OBSERVABILITY AND CONTROLLABILITY

5.1: Continuous-time observability: Where am I?

- We describe dual ideas called observability and controllability.
- Both have precise (binary) mathematical descriptions, but we need to be careful in interpreting the result.
- We develop some other techniques to help quantify the concepts.

Continuous-time observability

- If a system is observable, we can determine the initial condition of the state vector $x(0)$ via processing the input to the system $u(t)$ and the output of the system $y(t)$.
- Since we can simulate the system if we know $x(0)$ and $u(t)$ this also implies that we can determine $x(t)$ for $t \geq 0$. e.g., for LTI,

$$x(t) = e^{At}x(0) + \int_0^t e^{A(t-\tau)}Bu(\tau)\,\mathrm{d}\tau.$$

- Consider the LTI SISO system LCCODE

$$\dddot{y}(t) + a_1\ddot{y}(t) + a_2\dot{y}(t) + a_3y(t) = b_0\dddot{u}(t) + b_1\ddot{u}(t) + b_2\dot{u}(t) + b_3u(t).$$

- If we have a realization of this system in state-space form

$$\dot{x}(t) = Ax(t) + Bu(t)$$

$$y(t) = Cx(t) + Du(t),$$

and we have initial conditions $y(0)$, $\dot{y}(0)$, $\ddot{y}(0)$, how do we find $x(0)$?

$$
\begin{aligned}
y(0) &= Cx(0) + Du(0) \\
\dot{y}(0) &= C(\underbrace{Ax(0) + Bu(0)}_{\dot{x}(0)}) + D\dot{u}(0) \\
&= CAx(0) + CBu(0) + D\dot{u}(0) \\
\ddot{y}(0) &= CA^2x(0) + CABu(0) + CB\dot{u}(0) + D\ddot{u}(0).
\end{aligned}
$$

- In general,

$$y^{(k)}(0) = CA^kx(0) + CA^{k-1}Bu(0) + \cdots + CBu^{(k-1)}(0) + Du^{(k)}(0),$$

or,

$$
\begin{bmatrix} y(0) \\ \dot{y}(0) \\ \ddot{y}(0) \end{bmatrix} = \underbrace{\begin{bmatrix} C \\ CA \\ CA^2 \end{bmatrix}}_{\mathcal{O}(C,A)} x(0) + \underbrace{\begin{bmatrix} D & 0 & 0 \\ CB & D & 0 \\ CAB & CB & D \end{bmatrix}}_{\mathcal{T}} \begin{bmatrix} u(0) \\ \dot{u}(0) \\ \ddot{u}(0) \end{bmatrix},
$$

where $\mathcal{T}$ is a (block) "Toeplitz matrix".

- Thus, if $\mathcal{O}(C, A)$ is invertible, then

$$
x(0) = \mathcal{O}^{-1} \left\{ \begin{bmatrix} y(0) \\ \dot{y}(0) \\ \ddot{y}(0) \end{bmatrix} - \mathcal{T} \begin{bmatrix} u(0) \\ \dot{u}(0) \\ \ddot{u}(0) \end{bmatrix} \right\}.
$$

- We say that $\{C, A\}$ is an observable pair if $\mathcal{O}$ is nonsingular.

CONCLUSION: For a SISO system, if $\mathcal{O}$ is nonsingular, then we can determine/estimate the initial state of the system $x(0)$ using only $u(t)$ and $y(t)$ (and therefore, we can estimate $x(t)$ for all $t \geq 0$).

EXTENSION: For a MIMO system, if $\mathcal{O}$ is full rank, then we can determine/estimate the initial state of the system $x(0)$ using only $u(t)$ and $y(t)$ (and therefore, we can estimate $x(t)$ for all $t \geq 0$).

EXAMPLE: Observability canonical form:

$$\dot{x}(t) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -a_3 & -a_2 & -a_1 \end{bmatrix} x(t) + \begin{bmatrix} \beta_1 \\ \beta_2 \\ \beta_3 \end{bmatrix} u(t)$$

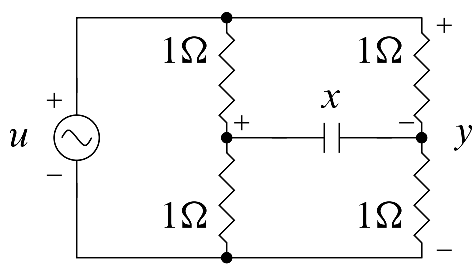
$$y(t) = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} x(t).$$

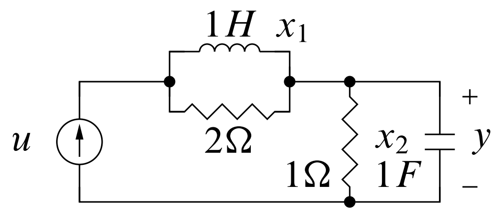
- Then

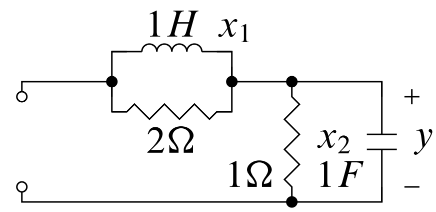
$$\mathcal{O} = \begin{bmatrix} C \\ CA \\ CA^2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & 1 \end{bmatrix} = I_n.$$

- This is why it is called observability form!

EXAMPLE: Two unobservable networks







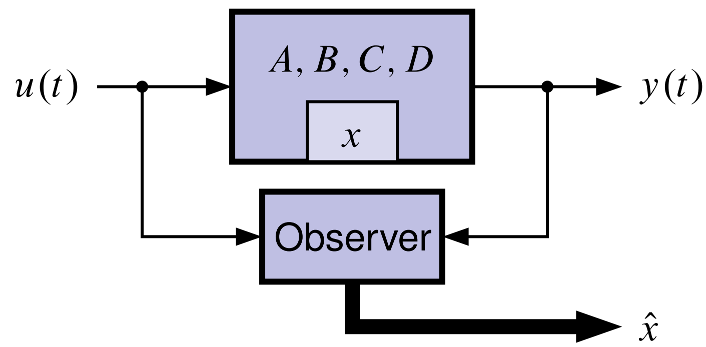
(Redrawn)

- In the first, if $u(t) = 0$ then $y(t) = 0 \quad \forall\, t$. Cannot determine $x(0)$.
- In the second, if $u(t) = 0$, $x_1(0) \neq 0$ and $x_2(0) = 0$, then $y(t) = 0$ and we cannot determine $x_1(0)$. [circuit redrawn for $u(t) = 0$].

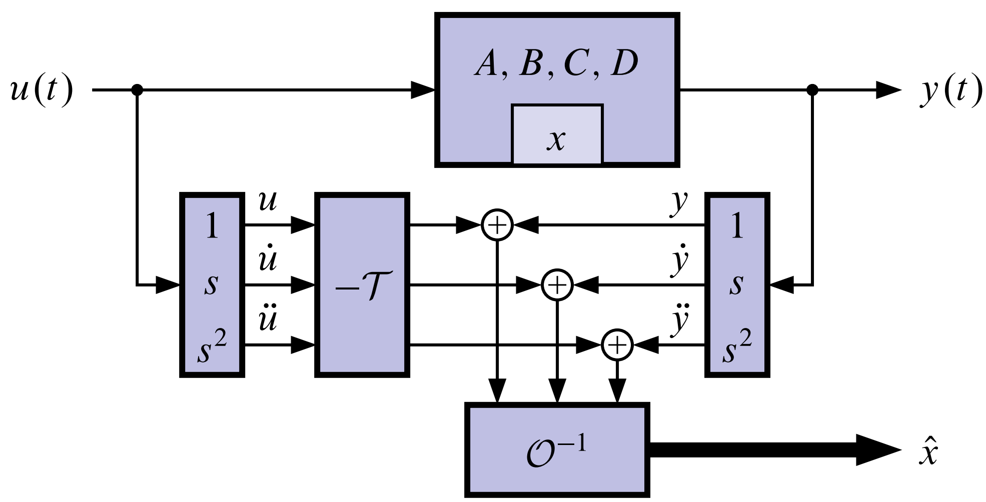
Observers

- An observer is a device that has as inputs $u(t)$ and $y(t)$—the input and output of a linear system. The output of the observer is the (estimated) state of the linear system.

- The observer "observes" the internal state x (estimated as $\hat{x}$) from external signals u and y.



- Note that our equations yield an observer:



- Later, we'll design more practical observers that don't use differentiators.

5.2: Continuous-time controllability: Can I get there from here?

- Can we generate an input $u(t)$ to set an initial condition quickly?

$$\dot{x}(t) = Ax(t) + Bu(t)$$

$$y(t) = Cx(t) + Du(t).$$

- If $u(t) = \delta(t)$ and $x(0^-) = 0$, then

$$X(s) = (sI - A)^{-1}BU(s) = (sI - A)^{-1}B.$$

- So, via the Laplace initial-value theorem,

$$\begin{aligned} x(0^+) &= \lim_{s\to\infty} sX(s) \\ &= \lim_{s\to\infty} s(sI - A)^{-1}B \\ &= \lim_{s\to\infty} \left(I - \frac{A}{s}\right)^{-1} B \\ &= B. \end{aligned}$$

- Thus, an impulse input brings the state to B from 0.
- What if $u(t) = \delta^{(k)}(t)$?
- Then

$$\begin{aligned} X(s) &= (sI - A)^{-1}Bs^k = \frac{1}{s}\left(I - \frac{A}{s}\right)^{-1} Bs^k \\ &= \frac{1}{s}\underbrace{\left(I + \frac{A}{s} + \frac{A^2}{s^2} + \dots\right)}_{\text{holds for large } s} Bs^k \\ &= Bs^{k-1} + ABs^{k-2} + A^2Bs^{k-3} + \dots + \frac{A^kB}{s} + \frac{A^{k+1}B}{s^2} + \cdots \end{aligned}$$

- The first terms are impulsive: they have zero value for $t > 0$.

- Thus,

$$\begin{aligned} x(0^+) &= \lim_{s\to\infty} s\left(\frac{A^kB}{s} + \frac{A^{k+1}B}{s^2} + \cdots\right) \\ &= A^kB. \end{aligned}$$

So, if $u(t) = \delta^{(k)}(t)$ then $x(0^+) = A^kB$.

- Now, consider the input

$$u(t) = g_1\delta(t) + g_2\dot{\delta}(t) + \cdots g_n\delta^{(n)}(t).$$

Since $x(0^-) = 0$, $x(0^+) = g_1B + g_2AB + \cdots + g_nA^{n-1}B$, or

$$x(0^+) = \underbrace{\begin{bmatrix} B & AB & \cdots & A^{n-1}B \end{bmatrix}}_{\mathcal{C}} \begin{bmatrix} g_1 \\ g_2 \\ g_3 \end{bmatrix}$$

where $\mathcal{C}$ is called the "controllability matrix."

CONCLUSION: For a SISO system, if $\mathcal{C}$ is nonsingular, then there is an impulsive input u such that $x(0^+)$ is any desired vector if $x(0^-) = 0$.

EXTENSION: For a MIMO system, if $\mathcal{C}$ is full rank, then there is an impulsive input u such that $x(0^+)$ is any desired vector if $x(0^-) = 0$.

- In fact, we may use

$$u(t) = \sum_{i=0}^{n-1} g_i\delta^{(i)}(t)$$

where

$$\begin{bmatrix} g_1 \\ g_2 \\ g_3 \end{bmatrix} = \begin{bmatrix} B & AB & \cdots & A^{n-1}B \end{bmatrix}^{-1} x_d$$

where x_d is the desired $x(0^+)$ vector.

- If $\mathcal{C}$ is nonsingular, we say $\{A, B\}$ is a controllable pair and the system is controllable.

EXAMPLE: Controllability canonical form:

$$\dot{x}(t) = \begin{bmatrix} 0 & 0 & -a_3 \\ 1 & 0 & -a_2 \\ 0 & 1 & -a_1 \end{bmatrix} x(t) + \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} u(t)$$

$$y(t) = \begin{bmatrix} \beta_1 & \beta_2 & \beta_3 \end{bmatrix} x(t).$$

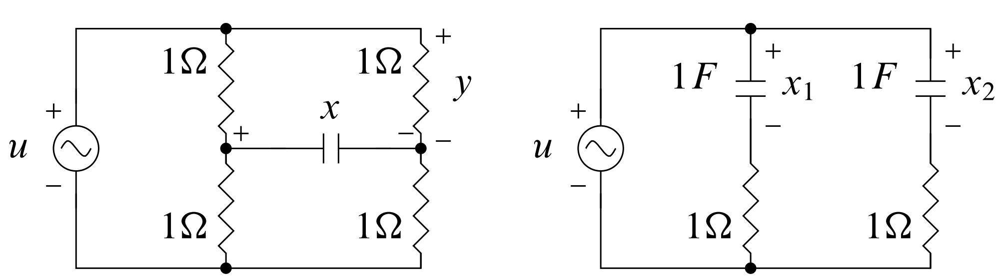
- Then

$$\begin{aligned} \mathcal{C} &= [B \quad AB \quad \cdots \quad A^{n-1}B] \\ &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & 1 \end{bmatrix} = I_n. \end{aligned}$$

- This is why it is called controllability form!
- If a system is controllable, we can instantaneously move the state from any known state to any other state, using impulse-like inputs.
- Later, we'll see that smooth inputs can effect the state transfer (not instantaneously, though!).

DUALITY: $\{A, B, C, D\}$ controllable $\Longleftrightarrow$ $\{A^T, C^T, B^T, D^T\}$ observable.

EXAMPLE: Two uncontrollable networks.



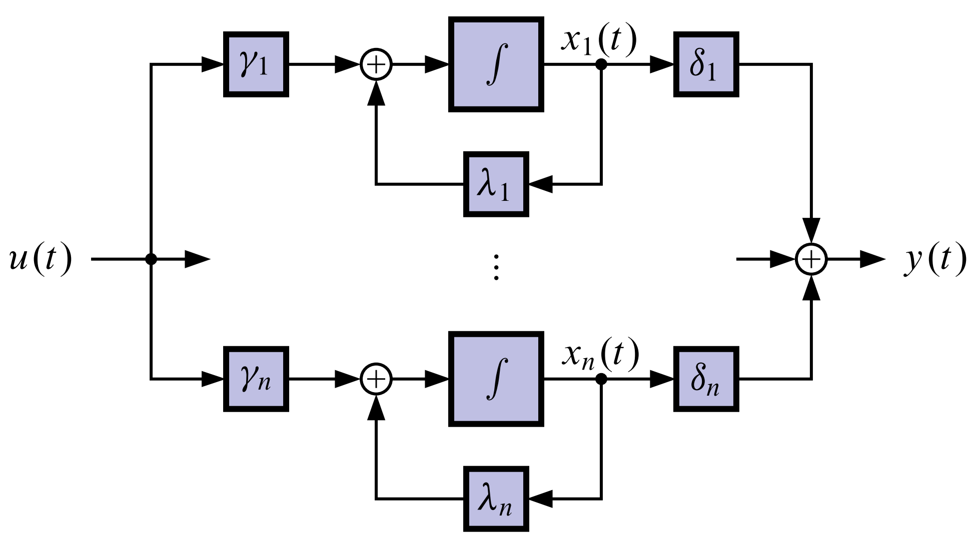
- In the first one, if $x(0) = 0$ then $x(t) = 0 \quad \forall\, t$. Cannot influence state!
- In the second one, if $x_1(0) = x_2(0)$ then $x_1(t) = x_2(t) \quad \forall\, t$. Cannot independently alter state.

Diagonal systems, controllability and observability

- Recall the diagonal form

$$\dot{x}(t) = \begin{bmatrix} \lambda_1 & & & 0 \\ & \lambda_2 & & \\ & & \ddots & \\ 0 & & & \lambda_n \end{bmatrix} x(t) + \begin{bmatrix} \gamma_1 \\ \gamma_2 \\ \vdots \\ \gamma_n \end{bmatrix} u(t)$$

$$y(t) = \begin{bmatrix} \delta_1 & \delta_2 & \cdots & \delta_n \end{bmatrix} x(t) + \begin{bmatrix} 0 \end{bmatrix} u(t).$$



- When controllable? When observable?

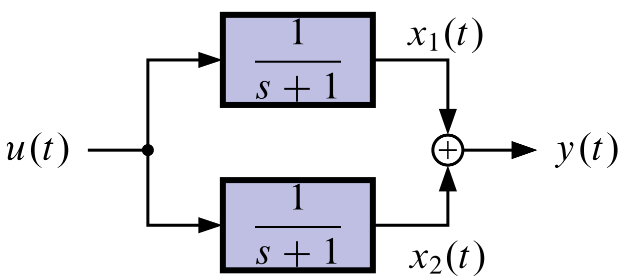
$$\mathcal{O} = \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix} = \begin{bmatrix} \delta_1 & \delta_2 & \cdots & \delta_n \\ \lambda_1\delta_1 & \lambda_2\delta_2 & \cdots & \lambda_n\delta_n \\ & & \ddots & \\ \lambda_1^{n-1}\delta_1 & \lambda_2^{n-1}\delta_2 & \cdots & \lambda_n^{n-1}\delta_n \end{bmatrix}$$

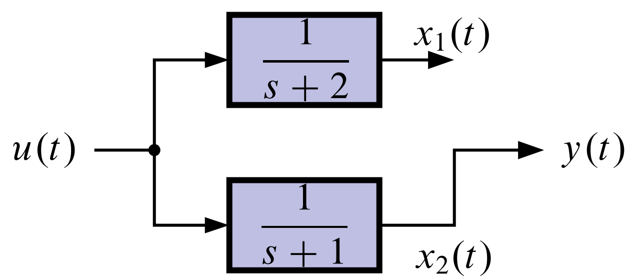
$$= \underbrace{\begin{bmatrix} 1 & 1 & \cdots & 1 \\ \lambda_1 & \lambda_2 & \cdots & \lambda_n \\ & & \ddots & \\ \lambda_1^{n-1} & \lambda_2^{n-1} & \cdots & \lambda_n^{n-1} \end{bmatrix}}_{\text{Vandermonde matrix } \mathcal{V}} \begin{bmatrix} \delta_1 & & & 0 \\ & \delta_2 & & \\ & & \ddots & \\ 0 & & & \delta_n \end{bmatrix}.$$

- Singular?

$$\det\{\mathcal{O}\} = (\delta_1 \cdots \delta_n)\det\{\mathcal{V}\} = (\delta_1 \cdots \delta_n)\prod_{i<j}(\lambda_j - \lambda_i).$$

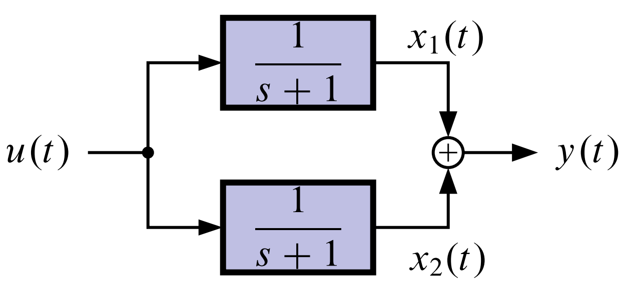
CONCLUSION: Observable $\iff \lambda_i \neq \lambda_j,\ i \neq j$ and $\delta_i \neq 0\ i = 1, \cdots, n$.

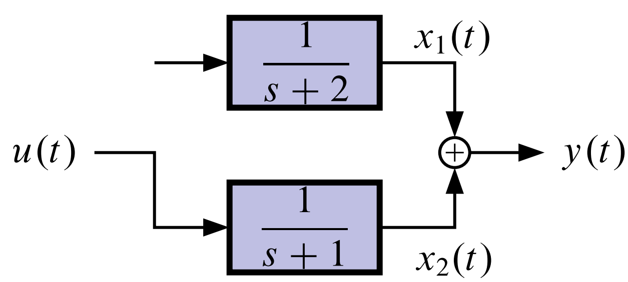




- If $\lambda_1 = \lambda_2$ then not observable. Can only "observe" the sum $x_1 + x_2$.
- If $\delta_k = 0$ then cannot observe mode k.
- What about controllability? Use duality and switch δs and γs.

CONCLUSION: Controllable $\iff \lambda_i \neq \lambda_j,\ i \neq j$ and $\gamma_i \neq 0\ i = 1, \cdots, n$.





- If $\lambda_1 = \lambda_2$ then not controllable. Can only "control" the sum $x_1 + x_2$.
- If $\gamma_k = 0$ then cannot control mode k.

5.3: Discrete-time controllability and observability

Discrete-time controllability

- Similar concept for discrete-time.
- Consider the problem of driving a system to some arbitrary state $x[n]$

$$\begin{aligned} x[k+1] &= Ax[k] + Bu[k] \\ x[1] &= Ax[0] + Bu[0] \\ x[2] &= A\left[Ax[0] + Bu[0]\right] + Bu[1] \\ x[3] &= A\left[A^2x[0] + ABu[0] + Bu[1]\right] + Bu[2] \\ &\vdots \\ x[n] &= A^n x[0] + \underbrace{\begin{bmatrix} B & AB & A^2B & \cdots & A^{n-1}B \end{bmatrix}}_{\mathcal{C}} \begin{bmatrix} u[n-1] \\ \vdots \\ u[0] \end{bmatrix}. \end{aligned}$$

- Which leads to

$$\begin{bmatrix} u[n-1] \\ \vdots \\ u[0] \end{bmatrix} = \mathcal{C}^{-1}\left[x[n] - A^n x[0]\right].$$

  If $\mathcal{C}$ has no inverse ($\det(\mathcal{C}) = 0$, $\mathcal{C}$ is not full-rank) then these control signals don't exist. In that case, the input is only *partially* effective in influencing the state.

- If $\mathcal{C}$ is full-rank, then the input can move the system to any *arbitrary* state for *any* $x[0]$.

NOTE I: State transition is not instantaneous. Takes n time steps.

NOTE II: In continuous-time, we used input $u(t) = g_0\delta(t) + g_1\dot{\delta}(t) + \cdots$, a signal we could only approximate in practice. Here, the input is a perfectly good input signal.

Discrete-time reachability

- In the literature, there are three different controllability definitions:
  1. Transfer any state to any other state.
  2. Transfer any state to zero, called *controllability to the origin*.
  3. Transfer the zero state to any state, called *controllability from the origin*, or *reachability*.
- In continuous time, because e^{At} is nonsingular, the three definitions are equivalent.
- In discrete time, if A is nonsingular, the three definitions are also equivalent.
- However, if A is singular, (1) and (3) are equivalent but not (2) and (3).

EXAMPLE:

$$x[k+1] = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} x[k] + \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} u[k].$$

- Its controllability matrix has rank 0 and the equation is not controllable in (1) or (3).
- However, $A^k = 0$ for $k \geq 3$ so $x[3] = A^3 x[0] = 0$ for any initial state $x[0]$ and any input $u[k]$.
- Thus, the system is *controllable to the origin* but not *controllable from the origin* or *reachable.*

- Definition (1) encompasses the other two definitions, so is used as our definition of controllable.

Discrete-time observability

- Can we reconstruct the state $x[0]$ from the output $y[k]$ and input $u[k]$?

$$
\begin{aligned}
y[k] &= Cx[k] + Du[k] \\
y[0] &= Cx[0] + Du[0] \\
y[1] &= C\left[Ax[0] + Bu[0]\right] + Du[1] \\
y[2] &= C\left[A^2x[0] + ABu[0] + Bu[1]\right] + Du[2] \\
&\ \vdots \\
y[n-1] &= C\left[A^{n-1}x[0] + A^{n-2}Bu[0] + \cdots + Bu[n-1]\right] + Du[n-1].
\end{aligned}
$$

- In vector form, we can write

$$
\begin{bmatrix} y[0] \\ y[1] \\ \vdots \\ y[n-1] \end{bmatrix} = \underbrace{\begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix}}_{\mathcal{O}} x[0] + \underbrace{\begin{bmatrix} D & 0 & \cdots & 0 \\ CB & D & \cdots & 0 \\ CAB & CB & \cdots & 0 \\ \vdots & \vdots & \ddots & D \end{bmatrix}}_{\mathcal{T}} \begin{bmatrix} u[0] \\ u[1] \\ \vdots \\ u[n-1] \end{bmatrix}.
$$

- So,

$$
x[0] = \mathcal{O}^{-1}\left[\begin{bmatrix} y[0] \\ \vdots \\ y[n-1] \end{bmatrix} - \mathcal{T}\begin{bmatrix} u[0] \\ \vdots \\ u[n-1] \end{bmatrix}\right].
$$

- If $\mathcal{O}$ is full-rank or nonsingular, $x[0]$ may be reconstructed with any $y[k]$, $u[k]$. We say that $\{C, A\}$ form an "observable pair."

- Do more measurements of $y[n],\ y[n+1],\ldots$ help in reconstructing $x[0]$? No! (Caley–Hamilton theorem: next section).
- So, if the original state is not "observable" with n measurements, then it will not be observable with more than n measurements either.
- Since we know $u[k]$ and the dynamics of the system, if the system is observable we can determine the entire state sequence $x[k],\ k \ge 0$ once we determine $x[0]$

$$\begin{aligned} x[n] &= A^n x[0] + \sum_{i=0}^{n-1} A^{n-1-i} B u[k] \\ &= A^n \mathcal{O}^{-1} \left[\begin{bmatrix} y[0] \\ \vdots \\ y[n-1] \end{bmatrix} - \mathcal{T} \begin{bmatrix} u[0] \\ \vdots \\ u[n-1] \end{bmatrix} \right] + \mathcal{C} \begin{bmatrix} u[n-1] \\ \vdots \\ u[0] \end{bmatrix}. \end{aligned}$$

- A perfectly good observer (no differentiators...)

5.4: Cayley–Hamilton Theorem

- A square matrix A satisfies its own characteristic equation. That is, if

$$\chi(\lambda) = \det(\lambda I - A) = 0$$

then

$$\chi(A) = 0.$$

- We can easily show this if A is diagonalizable. Let

$$A = V^{-1}\Lambda V.$$

- Then

$$\begin{aligned} A^2 &= V^{-1}\Lambda V V^{-1}\Lambda V \\ &= V^{-1}\Lambda^2 V \\ A^k &= V^{-1}\Lambda^k V. \end{aligned}$$

- The characteristic polynomial is:

$$\chi(\lambda) = \lambda^n + a_{n-1}\lambda^{n-1} + \cdots + a_1$$

so if we replace λ with A we get

$$\begin{aligned} \chi(A) &= A^n + a_{n-1}A^{n-1} + \cdots + a_1 I \\ &= V^{-1}\left[\Lambda^n + a_{n-1}\Lambda^{n-1} + \cdots + a_1 I\right]V. \end{aligned}$$

- To "prove" the Cayley–Hamilton theorem, we just need to show that the quantity inside the brackets is zero.
- It *is* a diagonal matrix, and each diagonal element has the form

$$\lambda_i^n + a_{n-1}\lambda_i^{n-1} + \cdots + a_1 = 0$$

because λ_i is an eigenvalue of A.

- So, each diagonal element is zero, and we have shown the proof.
- If A is not diagonalizable, the same proof may be repeated using the Jordan form and Jordan blocks:

$$A = T^{-1}JT.$$

- Consider a sketch of the proof for a Jordan block of size 2 and

$$\chi(\lambda_i) = \lambda_i^3 + a_2\lambda_i^2 + a_1\lambda_i + a_0 = 0.$$

- Then

$$J_i = \begin{bmatrix} \lambda_i & 1 \\ 0 & \lambda_i \end{bmatrix}$$

$$\chi(J_i) = \begin{bmatrix} \lambda_i^3 & 3\lambda_i^2 \\ 0 & \lambda_i^3 \end{bmatrix} + a_2 \begin{bmatrix} \lambda_i^2 & 2\lambda_i \\ 0 & \lambda_i^2 \end{bmatrix} + a_1 \begin{bmatrix} \lambda_i & 1 \\ 0 & \lambda_i \end{bmatrix} + a_0 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

- We can easily see that the diagonal and lower-diagonal components are zero so

$$\chi(J_i) = \begin{bmatrix} 0 & \alpha \\ 0 & 0 \end{bmatrix}$$

where

$$\alpha = 3\lambda_i^2 + 2a_2\lambda_i + a_1$$

but $\alpha = \dfrac{\mathrm{d}}{\mathrm{d}\lambda}\chi(\lambda) = 0$ which completes the sketch.

SIGNIFICANCE: The Cayley–Hamilton theorem shows us that A^n is a function of matrix powers A^{n-1} down to A^0. Therefore, to compute any polynomial of A it suffices to compute only powers of A up to A^{n-1} and appropriately weight their sum. A lot of proofs use the Cayley–Hamilton theorem.

- As we just saw with the section on discrete-time observability, the Cayley–Hamilton theorem implies that if we cannot observe the state with n measurements, we cannot observe it with more measurements either.

EXAMPLE: With $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ we have $\chi(\lambda) = \det(\lambda I - A)$, so

$$\begin{aligned}
\chi(\lambda) &= \det\left(\begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} - A\right) \\
\chi(\lambda) &= \lambda^2 - 5\lambda - 2 \\
\chi(A) &= A^2 - 5A - 2I \\
&= \begin{bmatrix} 7 & 10 \\ 15 & 22 \end{bmatrix} - 5\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} - 2\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \\
&= \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}.
\end{aligned}$$

Disclaimer: Does observability/controllability matter, practically?

- The singularity of $\mathcal{C}$ has only one "bit" of information: Is the realization mathematically controllable or not? This may not tell the whole story.

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 1+\epsilon \end{bmatrix}, \qquad B = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \qquad \mathcal{C} = \begin{bmatrix} 1 & 1 \\ 1 & 1+\epsilon \end{bmatrix}$$

- $\{A, B\}$ are a controllable pair, but barely.

EXAMPLE: Controlling an airplane. (Ideas only, no details). System state

$$x \triangleq \begin{bmatrix} \theta & \dot\theta & \phi & \dot\phi \end{bmatrix}^T, \qquad \theta = \text{Pitch}, \ \phi = \text{Roll}.$$

- Control with elevator?

$$\dot{x} = \begin{bmatrix} F_\theta & \epsilon \\ 0 & F_\phi \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} \delta_e$$

where δ_e is the elevator angle. $\mathcal{C}$ is singular ➠ can't influence roll with elevators.

- Control with ailerons?

$$\dot{x} = \begin{bmatrix} F_\theta & \epsilon \\ 0 & F_\phi \end{bmatrix} x + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \delta_a$$

where δ_a is the aileron angle. $\mathcal{C}$ is nonsingular! So, we can control both pitch *AND* roll with ailerons.

- *THIS IS NONSENSE*! Physically think of the system! Do you want to roll plane over every time you need to pitch down?
- Physical intuition can be better than finding $\mathcal{C}$. Other tools can help. . .

5.5: Continuous-time Gramians

Continuous-time controllability Gramian

- If a continuous-time system is controllable, then

$$W_c(t) = \int_0^t e^{A\tau} B B^T e^{A^T \tau}\,\mathrm{d}\tau$$

is nonsingular for $t > 0$.

SIGNIFICANCE: Consider

$$x(t_1) = e^{At_1} x(0) + \int_0^{t_1} e^{A(t_1-\tau)} B u(\tau)\,\mathrm{d}\tau.$$

- We claim that for any $x(0) = x_0$ and any $x(t_1) = x_1$ the input

$$u(t) = -B^T e^{A^T(t_1-t)} W_c^{-1}(t_1)\left[e^{At_1} x_0 - x_1\right]$$

will transfer x_0 to x_1 at time t_1.

PROOF: Substitute the expression for $u(t)$ into the convolution expression:

$$\begin{aligned}
x(t_1) &= e^{At_1} x(0) - \int_0^{t_1} e^{A(t_1-\tau)} B B^T e^{A^T(t_1-\tau)}\,\mathrm{d}\tau\ W_c^{-1}(t_1)\left[e^{At_1} x_0 - x_1\right] \\
&= e^{At_1} x(0) - \int_0^{t_1} e^{A\beta} B B^T e^{A^T \beta}\,\mathrm{d}\beta\ W_c^{-1}(t_1)\left[e^{At_1} x_0 - x_1\right] \\
&= e^{At_1} x(0) - W_c(t_1) W_c^{-1}(t_1)\left[e^{At_1} x_0 - x_1\right] \\
&= e^{At_1} x(0) - e^{At_1} x_0 + x_1 = x_1.
\end{aligned}$$

- Therefore, we can *compute* the input $u(t)$ required to transfer the state of the system from one state to another over an arbitrary interval of time. The solution is also the minimum-energy solution.

EXAMPLE: Consider the system in diagonal form

$$\dot{x}(t) = \begin{bmatrix} -0.5 & 0 \\ 0 & -1 \end{bmatrix} x(t) + \begin{bmatrix} 0.5 \\ 1 \end{bmatrix} u(t).$$

- The controllability matrix is:

$$\mathcal{C} = \begin{bmatrix} 0.5 & -0.25 \\ 1 & -1 \end{bmatrix}$$

which has rank 2, so the system is controllable.

- Consider the input required to move the system state from $x(0) = [10 \;\; -1]^T$ to zero in two seconds.

$$\begin{aligned} W_c(2) &= \int_0^2 \left(\begin{bmatrix} e^{-0.5\tau} & 0 \\ 0 & e^{-\tau} \end{bmatrix} \begin{bmatrix} 0.5 \\ 1 \end{bmatrix} \begin{bmatrix} 0.5 & 1 \end{bmatrix} \begin{bmatrix} e^{-0.5\tau} & 0 \\ 0 & e^{-\tau} \end{bmatrix} \right) \mathrm{d}\tau \\ &= \begin{bmatrix} 0.2162 & 0.3167 \\ 0.3167 & 0.4908 \end{bmatrix}, \end{aligned}$$

and

$$\begin{aligned} u(t) &= -\begin{bmatrix} 0.5 & 1 \end{bmatrix} \begin{bmatrix} e^{-0.5(2-t)} & 0 \\ 0 & e^{-(2-t)} \end{bmatrix} W_c(2)^{-1} \begin{bmatrix} e^{-1} & 0 \\ 0 & e^{-2} \end{bmatrix} \begin{bmatrix} 10 \\ -1 \end{bmatrix} \\ &= -58.82e^{0.5t} + 27.96e^{t}. \end{aligned}$$

- If a continuous-time system is controllable, and if it is also stable, then

$$W_c = \int_0^\infty e^{A\tau} B B^T e^{A^T \tau} \, \mathrm{d}\tau$$

can be found by solving for the unique (positive-definite) solution to the (Lyapunov) equation

$$A W_c + W_c A^T = -B B^T.$$

W_c is called the *controllability Gramian*.

PROOF: We proved this identity when considering Lyapunov stability.

- W_c measures the minimum energy required to reach a desired point x_1 starting at $x(0) = 0$ (with no limit on t)

$$\min\left\{\int_0^t \|u(\tau)\|^2 \, \mathrm{d}\tau \;\middle|\; x(0) = 0,\; x(t) = x_1\right\} = x_1^T W_c^{-1} x_1.$$

- In fact, for any specific "t", the minimum energy is $x_1^T W_c^{-1}(t) x_1$.
- If A is stable, $W_c^{-1} > 0$ which implies "we can't get anywhere for free".
- If A is unstable, then W_c^{-1} can have a nonzero nullspace $W_c^{-1} z = 0$ for some $z \neq 0$ which means that we can get to z using u's with energy as small as you like! (u just gives a little kick to the state; the instability carries it out to z efficiently).
- W_c may be a better indicator of controllability than $\mathcal{C}$.

Continuous-time observability Gramian

- If a system is observable, $W_o(t)$ is nonsingular for $t > 0$ where

$$W_o(t) = \int_0^t e^{A^T \tau} C^T C e^{A\tau} \, \mathrm{d}\tau.$$

SIGNIFICANCE: We can prove that

$$x(0) = W_o^{-1}(t_1) \int_0^{t_1} e^{A^T t} C^T \bar{y}(t) \, \mathrm{d}t$$

where

$$\bar{y}(t) = y(t) - C \int_0^t e^{A(t-\tau)} B u(\tau) \, \mathrm{d}\tau - D u(t) = C e^{At} x(0).$$

- Therefore, we can determine the initial state $x(0)$ given a finite observation period (and not use differentiators!).

PROOF: We prove that the above equations are correct by substitution:

$$
\begin{aligned}
W_o^{-1}(t_1)\int_0^{t_1} e^{A^T t} C^T \bar{y}(t)\,\mathrm{d}t &= W_o^{-1}(t_1)\int_0^{t_1} e^{A^T t} C^T C e^{At}\,\mathrm{d}t\ x(0)\\
&= W_o^{-1}(t_1) W_o(t_1) x(0)\\
&= x(0).
\end{aligned}
$$

- If a continuous-time system is observable, and if it is also stable, then

$$W_o = \int_0^{\infty} e^{A^T \tau} C^T C e^{A\tau}\,\mathrm{d}\tau$$

  is the unique (positive-definite) solution to the (Lyapunov) equation

$$A^T W_o + W_o A = -C^T C.$$

  W_o is called the *observability Gramian*.

- This relationship can be proven in the same way we proved the similar relationship for the controllability gramian.
- If measurement (sensor) noise is IID $\mathcal{N}(0, \sigma^2 I)$ then W_o is a measure of error covariance in measuring $x(0)$ from u and y over longer and longer periods

$$\lim_{t\to\infty} \mathbb{E}\,\|\hat{x}(0) - x(0)\|^2 = \sigma x(0)^T W_o^{-1} x(0).$$

- If A is stable, then $W_o^{-1} > 0$ and we can't estimate the initial state perfectly even with an infinite number of measurements $u(t)$ and $y(t)$ for $t \ge 0$ (since memory of $x(0)$ fades).
- If A is not stable then W_o^{-1} can have a nonzero nullspace $W_o^{-1}x(0) = 0$ which means that the covariance goes to zero as $t \to \infty$.
- W_o may be a better indicator of observability than $\mathcal{O}$.

5.6: Discrete-time Gramians

Discrete-time controllability Gramian

- In discrete-time, if a system is controllable, then

$$W_{dc}[n-1] = \sum_{m=0}^{n-1} A^m BB^T (A^T)^m$$

is nonsingular. In particular,

$$W_{dc} = \sum_{m=0}^{\infty} A^m BB^T (A^T)^m$$

is called the *discrete-time controllability Gramian* and is the unique positive-definite solution to the Lyapunov equation

$$W_{dc} - AW_{dc}A^T = BB^T.$$

- As with continuous-time, W_{dc} measures the minimum energy required to reach a desired point x_1 starting at $x[0] = 0$ (with no limit on m)

$$\min \left\{ \sum_{k=0}^{m} \|u[k]\|^2 \;\middle|\; x[0] = 0,\ x[m] = x_1 \right\} = x_1^T W_{dc}^{-1} x_1.$$

ASIDE: When considering discrete-time stability, we showed that this form of equation is indeed a Lyapunov equation.

Discrete-time observability Gramian

- In discrete-time, if a system is observable, then

$$W_{do}[n-1] = \sum_{m=0}^{n-1} (A^T)^m C C^T A^m$$

is nonsingular. In particular,

$$W_{do} = \sum_{m=0}^{\infty} (A^T)^m C C^T A^m$$

is called the *discrete-time observability Gramian* and is the unique positive-definite solution to the Lyapunov equation

$$W_{do} - A^T W_{do} A = C^T C.$$

- As with continuous-time, if measurement (sensor) noise is IID $\mathcal{N}(0, \sigma I)$ then W_{do} is a measure of error covariance in measuring $x[0]$ from u and y over longer and longer periods

$$\lim_{t\to\infty} \mathbb{E} \left\| \hat{x}[0] - x[0] \right\|^2 = \sigma x[0]^T W_{do}^{-1} x[0].$$

5.7: Computing transformation matrices

Transformation to controllability form

- Given a system

$$\dot{x}(t) = Ax(t) + Bu(t)$$
$$y(t) = Cx(t) + Du(t),$$

can we find a transformation (similarity) matrix T to transform this system into controllability form? Recall, this looks like:

$$\dot{x}_{co}(t) = \begin{bmatrix} 0 & \cdots & 0 & -a_n \\ 1 & & \vdots & -a_{n-1} \\ \vdots & \ddots & & \vdots \\ 0 & \cdots & 1 & -a_1 \end{bmatrix} x_{co}(t) + \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} u(t)$$
$$y(t) = \begin{bmatrix} \beta_1 & \beta_2 & \cdots & \beta_n \end{bmatrix} x_{co}(t) + Du(t).$$

- Note that $\mathcal{C}_{co} = I$ so it is controllable. Thus, our original system must be controllable.
- The transformation is accomplished via

$$x = Tx_{co}.$$

- Thus

$$T^{-1}AT = A_{co}, \qquad T^{-1}B = B_{co}$$
$$CT = C_{co}, \qquad D = D_{co}.$$

- Let's find T explicitly; let

$$T = [t_1, \ldots, t_n].$$

- Note that $B = TB_{co}$ or

$$B = T\begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} = t_1.$$

- So, $t_1 = B$. From $AT = TA_{co}$ we have

$$A[t_1, \ldots, t_n] = [t_1, \ldots, t_n]\begin{bmatrix} 0 & \cdots & 0 & -a_n \\ 1 & & \vdots & -a_{n-1} \\ \vdots & \ddots & & \vdots \\ 0 & \cdots & 1 & -a_1 \end{bmatrix}.$$

- So

$$[At_1, \ldots At_n] = \left[t_2, \ldots, t_n, -\sum_{k=1}^{n} a_k t_{n-k+1}\right].$$

- By induction,

$$At_1 = t_2 = AB$$

$$At_2 = t_3 = A^2B, \text{ and so forth} \ldots$$

so

$$T = [B\ AB\ \cdots\ A^{n-1}B] = \mathcal{C}.$$

CONCLUSION: A system $\{A, B, C, D\}$ can be transformed to controllability canonical form if and only if it is controllable, in which case the change of coordinates is

$$x = \mathcal{C}x_{co}.$$

EXTENSION I: If $x_{\text{old}} = Tx_{\text{new}}$ then $T = \mathcal{C}_{\text{old}}\mathcal{C}_{\text{new}}^{-1}$. That is, to convert between any two realizations, T is a combination of the controllability matrices of the two different realizations.

$$\begin{aligned}\mathcal{C}_{\text{new}} &= \begin{bmatrix} B_{\text{new}} & A_{\text{new}}B_{\text{new}} \cdots A_{\text{new}}^{n-1}B_{\text{new}} \end{bmatrix} \\ &= \begin{bmatrix} T^{-1}B_{\text{old}} & T^{-1}A_{\text{old}}T\,T^{-1}B_{\text{old}} \cdots T^{-1}A_{\text{old}}^{n-1}T\,T^{-1}B_{\text{old}} \end{bmatrix} \\ &= T^{-1}\mathcal{C}_{\text{old}},\end{aligned}$$

or

$$T = \mathcal{C}_{\text{old}}\mathcal{C}_{\text{new}}^{-1}.$$

EXTENSION II: If $x_{\text{old}} = T x_{\text{new}}$ then $T = \mathcal{O}_{\text{old}}^{-1}\mathcal{O}_{\text{new}}$. This can be shown in a similar way.

5.8: Canonical (Kalman) decompositions

- What happens if $\{A, B\}$ not controllable or if $\{A, C\}$ not observable?
  - Is "part" of the system controllable?
  - Is "part" of the system observable?
- Given a system with $\{A, B\}$ not controllable, $\dot{x}(t) = Ax(t) + Bu(t)$, $y(t) = Cx(t)$, let's try to transform to controllability canonical form.
- Let $t_1 = B$, $t_2 = AB, \ldots t_r = A^{r-1}B$, and suppose $t_1, t_2, \ldots, t_r$ are independent but

$$t_{r+1} = A^r B = -\alpha_r t_1 - \cdots - \alpha_1 t_r$$

  for some constants $\alpha_1, \ldots, \alpha_r$.
- Then $\text{rank}(\mathcal{C}) = r$ since the vectors $A^r B,\ A^{r+1}B, \ldots, A^{n-1}B$ can all be expressed as a linear combination of $t_1,\ t_2, \ldots, t_r$.
- Let $s_{r+1}, \ldots, s_n$ be your favorite vectors for which

$$\bar{\mathcal{C}}_{\text{old}} = \left[\begin{array}{ccc:ccc} t_1 & \cdots & t_r & s_{r+1} & \cdots & s_n \end{array}\right]$$

  is invertible.
  - If we were able to change coordinates to controllability form via $x_{\text{old}} = Tx_{\text{new}}$, we would use $T = \mathcal{C}_{\text{old}}\mathcal{C}_{\text{new}}^{-1}$.
  - But, the system is not controllable, so we use $T = \bar{\mathcal{C}}_{\text{old}}\mathcal{C}_{\text{new}}^{-1} = \bar{\mathcal{C}}_{\text{old}}$.
- We get (in the new coordinate system)

$$\left[\begin{array}{c} \dot{x}_c(t) \\ \hdashline \dot{x}_{\bar{c}}(t) \end{array}\right] = \left[\begin{array}{c:c} A_c & A_{12} \\ \hdashline 0 & A_{\bar{c}} \end{array}\right] \left[\begin{array}{c} x_c(t) \\ \hdashline x_{\bar{c}}(t) \end{array}\right] + \left[\begin{array}{c} B_c \\ \hdashline 0 \end{array}\right] u(t)$$

$$y(t) = \left[\begin{array}{c:c} C_c & C_{\bar{c}} \end{array}\right] \left[\begin{array}{c} x_c(t) \\ \hdashline x_{\bar{c}}(t) \end{array}\right] + Du(t).$$

- A_c is a right-companion matrix and B_c is of the controllability-canonical form.
- We see that the uncontrollable modes $x_{\bar{c}}$ are completely decoupled from $u(t)$.

EXAMPLE: Consider

$$\frac{1}{s+1} = \frac{s-1}{(s+1)(s-1)} = \frac{s-1}{s^2-1}.$$

- In observer-canonical form,

$$\dot{x}(t) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} x(t) + \begin{bmatrix} 1 \\ -1 \end{bmatrix} u(t)$$

$$y(t) = \begin{bmatrix} 1 & 0 \end{bmatrix} x(t).$$

- So,

$$t_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix} \qquad \text{and} \qquad t_2 = At_1 = \begin{bmatrix} -1 \\ 1 \end{bmatrix} = -t_1.$$

- So, $\text{rank}(\mathcal{C}) = 1$. Let $s_1 = [1 \quad 0]^T$. The converted state-space form is

$$\dot{\bar{x}}(t) = \begin{bmatrix} -1 & -1 \\ 0 & 1 \end{bmatrix} \bar{x}(t) + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u(t)$$

$$y(t) = \begin{bmatrix} 1 & 1 \end{bmatrix} \bar{x}(t).$$

- Now suppose that we desire to transform a system that is not observable. The dual form separates out the unobservable states.
- Let $t_1 = C,\ t_2 = CA, \ldots t_r = CA^{r-1}$, and suppose $t_1, t_2, \ldots, t_r$ are independent but

$$t_{r+1} = CA^r = -\alpha_r t_1 - \cdots - \alpha_1 t_r$$

for some constants $\alpha_1, \ldots, \alpha_r$.

- Then, let $s_{r+1}, \ldots, s_n$ be your favorite row vectors for which

$$\bar{\mathcal{O}}_{\text{old}} = \begin{bmatrix} t_1 \\ \vdots \\ t_r \\ \hdashline s_{r+1} \\ \vdots \\ s_n \end{bmatrix}$$

is invertible.

  - If we were able to change coordinates to observability form via $x_{\text{old}} = T x_{\text{new}}$, we would use $T = \mathcal{O}_{\text{old}}^{-1}\mathcal{O}_{\text{new}}$.
  - But, the system is not observable, so we use $T = \bar{\mathcal{O}}_{\text{old}}^{-1}\mathcal{O}_{\text{new}} = \bar{\mathcal{O}}_{\text{old}}^{-1}$.

- Then, we get the Kalman decomposition

$$\begin{bmatrix} \dot{x}_o(t) \\ \hdashline \dot{x}_{\bar{o}}(t) \end{bmatrix} = \begin{bmatrix} A_o & 0 \\ \hdashline A_{21} & A_{\bar{o}} \end{bmatrix} \begin{bmatrix} x_o(t) \\ \hdashline x_{\bar{o}}(t) \end{bmatrix} + \begin{bmatrix} B_o \\ \hdashline B_{\bar{o}} \end{bmatrix} u(t)$$

$$y(t) = \begin{bmatrix} C_o & 0 \end{bmatrix} \begin{bmatrix} x_o(t) \\ \hdashline x_{\bar{o}}(t) \end{bmatrix} + Du(t).$$

- Note: No path from $x_{\bar{o}}$ to y!

Full Kalman decomposition

- We can do both transformations in sequence to get full Kalman decomposition. See text chapter 16.

5.9: Popov–Belevitch–Hautus controllability/observability tests

PBH EIGENVECTOR TEST: $\{C, A\}$ is an unobservable pair iff a non-zero eigenvector v of A satisfies $Cv = 0$. (*i.e.*, C and v are perpendicular).

PROOF: $\Rightarrow$ Suppose $Av = \lambda v$ and $Cv = 0$ for $v \neq 0$. Then $CAv = \lambda Cv = 0$ and so forth up to $CA^{n-1}v = \lambda^{n-1}Cv = 0$. So, $\mathcal{O}v = 0$ and since $v \neq 0$ this means that $\mathcal{O}$ is not full rank and $\{C, A\}$ is not observable.

- $\Leftarrow$ Now suppose that $\mathcal{O}$ is not full rank ($\{C, A\}$ unobservable). Let's extract the unobservable part. That is, find T such that

$$T^{-1}AT = \begin{bmatrix} A_o & 0 \\ A_{21} & A_{\bar{o}} \end{bmatrix}$$

$$CT = \begin{bmatrix} C_o & 0 \end{bmatrix},$$

where A_o is size r where $r = \mathrm{rank}(\mathcal{O})$ and therefore $A_{\bar{o}}$ is size $n - r$.

- Let $v_2 \neq 0$ be an eigenvector of $A_{\bar{o}}$. Then

$$T^{-1}AT \begin{bmatrix} 0 \\ v_2 \end{bmatrix} = \begin{bmatrix} A_o & 0 \\ A_{21} & A_{\bar{o}} \end{bmatrix} \begin{bmatrix} 0 \\ v_2 \end{bmatrix} = \lambda \begin{bmatrix} 0 \\ v_2 \end{bmatrix}$$

so we have found an eigenvector z of A

$$Az = \lambda z \qquad \text{where} \qquad z = T \begin{bmatrix} 0 \\ v_2 \end{bmatrix}.$$

- Now, we just need to show that $Cz = 0$ (note: $z \neq 0$).

$$Cz = \underbrace{CT}_{\bar{C}} \begin{bmatrix} 0 \\ v_2 \end{bmatrix} = \begin{bmatrix} C_o & 0 \end{bmatrix} \begin{bmatrix} 0 \\ v_2 \end{bmatrix} = 0$$

and we are done.

DUAL: $\{A, B\}$ is uncontrollable iff there is a left eigenvector w^T of A such that $w^T B = 0$.

INTERPRETATION: In modal coordinates, homogeneous response is

$$x(t) = \sum_{i=1}^{n} e^{\lambda_i t} v_i (w_i^T x(0)) \qquad \text{and} \qquad y(t) = Cx(t).$$

Or,

$$y(t) = \sum_{i=1}^{n} e^{\lambda_i t} C v_i (w_i^T x(0)).$$

If $\{C, A\}$ is unobservable, then it has an *unobservable mode*, where $Av_i = \lambda_i v_i$ and $Cv_i = 0$.

- If $\{A, B\}$ is uncontrollable, then it has an *uncontrollable mode*, namely the coefficients of the state along that mode is independent of the input $u(t)$.
  - The coefficients of x in the mode associated with λ are $w^T x$.
  $$\frac{\mathrm{d}}{\mathrm{d}t}(w^T x) = w^T(Ax + Bu) = \lambda(w^T x)$$
  or
  $$w^T x(t) = e^{\lambda t}(w^T x(0))$$
  regardless of the input $u(t)$!

PBH RANK TESTS: The following two tests are often easier to perform

1. $\{A, B\}$ controllable iff $\operatorname{rank}\left[\begin{array}{c:c} sI - A & B \end{array}\right] = n$ for all $s \in \mathbb{C}$.

   $\Rightarrow$ If $\operatorname{rank}\left[\begin{array}{c:c} sI - A & B \end{array}\right] = n$ for all $s \in \mathbb{C}$ then there can be no nonzero vector v such that $v^T \left[\begin{array}{c:c} sI - A & B \end{array}\right] = \left[\begin{array}{c:c} v^T(sI - A) & v^T B \end{array}\right] = 0$.

- Consequently, there is no nonzero vector v such that $v^T s = v^T A$ and $v^T B = 0$. By the PBH eigenvector test, the system will therefore be controllable

$\Leftarrow$ If the system is controllable, then there is no nonzero vector v such that $v^T s = v^T A$ and $v^T B = 0$ by the PBH eigenvector test.

- Therefore, $\operatorname{rank}\left[\begin{array}{c:c} sI - A & B \end{array}\right] = n$ for all $s \in \mathbb{C}$.

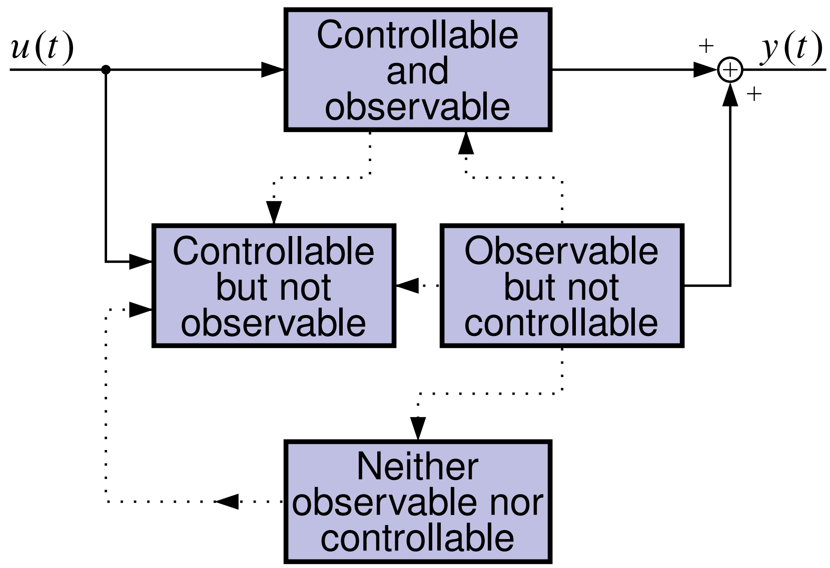
2. $\{C, A\}$ observable iff $\operatorname{rank}\left[\begin{array}{c} C \\ \hdashline sI - A \end{array}\right] = n$ for all $s \in \mathbb{C}$. Proof similar.

COMMENTS: $\operatorname{rank}\left[\begin{array}{c:c} sI - A & B \end{array}\right] = n$ for all s not eigenvalues of A, so the test is really $\{A, B\}$ controllable iff $\operatorname{rank}\left[\begin{array}{c:c} \lambda_i I - A & B \end{array}\right] = n$ for λ_i, $i = 1, \ldots, n$ the eigenvalues of A. (Dual argument for observability).

- If $\left[\begin{array}{c:c} sI - A & B \end{array}\right]$ drops rank at $s = \lambda$ then there is an uncontrollable mode with exponent (frequency) λ.
- If $\left[\begin{array}{c} C \\ \hdashline sI - A \end{array}\right]$ drops rank at $s = \lambda$ then there is an unobservable mode with exponent (frequency) λ.

Summary

- Therefore, we can label *individual modes* of a system as either controllable or not, or observable or not.
- The overall picture is:



■ Some other definitions:

STABILIZABLE: A system whose unstable modes are controllable.

DETECTABLE: A system whose unstable modes are observable.

5.10: Minimal realizations—Why a system isn't control/observ-able

- To realize $H(s) = \dfrac{1}{s+1}$ we could use

1. $\dot{x}(t) = -x(t) + u(t)$, $y(t) = x(t)$. This gives
   $A = [-1]$, $B = [1]$, $C = [1]$, $D = [0]$.
   $$\frac{C\,\mathrm{adj}(sI - A)B}{\det(sI - A)} = \frac{1}{s+1}.$$
   This realization is both controllable and observable.

2. Observer realization of $\dfrac{1}{s+1}\dfrac{s-1}{s-1} = \dfrac{s-1}{s^2-1}$. This gives
   $A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$, $C = \begin{bmatrix} 1 & 0 \end{bmatrix}$, $D = [0]$.
   $$\frac{C\,\mathrm{adj}(sI - A)B}{\det(sI - A)} = \frac{s-1}{s^2-1} = \frac{1}{s+1}.$$
   $\mathcal{O} = \begin{bmatrix} C \\ CA \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$. Observable.
   $\mathcal{C} = \begin{bmatrix} B & AB \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$. Not controllable.

3. Controller realization of $\dfrac{1}{s+1}\dfrac{s+10}{s+10} = \dfrac{s+10}{s^2+11s+10}$.
   $A = \begin{bmatrix} -11 & -10 \\ 1 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $C = \begin{bmatrix} 1 & 10 \end{bmatrix}$, $D = [0]$.
   $$\frac{C\,\mathrm{adj}(sI - A)B}{\det(sI - A)} = \frac{s+10}{s^2+11s+10} = \frac{1}{s+1}.$$
   $\mathcal{C} = \begin{bmatrix} B & AB \end{bmatrix} = \begin{bmatrix} 1 & -11 \\ 0 & 1 \end{bmatrix}$. Controllable.

$\mathcal{O} = \begin{bmatrix} C \\ CA \end{bmatrix} = \begin{bmatrix} 1 & 10 \\ -1 & -10 \end{bmatrix}$. Not observable.

TREND: Non-minimal realizations of transfer functions will either be uncontrollable, unobservable, or both.

- Four equivalent statements:

  I: There exist common roots of $C\,\mathrm{adj}(sI - A)B$ and $\det(sI - A)$.

  II: There exist eigenvalues of A which are not poles of $G(s)$, counting multiplicities.

  III: The system is either unobservable or uncontrollable.

  IV: There exist extra (unnecessary) states—non minimal.

DEFINITION: We say a system is minimal if no system with the same transfer function has fewer states.

PROOF: I$\Longleftrightarrow$II

I$\Longrightarrow$II: The transfer function

$$\begin{aligned} G(s) &= C(sI - A)^{-1}B + D \\ &= \frac{C\,\mathrm{adj}(sI - A)B + D\det(sI - A)}{\det(sI - A)}. \end{aligned}$$

If there are common roots in $C\,\mathrm{adj}(sI - A)B$ and $\det(sI - A)$ they will cancel out of the transfer function. But, eigenvalues of A are $\det(sI - A) = 0$, so poles of $G(s)$ will not contain all eigenvalues of A.

II$\Longrightarrow$I: Eigenvalues of A are $\det(sI - A) = 0$. Poles of $G(s)$, from above, are $\det(sI - A) = 0$ unless canceled. The only way to cancel a pole is to have common root in $C\,\mathrm{adj}(sI - A)B$.

PROOF: I$\Longleftrightarrow$IV $\{A, B, C, D\}$ is minimal iff $C\,\mathrm{adj}(sI - A)B$ and $\det(sI - A)$ are *coprime* (have no common roots).

I$\Longrightarrow$IV: Suppose $C\,\mathrm{adj}(sI - A)B$ and $\det(sI - A)$ have common roots.

- Cancel to get

$$G(s) = \frac{C\,\mathrm{adj}(sI - A)B}{\det(sI - A)} + D = \frac{b_r(s)}{a_r(s)}$$

  where $b_r(s)$ and $a_r(s)$ are coprime ("r" means "reduced").
- Because of cancellation, $k = \deg(a_r) < \deg(\det(sI - A)) = n$.
- Consider controller canonical form realization of $b_r(s)/a_r(s)$, for example.
- It has k states, but same transfer function as $\{A, B, C, D\}$, contradicting that $\{A, B, C, D\}$ minimal.

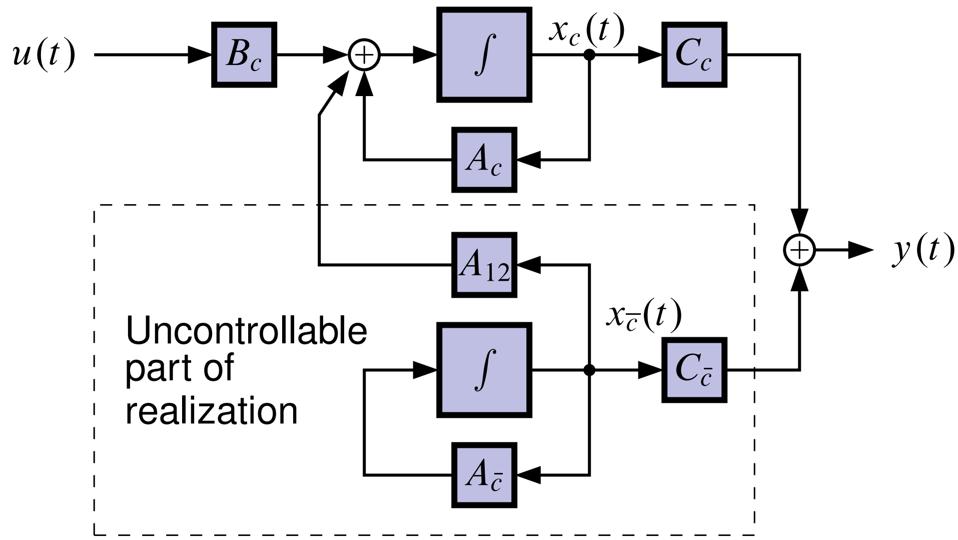
IV$\Longrightarrow$I: If there are extra states (non-minimal) then $n > k$ ($n =$# states, $k =$# poles in $G(s)$).

$$\begin{aligned} G(s) &= C(sI - A)^{-1}B + D \\ &= \frac{C\,\mathrm{adj}(sI - A)B + D\det(sI - A)}{\det(sI - A)} \end{aligned}$$

$G(s)$ has n poles unless some cancel with $C\,\mathrm{adj}(sI - A)B$. Therefore, if $n > k$, $C\,\mathrm{adj}(sI - A)B$ and $\det(sI - A)$ are not coprime.

PROOF: III$\Longleftrightarrow$IV Controllable and observable iff minimal.

III$\Longrightarrow$IV: Uncontrollable or unobservable $\Longrightarrow$ not minimal.
Perform Kalman decomposition to split system into co, $c\bar{o}$, $\bar{c}o$ and $\bar{c}\bar{o}$ parts.



$$\bar{A} = \begin{bmatrix} A_c & A_{12} \\ 0 & A_{\bar{c}} \end{bmatrix}, \bar{B} = \begin{bmatrix} B_c \\ 0 \end{bmatrix}, \bar{C} = \begin{bmatrix} C_c & C_{\bar{c}} \end{bmatrix}, \bar{D} = [0].$$

Same transfer function using A_c, B_c, C_c as $\bar{A}$, $\bar{B}$, $\bar{C}$. Therefore uncontrollable and/or unobservable means not minimal.

IV$\Longrightarrow$III: Non-minimal means uncontrollable or unobservable.

- Suppose $\{A, B, C, D\}$ is non-minimal.

$$\underbrace{C(sI - A)^{-1}B + D}_{n \text{ states}} = \underbrace{\bar{C}(sI - \bar{A})^{-1}\bar{B} + \bar{D}}_{r<n \text{ states}}$$

- $\dfrac{CB}{s} + \dfrac{CAB}{s^2} + \dfrac{CA^2B}{s^3} + \cdots = \dfrac{\bar{C}\bar{B}}{s} + \dfrac{\bar{C}\bar{A}\bar{B}}{s^2} + \dfrac{\bar{C}\bar{A}^2\bar{B}}{s^3} + \cdots$
- Consider

$$\mathcal{OC} = \begin{bmatrix} C \\ CA \\ CA^2 \\ \vdots \\ CA^{n-1} \end{bmatrix} \begin{bmatrix} B & AB & A^2B & \cdots & A^{n-1}B \end{bmatrix}$$

$$
\begin{aligned}
&= \begin{bmatrix} CB & \cdots & CA^{n-1}B \\ \vdots & & \vdots \\ CA^{n-1}B & \cdots & CA^{2n-2}B \end{bmatrix} = \begin{bmatrix} \bar{C}\bar{B} & \cdots & \bar{C}\bar{A}^{n-1}\bar{B} \\ \vdots & & \vdots \\ \bar{C}\bar{A}^{n-1}\bar{B} & \cdots & \bar{C}\bar{A}^{2n-2}\bar{B} \end{bmatrix} \\
&= n\left\updownarrow \begin{bmatrix} \begin{matrix} \bar{C} \\ \bar{C}\bar{A} \\ \vdots \\ \bar{C}\bar{A}^{n-2} \\ \bar{C}\bar{A}^{n-1} \end{matrix} & 0 \\ \underleftrightarrow{\quad r \quad} & \underleftrightarrow{\quad n-r \quad} \end{bmatrix}\right. \left. \begin{bmatrix} \bar{B} & \bar{A}\bar{B} & \cdots & \bar{A}^{n-1}\bar{B} \\ \hline & 0 & & \\ \multicolumn{4}{c}{\underleftrightarrow{\qquad n \qquad}} \end{bmatrix} \begin{matrix} \updownarrow r \\ \updownarrow n-r \end{matrix}\right.
\end{aligned}
$$

- Therefore $\det(\mathcal{O})\det(\mathcal{C}) = 0$, so the system is either unobservable, or uncontrollable, or both.

- The four equivalences have now been proven.

FACT: All minimal realization of $G(s)$ are related by a unique change of coordinates T. Can you prove this?

Where to from here?

- Have seen important topics of observability and controllability.
- Now understand how a system could be unobservable or uncontrollable.
  - If physically true, may need to add sensors or actuators.
- Important to work with minimal realizations, when possible.
- Now, time to consider: How do I build a controller?
  - Very powerful tools in next chapter.