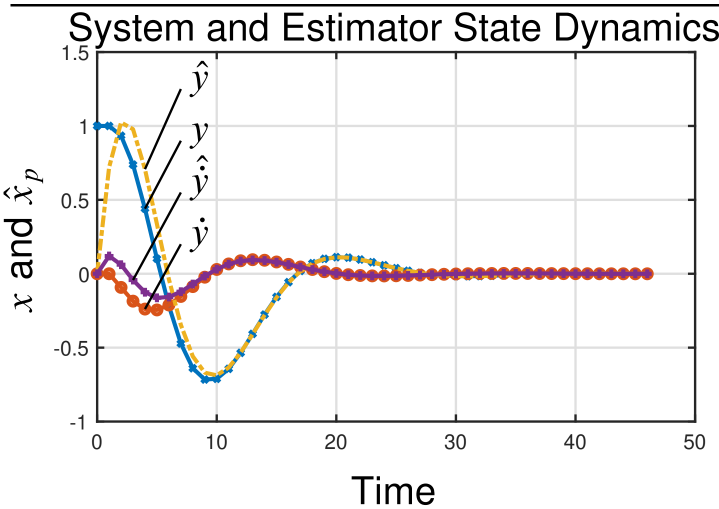
- Compare to prediction estimator.

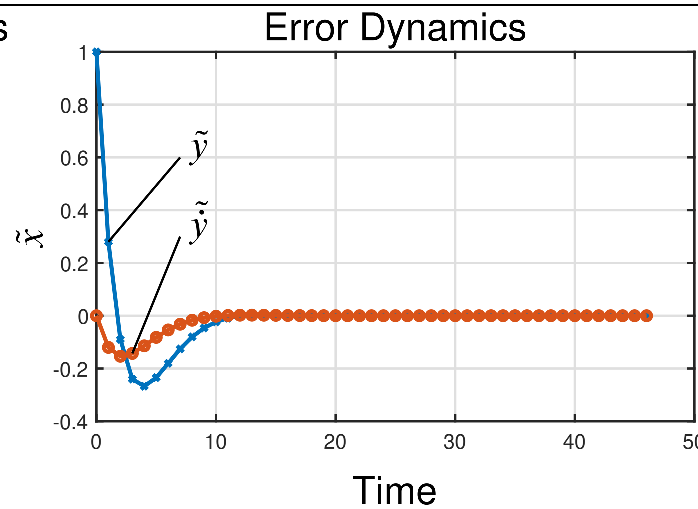
EXAMPLE: In MATLAB,

```
Q=[A  -B*K; Lc*C*A  A-B*K-Lc*C*A];
% get resp. of u, x, xhat to init. condition
[u,X]=dinitial(Q,zeros([4,1]),[0 0 -K],0,x0);
% get the estimate error.
xtilde=X(:,1:2)-X(:,3:4);
```



- Compare to prior result:


System and Estimator State Dynamics
$\hat{y}$
y
$\dot{\hat{y}}$
$\dot{y}$
x and $\hat{x}_p$
1.5
1
0.5
0
-0.5
-1
0
10
20
30
40
50
Time


Error Dynamics
$\tilde{y}$
$\dot{\tilde{y}}$
$\tilde{x}$
1
0.8
0.6
0.4
0.2
0
-0.2
-0.4
0
10
20
30
40
50
Time

7.8: Discrete-time reduced-order estimator

- Why construct the entire state vector when you are directly measuring a state? If there is little noise in your sensor, you get a great estimate by just letting

$$\hat{x}_1 = y \qquad (C = [1\ 0\ \ldots\ 0]).$$

- If there is noise in the measurement

$$y[k] = Cx[k] + v, \qquad v = \text{noise},$$

  then the estimate $\hat{y}_p$ or $\hat{y}_c$ can be a smoothed version of y!

- Consider partitioning the plant state into

$$x_a : \text{measured state}$$

$$x_b : \text{to be estimated.}$$

  So

$$y = Cx = x_a.$$

- (Note: This may require a transformation).
- Our partitioned system

$$\begin{bmatrix} x_a[k+1] \\ \hline x_b[k+1] \end{bmatrix} = \left[\begin{array}{c|c} A_{aa} & A_{ab} \\ \hline A_{ba} & A_{bb} \end{array}\right] \begin{bmatrix} x_a[k] \\ \hline x_b[k] \end{bmatrix} + \begin{bmatrix} B_a[k] \\ \hline B_b[k] \end{bmatrix} u[k]$$

$$y[k] = \left[\begin{array}{c|c} I & 0 \end{array}\right] \begin{bmatrix} x_a[k] \\ \hline x_b[k] \end{bmatrix}$$

- In order to design an estimator for x_b, we need to create a suitable state-space model of the dynamics of x_b such as

$$x_b[k+1] = A_{xb}x_b[k] + B_{xb}m_1[k]$$

$$m_2[k] = C_{xb}x_b[k] + D_{xb}m_1[k]$$

where $m_1[k]$ and $m_2[k]$ are some measurable inputs. Note that $m_1[k]$ and $m_2[k]$ will be combinations of $y[k] = x_a[k]$ and $u[k]$.

- Once we have this state-space form, we can create a standard prediction or current estimator.
- We start the derivation by finding an output equation for x_b. Consider the dynamics of the measured state:

$$x_a[k+1] = A_{aa}x_a[k] + A_{ab}x_b[k] + B_a u[k]$$

where $x_b[k]$ is the only unknown. Let

$$m_2[k] = x_a[k+1] - A_{aa}x_a[k] - B_a u[k].$$

- Then

$$m_2[k] = A_{ab}x_b[k],$$

where $m_2[k]$ is known/measurable and thus "C_{xb}" is equal to A_{ab}.

- This is our reduced-order estimator output relation.
- We now look for a state equation for x_b. Consider the dynamics of the estimated state:

$$x_b[k+1] = A_{ba}x_a[k] + A_{bb}x_b[k] + B_b u[k].$$

Let

$$B_{xb}m_1[k] = A_{ba}x_a[k] + B_b u[k]$$

so that the reduced-order recurrence relation is

$$x_b[k+1] = A_{bb}x_b[k] + B_{xb}m_1[k].$$

- This might be accomplished via (single-input system)

$$B_{xb}m_1[k] = \left[\begin{array}{c:c} A_{ba} & B_b \end{array}\right] \left[\begin{array}{c} x_a[k] \\ \hdashline u[k] \end{array}\right]$$

although the details of how this is done do not matter in the end.

- So, for the purpose of designing our estimator, the state-space equations are:

$$x_b[k+1] = A_{bb}x_b[k] + B_{xb}m_1[k]$$

$$m_2[k] = A_{ab}x_b[k].$$

- In terms of our standard estimator design procedures,

$$A \leftarrow A_{bb}; \qquad Bu[k] \leftarrow B_{xb}m_1[k]; \qquad C \leftarrow A_{ab}; \qquad y \leftarrow m_2[k].$$

- For example, we can design a *prediction* reduced-order estimator. Once we know L_r, we have the estimator equation

$$\hat{x}_b[k+1] = A_{bb}\hat{x}_b[k] + B_{xb}m_1[k] + L_r\left(m_2[k] - A_{ab}\hat{x}_b[k]\right).$$

- In terms of our known (measured) quantities,

$$\hat{x}_b[k+1] = A_{bb}\hat{x}_b[k] + A_{ba}x_a[k] + B_bu[k] +$$
$$L_r\left(x_a[k+1] - A_{aa}x_a[k] - B_au[k] - A_{ab}\hat{x}_b[k]\right).$$

- Note that when we look at the compensator, we will be able to make the $x_a[k+1]$ disappear.
- In order to design L_r we must find the equation that the error dynamics satisfy.

$$\tilde{x}_b[k+1] = x_b[k+1] - \hat{x}_b[k+1]$$

$$
\begin{aligned}
&= A_{bb}x_b[k] + B_{xb}m_1[k] - \\
&\qquad (A_{bb}\hat{x}_b[k] + B_{xb}m_1[k] + L_r(A_{ab}x_b[k] - A_{ab}\hat{x}_b[k])) \\
&= (A_{bb} - L_r A_{ab})\,\tilde{x}_b[k].
\end{aligned}
$$

- So we pick estimate error dynamics related to roots of

$$\chi_{r,des}(z) = \det(zI - A_{bb} + L_r A_{ab}) = 0.$$

- You might guess that arbitrary reduced-order estimator poles can be selected if $\{A_{ab}, A_{bb}\}$ forms an observable pair.

Design of L_r

- Relate coefficients of

$$\det(zI - A_{bb} + L_r A_{ab}) = \chi_{r,des}(z).$$

- Ackermann's formula with

$$A \leftarrow A_{bb}, \qquad C \leftarrow A_{ab},$$

$$L_r = \chi_{r,des}(A_{bb}) \begin{bmatrix} A_{ab} \\ A_{ab}A_{bb} \\ \vdots \\ A_{ab}A_{bb}^{n-1} \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}.$$

- In MATLAB,

```
Lr=acker(Abb',Aab',poles)';
Lr=place(Abb',Aab',poles)';
```

7.9: Discrete-time reduced-order prediction compensator

- We now determine (1) the closed-loop system dynamics using the reduced-order compensator, and (2) the reduced-order discrete-time compensator $D(z)$.
- We first combine the control law, the plant dynamics, and the estimator dynamics to find the closed-loop system dynamics.
- Control law:

$$\begin{aligned} u[k] &= -K_a x_a[k] - K_b \hat{x}_b[k] \\ &= -\begin{bmatrix} K_a & K_b \end{bmatrix} \hat{x}[k]. \end{aligned}$$

- Plant:

$$\begin{aligned} x[k+1] &= Ax[k] + Bu[k] \\ y[k] &= Cx[k] = x_a[k]. \end{aligned}$$

- Estimator:

$$\begin{aligned} \hat{x}_b[k+1] = {} & A_{bb}\hat{x}_b[k] + A_{ba}x_a[k] + B_b u[k] + \\ & L_r\left(x_a[k+1] - A_{aa}x_a[k] - B_a u[k] - A_{ab}\hat{x}_b[k]\right) \end{aligned}$$

using

$$\begin{aligned} u[k] &= -K_a C x[k] - K_b \hat{x}_b[k] \\ L_r x_a[k+1] &= L_r C x[k+1] \\ &= L_r C\left(Ax[k] + Bu[k]\right) \\ &= L_r C A x[k] - L_r B_a K_a C x[k] - L_r B_a K_b \hat{x}_b[k]. \end{aligned}$$

$$\begin{bmatrix} x[k+1] \\ \hline \hat{x}_b[k+1] \end{bmatrix} =$$

$$\left[\begin{array}{c|c} A - BK_aC & -BK_b \\ \hline L_rCA + A_{ba}C - B_bK_aC - L_rA_{aa}C & A_{bb} - B_bK_b - L_rA_{ab} \end{array}\right] \begin{bmatrix} x[k] \\ \hline \hat{x}_b[k] \end{bmatrix}$$

- We can also investigate the closed-loop dynamics in terms of the estimator error state

$$x[k+1] = Ax[k] - BK_ax_a[k] - BK_b(x_b[k] - \tilde{x}_b[k])$$

$$\tilde{x}_b[k+1] = (A_{bb} - L_rA_{ab})\,\tilde{x}_b[k],$$

or

$$\begin{bmatrix} x[k+1] \\ \hline \tilde{x}_b[k+1] \end{bmatrix} = \left[\begin{array}{c|c} A - BK & BK_b \\ \hline 0 & A_{bb} - L_rA_{ab} \end{array}\right] \begin{bmatrix} x[k] \\ \hline \tilde{x}_b[k] \end{bmatrix}$$

- So, we see that the separation principle holds when using the reduced-order estimator.
- We now proceed to find the compensator. We combine the estimator state equation and controller output equation.

$$\begin{aligned} \hat{x}_b[k+1] = {} & (A_{bb} - B_bK_b + L_rB_aK_b - L_rA_{ab})\,\hat{x}_b[k] \\ & + (A_{ba} + L_rB_aK_a - L_rA_{aa} - B_bK_a)\,y[k] \\ & + L_ry[k+1] \end{aligned}$$

$$\hat{x}_b[k+1] = \bar{A}\hat{x}_b[k] + \bar{B}y[k] + L_ry[k+1],$$

and

$$u[k] = -K_b\hat{x}_b[k] - K_ay[k],$$

so taking the z-transform

$$D(z)=\frac{U(z)}{Y(z)}=-K_a-K_b\left(zI-\bar{A}\right)^{-1}\left(\bar{B}+zL_r\right).$$

EXAMPLE: $G(s)=\dfrac{1}{s^2}$; $T=1.4$ seconds.

- System description

$$A=\begin{bmatrix}1 & 1.4\\ 0 & 1\end{bmatrix},\qquad B=\begin{bmatrix}1\\ 1.4\end{bmatrix},\qquad C=\begin{bmatrix}1 & 0\end{bmatrix},$$

- Measure $y[k]=x_1[k]$ directly; estimate $v[k]=x_2[k]$.
- Pick control poles the same as before: $\chi_{des}(z)=z^2-1.6z+0.7$.
- Choose "dead-beat" estimation of $v[k]$: $\chi_{r,des}(z)=z$.
- Control design: $K=\begin{bmatrix}0.05 & 0.25\end{bmatrix}$.
- Reduced-order estimator

$$\chi_r(z)=\det(zI-A_{bb}+L_rA_{ab})=z-1+L_rT$$

where

$$A=\begin{bmatrix}A_{aa} & A_{ab}\\ A_{ba} & A_{bb}\end{bmatrix}=\begin{bmatrix}1 & 1.4\\ 0 & 1\end{bmatrix}$$

so $L_r=\dfrac{1}{T}=0.714$ implies

$$\chi_r(z)=\det(z-1+0.714\cdot 1.4)=z.$$

- Reduced-order estimator:

$$A \text{ as above, } B_a=\frac{T^2}{2}=0.98;\quad B_b=T=1.4.$$

implies

$$\hat{v}[k+1]=\hat{v}[k]+Tu[k]+\frac{1}{T}\left(y[k+1]-y[k]-\frac{T^2}{2}u[k]-T\hat{v}[k]\right)$$

$$=\frac{y[k+1]-y[k]}{1.4}+0.7u[k].$$

- Control:

$$u[k] = -\frac{1}{20}y[k] - \frac{1}{4}\hat{v}[k],$$

so

$$\hat{v}[k+1] = -\frac{T}{8}\hat{v}[k] + \frac{1}{T}y[k+1] - \left(\frac{1}{T} + \frac{T}{40}\right)y[k]$$

$$u[k] = -\frac{1}{4}\hat{v}[k] - \frac{1}{20}y[k].$$

- Taking the transfer function:

$$D(z) = \frac{U(z)}{Y(z)} = -\frac{1}{20} - \frac{1}{4}\frac{\frac{1}{T}z - \frac{1}{T} - \frac{T}{40}}{z + \frac{T}{8}},$$

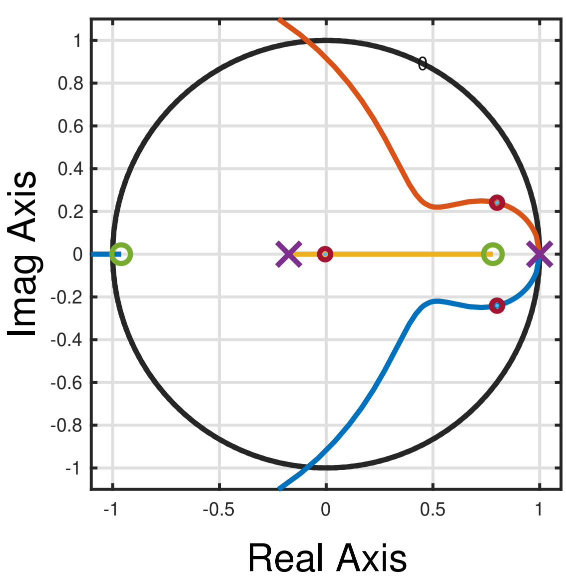
or

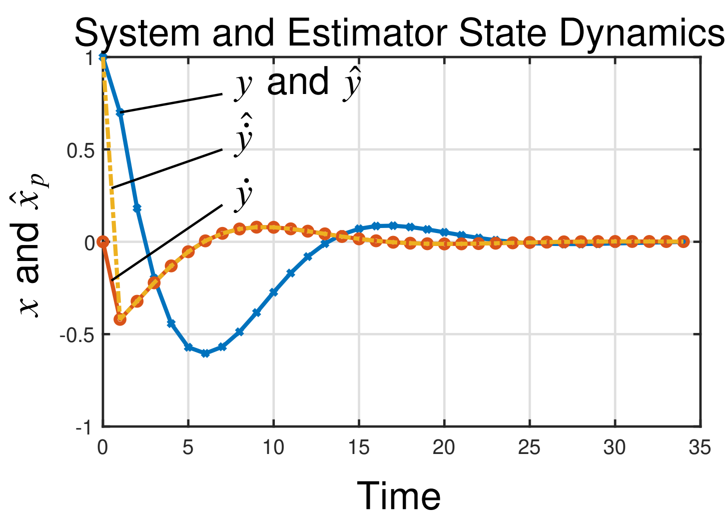
$$D(z) = -\frac{1}{20}\left(1 + \frac{5}{T}\right)\frac{z - \frac{\frac{5}{T}}{1+\frac{5}{T}}}{z + \frac{T}{8}}.$$

- In our case, $T = 1.4$,

$$D(z) = -0.229\frac{z - 0.781}{z + 0.175}$$

- Root locus and transient response





7.10: Continuous-time reduced-order estimator

- Here, we set x_a to be the measured portion of the state and x_b to be the unmeasured portion of the state. Then,

$$x(t) = \begin{bmatrix} x_a(t) \\ x_b(t) \end{bmatrix}, \quad \text{and} \quad \hat{x}(t) = \begin{bmatrix} x_a(t) \\ \hat{x}_b(t) \end{bmatrix}.$$

- The state equations that the system must satisfy are

$$\dot{x}(t) = \begin{bmatrix} A_{aa} & A_{ab} \\ A_{ba} & A_{bb} \end{bmatrix} \begin{bmatrix} x_a(t) \\ x_b(t) \end{bmatrix} + \begin{bmatrix} B_a \\ B_b \end{bmatrix} u(t)$$

$$y(t) = \begin{bmatrix} I & 0 \end{bmatrix} \begin{bmatrix} x_a(t) \\ x_b(t) \end{bmatrix}.$$

- We need to manipulate this form into a state-space system description of the unmeasurable $x_b(t)$ so that we can use standard tools to design an estimator $\hat{x}_b(t)$.
- First, we notice that the measured state $x_a(t)$ has dynamics

$$\dot{x}_a(t) = A_{aa}x_a(t) + A_{ab}x_b(t) + B_a u(t)$$

where $x_b(t)$ is the only unknown.
- Let

$$m_2(t) = \dot{x}_a(t) - A_{aa}x_a(t) - B_a u(t),$$

so that we can create an "output equation" of the x_b dynamics

$$m_2(t) = A_{ab}x_b(t)$$

where $m_2(t)$ is the measured/computed output value and x_b is the estimated value.

- Now, we proceed to find the "state equation" of the x_b dynamics. Note

$$\dot{x}_b(t) = A_{bb}x_b(t) + \left[A_{ba}x_a(t) + B_b u(t)\right],$$

where the items in the square brackets form a measured input signal.

- We design an estimator state equation as

$$\begin{aligned}\dot{\hat{x}}_b(t) &= A_{bb}\hat{x}_b(t) + A_{ba}x_a(t) + B_b u(t) + \\ &\qquad L_r(m_2(t) - A_{ab}\hat{x}_b(t)) \\ &= A_{bb}\hat{x}_b(t) + A_{ba}x_a(t) + B_b u(t) + \\ &\qquad L_r(\dot{x}_a(t) - A_{aa}x_a(t) - B_a u(t) - A_{ab}\hat{x}_b(t)).\end{aligned}$$

- We cannot implement this directly due to the offending $\dot{x}_a(t)$ term in the measurement update.
- To make this problem disappear, define $w(t) = \hat{x}_b(t) - L_r x_a(t)$

$$\begin{aligned}\dot{w}(t) &= \dot{\hat{x}}_b(t) - L_r\dot{x}_a(t) \\ &= A_{bb}\hat{x}_b(t) + A_{ba}x_a(t) + B_b u(t) - \\ &\qquad L_r A_{aa}x_a(t) - L_r B_a u(t) - L_r A_{ab}\hat{x}_b(t). \\ &= \left[A_{bb} - L_r A_{ab}\right]\left[\hat{x}_b(t) - L_r x_a(t)\right] + \\ &\qquad \left[A_{bb} - L_r A_{ab}\right] L_r x_a(t) + \\ &\qquad A_{ba}x_a(t) - L_r A_{aa}x_a(t) + \left[B_b - L_r B_a\right] u(t) \\ &= \left[A_{bb} - L_r A_{ab}\right] w(t) + \\ &\qquad \left[(A_{bb} - L_r A_{ab})L_r + A_{ba} - L_r A_{aa}\right] x_a(t) + \\ &\qquad \left[B_b - L_r B_a\right] u(t)\end{aligned}$$

$$\dot{w}(t) = \bar{A}w(t) + \bar{B}_1 x_a(t) + \bar{B}_2 u(t)$$

$$\hat{x}_b(t) = w(t) + L_r x_a(t).$$

- Therefore, the final state-space equations describing the dynamics of the estimator are:

$$\dot{w}(t) = [A_{bb} - L_r A_{ab}]\, w(t) + [(A_{bb} - L_r A_{ab})L_r + A_{ba} - L_r A_{aa}]\, x_a(t) + [B_b - L_r B_a]\, u(t)$$

$$\hat{x}_b(t) = w(t) + L_r x_a(t).$$

- The poles of the estimator are

$$\det(sI - (A_{bb} - L_r A_{ab})) = 0$$

and we can design L_r using, for example

```
Lr=acker(Abb',Aab',poles)';
```

7.11: Estimator pole placement

- As was the case for finding the control law, the design of an estimator (for single-output plants) simply consists of

  1. Selecting desired estimator error dynamics.
  2. Solving for the corresponding estimator gain.

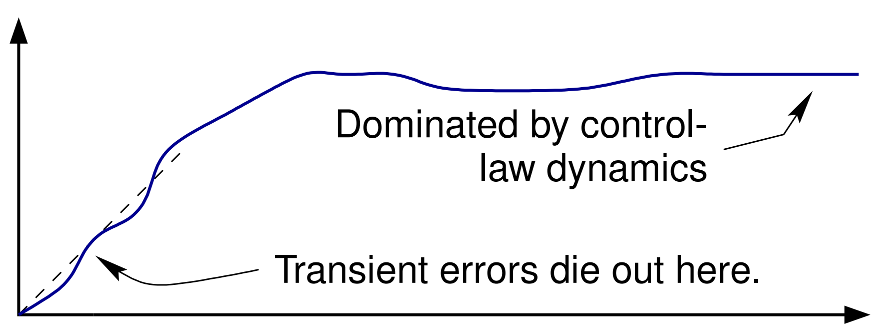
- In other words, we find L_p, L_c, or L_r by first selecting the roots of

$$\chi_{ob,des}(z) \quad \text{or} \quad \chi_{r,des}(z).$$

- So, what estimator poles do we choose?
- If possible, pick estimator poles that do not influence transient response of control-law poles.
- We know

$$\chi_{cl} = \chi_{des} \cdot \chi_{ob,des}.$$

- Since χ_{des} was chosen to meet transient specifications, try to pick $\chi_{ob,des}$ such that estimator dynamics die out before control-law dynamics.



- With no disturbance, the only job requirement for the estimator is to correct for the uncertainty of $x[0]$!
- This will be (near) immediate if we pick poles well inside unit circle.

- Pick estimator poles as fast as possible?
- In control law design we were concerned with

Fast response		Slower response
Large intermediate states	*versus*	Smaller intermediate states
Large control effort		Smaller control effort

- In estimator design, large intermediate states and large feedback signals (control effort) do not carry the same penalty since they are just computer signals!
- That is, $L_p y[k]$ is not limited by actuator hardware!
- Question: Why not pick very fast estimator poles?
- Answer: Sensor noise and uncertainty.
- Control law design: Transient response versus actuator effort.
- Estimator design: Sensor-noise rejection versus process-noise rejection.
- Consider the design of L_p: The plant is

$$x[k+1] = Ax[k] + Bu[k] + B_w w[k]$$

$$y[k] = Cx[k] + v[k].$$

$w[k]$: Process noise, plant disturbances, plant uncertainties
$v[k]$: Sensor noise, biases.

- Estimator: $\hat{x}_p[k+1] = A\hat{x}_p[k] + Bu[k] + L_p\left(y[k] - C\hat{x}_p[k]\right).$
- Only knowledge of v, w through $y[k]$.

- Estimator error dynamics:

$$\begin{aligned}\tilde{x}[k+1] &= Ax[k] + Bu[k] + B_w w[k] - A\hat{x}_p[k] - Bu[k] \\ &\quad -L_p\left(Cx[k] + v[k] - C\hat{x}_p[k]\right) \\ &= \left(A - L_pC\right)\tilde{x}[k] + B_w w[k] - L_p v[k].\end{aligned}$$

- If $v[k] = 0$ then
  - "Large" L_p produces fast poles of $A - L_pC$. Fast correction of $w[k]$ effects.
  - That is, we believe our sensor more than our model.
  - We rely heavily on feedback to keep $\tilde{x}[k]$ small.
  - High bandwidth to quickly compensate for disturbances.
- If $w[k] = 0$ then
  - "Small" L_p produces slow poles of $A - L_pC$. Less amplification of sensor noise.
  - We believe our model more than our sensor.
  - Rely on model to keep $\tilde{x}[k]$ small... open-loop estimation!
  - Low bandwidth for good noise rejection (smoothing).
- Therefore, we pick the estimator poles, or design the estimator gain by examining the tradeoff:
  - If $A - L_pC$ fast:
    - Small transient response effects.
    - Fast correction of model, disturbances.
    - Low noise rejection.
  - If $A - L_pC$ slow:

- Significant transient response effect.
- Slow correction of modeling errors, disturbances.
- High noise rejection.

- In general
  1. Place poles two to six times faster than controller poles and in well-damped locations. This will limit the estimator influence on output response.
  2. If the sensor noise is too big, the estimator poles can be placed as much as two times *slower* than controller poles.

- Notes about (2):
  - Controller may need to be redesigned since the estimator will strongly influence transient behavior.
  - These slow estimator dynamics will *not* be excited by our reference command if our signal is properly included!

MIMO observer design

- The MIMO observer design problem is the dual of the MIMO controller design problem.
- Therefore, if $\{A^T, C^T\}$ is "controllable," the controller design procedures return $L = K^T$.

Where to from here?

- We now know how to place closed-loop poles anywhere we want.
- Where do we want them? Is there an optimal set of locations?
- We next preview LQR design, which attempts to answer this question.