

# **STABILITY**

## **4.1: Vector norms and quadratic forms**

- Loosely speaking, a system is said to be stable if small perturbations to its initial state or input result in small changes to how the system's state or output evolves over time.
  - Sometimes, a system will be designed intentionally to be open-loop unstable (why?), and then require a controller to stabilize it.
- Mathematically, we find that there are a number of ways to describe system stability properties, and we will discuss many of them here.
- First, though, we need to review matrix norms: We need to be able to quantify gains that lead to “small” or “large” changes in a system's response.

**NORM:** A measure of length or gain.

- Properties of a vector norm, where  $x, y$  are vectors and  $\alpha$  is scalar:

$$\|x\| \geq 0$$

$$\|x\| = 0 \quad \text{iff } x = 0$$

$$\|\alpha x\| = |\alpha| \|x\|$$

$$\|x + y\| \leq \|x\| + \|y\|.$$

## Vector $p$ -norms

- The  $p$ -norm of a vector is defined as

$$\|x\|_p = \lim_{\alpha \rightarrow p} \alpha \sqrt[\alpha]{\sum_{k=1}^{n_x} |x_k|^\alpha}.$$

- One common example is the (Euclidean) 2-norm

$$\|x\| = \|x\|_2 = \sqrt{x_1^2 + \cdots + x_n^2} = \sqrt{x^T x}.$$

- $\|x\|$  measures length of vector (from origin).
- Another common vector norm is the  $\infty$ -norm

$$\|x\|_\infty = \lim_{\alpha \rightarrow \infty} \alpha \sqrt[\alpha]{\sum_{k=1}^{n_x} |x_k|^\alpha} = \max_k |x_k|.$$

- A third common vector norm is the 1-norm

$$\|x\|_1 = \lim_{\alpha \rightarrow 1} \alpha \sqrt[\alpha]{\sum_{k=1}^{n_x} |x_k|^\alpha} = \sum_{k=1}^{n_x} |x_k|.$$

## Quadratic forms

- The machinery required to understand *matrix* norms is a little more complicated.
- We spend some time to studying some relevant linear algebra first.

### *Eigenvalues and eigenvectors of symmetric matrices (i.e., $A = A^T$ )*

**FACT:** The eigenvalues of symmetric matrix  $A \in \mathbb{R}^{n \times n}$  are real.

- To see this, suppose  $Av = \lambda v$ ,  $v \neq 0$ ,  $v \in \mathbb{C}^n$ .

- Then,  $(v^*)^T Av = (v^*)^T (Av) = \lambda (v^*)^T v = \lambda \sum_{i=1}^n |v_i|^2 = \lambda \|v\|_2^2$ .
- Also,  $(v^*)^T Av = ((Av)^*)^T v = ((\lambda v)^*)^T v = \lambda^* \sum_{i=1}^n |v_i|^2 = \lambda^* \|v\|_2^2$ .
- So,  $\lambda = \lambda^*$ , which means that  $\lambda \in \mathbb{R}$  (hence, we can assume  $v \in \mathbb{R}^n$ ).

**FACT:** There is a set of orthonormal eigenvectors of  $A$ . *i.e.*,  $q_1, \dots, q_n$ , such that  $Aq_i = \lambda_i q_i$  and  $q_i^T q_j = \delta_{ij}$ . We'll show this for  $\lambda_i$  distinct.

- Suppose  $v_1, \dots, v_n$  is a set of linearly independent eigenvectors of  $A$ .
- That is, suppose  $Av_i = \lambda_i v_i$ ,  $\|v_i\| = 1$ .
- Then, we have  $v_i^T (Av_j) = \lambda_j v_i^T v_j = (Av_i)^T v_j = \lambda_i v_i^T v_j$ .
- This gives  $(\lambda_i - \lambda_j) v_i^T v_j = 0$  for  $i \neq j$  and  $\lambda_i \neq \lambda_j$ . Hence,  $v_i^T v_j = 0$ .
  - In this case, we can say that the eigenvectors *are* orthogonal.
  - In the general case ( $\lambda_i$  not distinct) we must say that the eigenvectors *can be chosen* to be orthogonal.
- Grouping the eigenvectors into an (orthogonal) matrix  $Q$  allows us to write  $Q^{-1}AQ = Q^T AQ = \Lambda$ .

- Hence, we can express  $A$  as  $A = Q\Lambda Q^T = \sum_{i=1}^n \lambda_i q_i q_i^T$ .

### Quadratic forms

- A function  $f : \mathbb{R}^n \rightarrow \mathbb{R}$  of the form

$$f(x) = x^T Ax = \sum_{i,j} A_{ij} x_i x_j$$

is called a quadratic form.

- In a quadratic form, we may assume  $A$  is symmetric since  $y = y^T$ , so

$$y = x^T A x = x^T A^T x = x^T ((A + A^T)/2)x,$$

where  $(A + A^T)/2$  is called the symmetric part of  $A$ .

- Suppose that  $A = A^T$  and  $A = Q\Lambda Q^T$  with eigenvalues sorted so  $\lambda_1 \geq \dots \geq \lambda_n$ . Then,

$$\begin{aligned} x^T A x &= x^T Q\Lambda Q^T x \\ &= (Q^T x)^T \Lambda (Q^T x) \\ &= \sum_{i=1}^n \lambda_i (q_i^T x)^2 \\ &\leq \lambda_1 \sum_{i=1}^n (q_i^T x)^2 = \lambda_1 [(Q^T x)^T I (Q^T x)] = \lambda_1 x^T x \\ &= \lambda_1 \|x\|^2. \end{aligned}$$

- That is, we have  $x^T A x \leq \lambda_1 x^T x$ .
- By a similar argument, we can find that  $x^T A x \geq \lambda_n \|x\|^2$ , so we have

$$\lambda_n x^T x \leq x^T A x \leq \lambda_1 x^T x.$$

- Sometimes  $\lambda_1$  is called  $\lambda_{\max}$  and  $\lambda_n$  is called  $\lambda_{\min}$ .
- Note also that  $q_1^T A q_1 = \lambda_1 \|q_1\|^2$  and  $q_n^T A q_n = \lambda_n \|q_n\|^2$ , so the inequalities are tight.

## 4.2: Matrix gain

- Suppose now that  $A \in \mathbb{R}^{m \times n}$  (not necessarily square or symmetric).
- For  $x \in \mathbb{R}^n$ ,  $\|Ax\| / \|x\|$  gives the amplification factor or gain of  $A$  in the direction  $x$ .
- Since this gain varies with direction of the input  $x$ , we might be interested in knowing:
  - What is the maximum gain and corresponding gain direction of  $A$ ?
  - What is the minimum gain of  $A$  (and corresponding gain direction)?
  - How does gain of  $A$  vary with direction?
- The max. gain is called the matrix norm of  $A$ , and is denoted  $\|A\|$ .

- In particular, the matrix  $p$ -norm is defined as

$$\|A\|_p = \sup_{x \neq 0} \frac{\|Ax\|_p}{\|x\|_p}.$$

- For  $p = 2$ , we have already developed the machinery to evaluate the norm. Consider first the norm squared:

$$\sup_{x \neq 0} \frac{\|Ax\|_2^2}{\|x\|_2^2} = \sup_{x \neq 0} \frac{x^T A^T A x}{\|x\|_2^2} = \lambda_{\max}(A^T A),$$

so we have  $\|A\|_2 = \sqrt{\lambda_{\max}(A^T A)}$ .

- Similarly, the min. gain is given by  $\inf_{x \neq 0} \|Ax\|_2 / \|x\|_2 = \sqrt{\lambda_{\min}(A^T A)}$ .
- Note that  $A^T A \in \mathbb{R}^{n \times n}$  is symmetric and  $A^T A \geq 0$  so  $\lambda_{\min}$  and  $\lambda_{\max} \geq 0$ .
  - The “max gain” input direction is  $x = q_1$ , the eigenvector of  $A^T A$  associated with  $\lambda_{\max}$ .
  - The “min gain” input direction is  $x = q_n$ , the eigenvector of  $A^T A$  associated with  $\lambda_{\min}$ .

- For  $p = 1$ , it can be shown that

$$\|A\|_1 = \max_{1 \leq j \leq n} \sum_{i=1}^m |a_{ij}|,$$

which is simply the maximum absolute column sum of the matrix.

- For  $p = \infty$ , it can be shown that

$$\|A\|_\infty = \max_{1 \leq i \leq m} \sum_{j=1}^n |a_{ij}|,$$

which is the maximum absolute row sum of the matrix.

## Matrix 2-norms via SVD

- Finding matrix 1- and  $\infty$ -norms is quite simple. Finding the 2-norm is more difficult because of the need to find eigenvalues, eigenvectors.
- We can replace this step with one that is computationally simpler by using the singular value decomposition (SVD) of  $A \in \mathbb{R}^{m \times n}$ , where  $\text{rank}(A) = r$ :

$$A = U \Sigma V^T.$$

- $U = [u_1, \dots, u_r] \in \mathbb{R}^{m \times r}$ , and  $U^T U = I$ , and  $u_i$  are the left or output singular vectors of  $A$ .
  - $V = [v_1, \dots, v_r] \in \mathbb{R}^{n \times r}$ , and  $V^T V = I$ , and  $v_i$  are the right or input singular vectors of  $A$ .
  - $\Sigma = \text{diag}(\sigma_1, \dots, \sigma_r)$  where  $\sigma_1 \geq \dots \geq \sigma_r > 0$ , and  $\sigma_i$  are the (nonzero) singular values of  $A$ .
- The above is called a compact SVD. Most often, we compute a full SVD, where

- $U = [u_1, \dots, u_m] \in \mathbb{R}^{m \times m}$ , and  $U^T U = I$ ,
- $V = [v_1, \dots, v_n] \in \mathbb{R}^{n \times n}$ , and  $V^T V = I$ ,
- The matrix  $\Sigma \in \mathbb{R}^{m \times n}$  is “diagonal”

$$\Sigma = \begin{bmatrix} \sigma_1 & & 0 & 0 \\ & \ddots & & \\ 0 & & \sigma_m & 0 \end{bmatrix} \text{ or } \Sigma = \begin{bmatrix} \sigma_1 & & 0 \\ & \ddots & \\ 0 & & \sigma_n \end{bmatrix} \text{ or } \Sigma = \begin{bmatrix} \sigma_1 & & 0 \\ & \ddots & \\ 0 & & \sigma_n \\ 0 & 0 & 0 \end{bmatrix}$$

when  $m < n$ ,  $m = n$  and  $m > n$ , respectively.

- In this case,  $\sigma_1 \geq \dots \geq \sigma_r > 0$ , and  $\sigma_i = 0$  for  $i > r$ .
  - In MATLAB, `svd.m` and `svds.m`
- We often write the full SVD as partitioned:

$$A = \begin{bmatrix} U_1 & \vdots & U_2 \end{bmatrix} \begin{bmatrix} \Sigma_1 & \vdots & 0_{r \times (n-r)} \\ \hline 0_{(m-r) \times r} & \vdots & 0_{(m-r) \times (n-r)} \end{bmatrix} \begin{bmatrix} V_1^T \\ \hline V_2^T \end{bmatrix},$$

where  $A = U_1 \Sigma_1 V_1^T$  is the compact SVD.

- The matrices  $U$  and  $V$  form orthonormal bases for the four fundamental subspaces
- The first  $r$  columns of  $V$  form a basis for the row space of  $A$ ;
    - ◆ The row space of a matrix is the set of all possible linear combinations of its row vectors. It's the span of the input vectors  $x$  to  $y = Ax$  that result in nonzero  $y$ .
  - The last  $n - r$  columns of  $V$  form a basis for the nullspace of  $A$ ;
    - ◆ The null space of a matrix is the set of all possible vectors  $x$  such that  $y = Ax = 0$ .
  - The first  $r$  columns of  $U$  form a basis for the column space of  $A$ ;

- ◆ The column space is also the range of the matrix. It's the set of all possible outputs  $y$  generated as  $y = Ax$ .
- The last  $m - r$  columns of  $U$  form a basis for the nullspace of  $A^T$ .
  - ◆ The nullspace of  $A^T$  is the set of all outputs  $y$  that *cannot* be generated as  $y = Ax$ .
- So, immediately we see some value to the SVD: It gives us the rank of the matrix, and these four bases, which can give us range space, nullspace, etc.
- Also, the SVD is computationally much simpler than an eigensystem calculation, which makes finding matrix 2-norms far simpler:

$$A^T A = (U \Sigma V^T)^T (U \Sigma V^T) = V \Sigma^2 V^T$$

so  $\sigma_i = \sqrt{\lambda_i(A^T A)}$  and hence  $\|A\|_2 = \sigma_1$ .

### Positive (semi) definite; negative (semi) definite

- Finally, before continuing to study system stability, we consider some properties of the gain matrix in a quadratic form.
- We say that symmetric matrix  $Q$  is:
  - Positive definite if  $x^T Q x > 0$ ,  $\forall x \neq 0$ .
  - Positive semidefinite if  $x^T Q x \geq 0$ ,  $\forall x \neq 0$ .
  - Negative definite if  $x^T Q x < 0$ ,  $\forall x \neq 0$ .
  - Negative semidefinite if  $x^T Q x \leq 0$ ,  $\forall x \neq 0$ .

For positive-definite matrix  $Q$ ,

$$0 < \lambda_{\min}[Q] \|x\|^2 \leq x^T Q x \leq \lambda_{\max}[Q] \|x\|^2, \quad \forall x \neq 0.$$



### 4.3: Lyapunov stability

- Consider a continuous-time LTV system

$$\dot{x}(t) = A(t)x(t) + B(t)u(t)$$

$$y(t) = C(t)x(t) + D(t)u(t).$$

- The system is (marginally) stable in the sense of Lyapunov or internally stable if, for every  $x(t_0) = x_0$ , the homogeneous response

$$x(t) = \Phi(t, t_0)x_0, \quad t \geq 0$$

is uniformly bounded (*i.e.*, all trajectories can be bounded by a single constant vector).

- The effect of the initial condition can not grow unbounded over time, but can grow temporarily during a transient phase.
- It is additionally asymptotically stable if, for every  $x(t_0) = x_0$ ,

$$x(t) \rightarrow 0 \quad \text{as} \quad t \rightarrow \infty.$$

- The effect of initial conditions eventually disappear completely.
- It is additionally exponentially stable if there exist constants  $c, \lambda$  such that for every initial condition  $x(t_0) = x_0$

$$\|x(t)\| \leq c e^{\lambda(t-t_0)} \|x_0\|, \quad t \geq 0.$$

- Finally, it is unstable if it is not marginally stable.

- The state may diverge over time, depending on specific  $x_0$ .
- Notice that  $B(t)$ ,  $C(t)$ , and  $D(t)$  have no influence over Lyapunov stability. All that matters is  $A(t)$ .
  - So, we often simply talk about the Lyapunov stability of the homogeneous system  $\dot{x}(t) = A(t)x(t)$ ,  $t \geq 0$ .

## Eigenvalue conditions for LTI systems

- For LTI systems, it is relatively simple to transform these definitions of stability into tests for stability.
- We know matrix-exponential properties and that  $\Phi(t, t_0) = e^{A(t-t_0)}$ .
- Therefore, a continuous-time LTI system is:
  - (Marginally) stable iff all eigenvalues of  $A$  have negative or zero real parts and all Jordan blocks for eigenvalues having zero real parts are  $1 \times 1$  in dimension.
  - Asymptotically stable iff all eigenvalues have negative real parts.
  - Exponentially stable iff all eigenvalues have negative real parts.
  - Unstable iff at least one eigenvalue has pos. real part, or a Jordan block for an eigenvalue having zero real part is bigger than  $1 \times 1$ .
- When all eigenvalues have negative real part, we can find  $c, \lambda$  so that

$$\|e^{At}\| \leq ce^{-\lambda t}.$$

- Therefore,  $\|x(t)\| = \|e^{A(t-t_0)}x_0\| \leq \|e^{A(t-t_0)}\| \|x_0\| \leq ce^{-\lambda(t-t_0)} \|x_0\|$ .
  - Asymptotic and exponential stability are equivalent for LTI systems.
- Note: This test does *not* work for LTV systems in general. It's possible for  $\mathbb{R}(\lambda[A(t)]) < 0$  for all time and still have an unstable system.

## Lyapunov stability theorem

- The eigenvalue test will often be sufficient for evaluating Lyapunov stability, but some extra properties can be helpful, especially when extending to understanding stability of nonlinear systems.

- The Lyapunov stability theorem provides an alternative condition to check whether a continuous-time homogeneous LTI system is asymptotically stable.
- The theorem states that the following five conditions are equivalent:
  1. The system is asymptotically stable.
  2. The system is exponentially stable.
  3. All the eigenvalues of  $A$  have strictly negative real parts.
  4. For every symmetric positive-definite matrix  $Q$ , there exists a unique solution  $P$  to the following Lyapunov equation

$$A^T P + PA = -Q.$$

Further,  $P$  is symmetric and positive-definite.

5. There exists a symmetric positive-definite matrix  $P$  for which the following Lyapunov matrix inequality holds

$$A^T P + PA < 0.$$

- We have already proven the equivalence between 1, 2, and 3.
- To prove the rest, we show that 2 implies 4, that 4 implies 5, and that 5 implies 2 (thus forming a complete circular set of implications).

## 4.4: Proof of the Lyapunov stability theorem

- We start by showing that the unique solution to the Lyapunov equation  $A^T P + PA = -Q$  is

$$P = \int_0^{\infty} e^{A^T t} Q e^{At} dt.$$

- We verify this in four steps. The first step is to show that the integral is well defined (finite).
  - This is true since the system is exponentially stable, so  $e^{At}$  and hence  $\|e^{A^T t} Q e^{At}\|$  converges to zero exponentially quickly as  $t \rightarrow \infty$ . Therefore, the integral is absolutely convergent.
- Second, we show that this integral solves the Lyapunov equation via direct substitution:

$$A^T P + PA = \int_0^{\infty} A^T e^{A^T t} Q e^{At} + e^{A^T t} Q e^{At} A dt.$$

- But (noting that  $e^{At} A = A e^{At}$  via infinite-series expansion of  $e^{At}$ ),

$$\frac{d}{dt}(e^{A^T t} Q e^{At}) = A^T e^{A^T t} Q e^{At} + e^{A^T t} Q e^{At} A.$$

- So,

$$\begin{aligned} A^T P + PA &= \int_0^{\infty} \frac{d}{dt}(e^{A^T t} Q e^{At}) dt = \left[ e^{A^T t} Q e^{At} \right]_0^{\infty} \\ &= \left( \lim_{t \rightarrow \infty} e^{A^T t} Q e^{At} \right) - e^{A^T 0} Q e^{A0}. \end{aligned}$$

- Because of asymptotic stability, the first term vanishes.
- Because  $e^{A^T 0} = I$ , the second term becomes  $-Q$ .
- So, the proposed solution solves the equation, but we don't yet know whether it is symmetric, positive-definite, and/or unique.

- Third, to show that it is symmetric we compute

$$\begin{aligned} P^T &= \int_0^\infty \left( e^{A^T t} Q e^{At} \right)^T dt \\ &= \int_0^\infty (e^{At})^T Q^T (e^{A^T t})^T dt \\ &= \int_0^\infty e^{A^T t} Q e^{At} dt = P. \end{aligned}$$

- To show that it is positive-definite, we pick an arbitrary constant vector  $z \neq 0$  and compute

$$\begin{aligned} z^T P z &= \int_0^\infty z^T \left( e^{A^T t} Q e^{At} \right) z dt \\ &= \int_0^\infty w(t)^T Q w(t) dt, \end{aligned}$$

which has positive-definite form since  $Q$  is positive definite.

- Further, we see that  $z^T P z = 0$  only when  $w(t) = e^{At} z = 0$ , which can only happen when  $z = 0$  for finite time.
- Therefore  $P$  is positive-definite.
- Fourth, to see that no other matrix solves this equation, we proceed using a proof by contradiction.

- That is, assume that there exists another solution  $\bar{P}$  to the Lyapunov equation. If this were so, we would have both

$$\begin{aligned} A^T P + P A &= -Q \\ A^T \bar{P} + \bar{P} A &= -Q. \end{aligned}$$

- Then,

$$A^T (P - \bar{P}) + (P - \bar{P}) A = 0.$$

- Multiplying on the left by  $e^{A^T t}$  and on the right by  $e^{At}$

$$e^{A^T t} A^T (P - \bar{P}) e^{At} + e^{A^T t} (P - \bar{P}) A e^{At} = 0.$$

- On the other hand,

$$\frac{d}{dt} \left( e^{A^T t} (P - \bar{P}) e^{At} \right) = e^{A^T t} A^T (P - \bar{P}) e^{At} + e^{A^T t} (P - \bar{P}) e^{At} A = 0.$$

- Therefore,  $e^{A^T t} (P - \bar{P}) e^{At}$  must be constant for all time, which requires that  $P = \bar{P}$  since  $e^{At}$  is nonsingular.

- We have now shown that condition 2 implies 4.
- It immediately follows that 4 implies 5 if we select  $Q = I$ .
- To show that condition 5 implies condition 2, let  $P$  be a symmetric positive-definite matrix that satisfies

$$A^T P + P A < 0$$

and let  $Q = -(A^T P + P A) > 0$ .

- Consider an arbitrary solution  $x(t)$  to the homogeneous system dynamics and define the scalar signal

$$v(t) = x^T(t) P x(t) \geq 0.$$

- This signal is a kind of weighted “energy of the state” for the homogeneous system.
- For a stable homogeneous system, we would expect this energy to dissipate over time, which is exactly what we will show.
  - ◆ This idea returns when talking about nonlinear stability, later.
- Taking derivatives of  $v(t)$ ,

$$\dot{v}(t) = \dot{x}^T(t) P x(t) + x^T(t) P \dot{x}(t).$$

- But, since  $\dot{x}(t) = Ax(t)$ ,

$$\dot{v}(t) = x^T(t) (A^T P + PA) x(t) = -x^T(t) Q x(t) \leq 0.$$

- Therefore  $v(t)$  is a nonincreasing signal and we conclude that

$$v(t) = x^T(t) P x(t) \leq v(0) = x^T(0) P x(0), \quad t \geq 0.$$

- But since  $v(t) = x^T(t) P x(t) \geq \lambda_{\min}[P] \|x(t)\|^2$ , we conclude that

$$\|x(t)\|_2^2 \leq \frac{x^T(t) P x(t)}{\lambda_{\min}[P]} = \frac{v(t)}{\lambda_{\min}[P]} \leq \frac{v(0)}{\lambda_{\min}[P]}, \quad t \geq 0.$$

- This means that the system is stable. To show that it is exponentially stable, we combine our known results that

$$x^T(t) Q x(t) \geq \lambda_{\min}[Q] \|x(t)\|_2^2$$

and that  $v(t) = x^T(t) P x(t) \leq \lambda_{\max}[P] \|x(t)\|_2^2$  and conclude

$$\dot{v}(t) = -x^T(t) Q x(t) \leq -\lambda_{\min}[Q] \|x(t)\|_2^2 \leq -\frac{\lambda_{\min}[Q]}{\lambda_{\max}[P]} v(t),$$

or that

$$\dot{v}(t) \leq \mu v(t).$$

- It can be shown (see text) that this converges more quickly than the exponential

$$v(t) \leq e^{\mu(t-t_0)} v(t_0).$$

- Since  $v(t)$  converges exponentially quickly, so too must  $\|x(t)\|_2$  since we have already shown that

$$\|x(t)\|_2^2 \leq \frac{v(t)}{\lambda_{\min}[P]}.$$

- This finishes our proof of the Lyapunov stability theorem for LTI continuous-time systems.

## Solving Lyapunov equations

- Lyapunov equations of the form

$$XA + BX = C,$$

crop up numerous times in control systems, for different reasons.

- Sometimes we need to solve for  $X$ .
- The equations can be solved by writing them out in terms of  $X_{ij}$ .
- Another way to do it uses the Kronecker product and vectorized matrices.

### KRONECKER PRODUCT:

$$A \otimes B = \begin{bmatrix} a_{11}B & a_{12}B & \cdots & a_{1n}B \\ \vdots & & & \vdots \\ a_{m1}B & a_{m2}B & \cdots & a_{mn}B \end{bmatrix}.$$

That is, the Kronecker product is a large matrix containing all possible permutations of  $A$  and  $B$ .

**VECTORIZED MATRICES:** We can convert a matrix into a column vector which stacks up each column of the matrix. That is, if

$$A = [a_1, a_2, \dots, a_n]$$

$$(A) = \begin{bmatrix} a_1 \\ \cdots \\ a_2 \\ \cdots \\ \vdots \\ \cdots \\ a_n \end{bmatrix}.$$

- A general Kronecker-product rule is:

$$(AXB) = [B^T \otimes A](X).$$



- Three specific special-case rules result:

$$(PMP^T) = [P \otimes P](M) \quad (\text{to be used later in course})$$

$$(XA) = [A^T \otimes I](X)$$

$$(BX) = [I \otimes B](X)$$

so, to solve the Lyapunov equation,

$$XA + BX = C$$

$$IXA + BXI = C$$

$$(IXA) + (BXI) = (C)$$

$$[A^T \otimes I](X) + [I \otimes B](X) = (C)$$

$$\{[A^T \otimes I] + [I \otimes B]\}(X) = (C)$$

$$(X) = \{[A^T \otimes I] + [I \otimes B]\}^{-1}(C).$$

$X$  is then found by un-vectorizing  $(X)$ .

- In MATLAB: `lyap( )`

## 4.5: Discrete-time Lyapunov stability

- Consider now the discrete-time LTV system

$$x[k + 1] = A[k]x[k] + B[k]u[k]$$

$$y[k] = C[k]x[k] + D[k]u[k].$$

- The system is (marginally) stable in the Lyapunov sense or internally stable if for every  $x[k_0] = x_0$  the homogeneous state response

$$x[k] = \Phi[k, k_0]x_0, \quad k_0 \geq 0$$

is uniformly bounded.

- It is asymptotically stable if, additionally, for every initial condition we have  $x[k] \rightarrow 0$  as  $k \rightarrow \infty$ .
- It is exponentially stable if, additionally, there exist constants  $c, \lambda$  such that for every  $x[k_0] = x_0$  we have

$$\|x[k]\| \leq c\lambda^{k-k_0}\|x_0\|, \quad k \geq k_0.$$

- It is unstable if it is not marginally stable in the Lyapunov sense.
- Note again that the matrices  $B[k]$ ,  $C[k]$ , and  $D[k]$  play no role in this definition, so we need consider only the homogeneous response when considering Lyapunov stability.

### Eigenvalue conditions

- A discrete-time LTI system is
  - Marginally stable iff all the eigenvalues of  $A$  have magnitude smaller than or equal to 1, and all the Jordan blocks corresponding to eigenvalues with magnitude equal to 1 are  $1 \times 1$ .

- Asymptotically *and* exponentially stable iff all the eigenvalues of  $A$  have magnitude strictly less than 1, or
- Unstable iff at least one eigenvalue of  $A$  has magnitude larger than 1 or magnitude equal to 1, but the corresponding Jordan block is larger than  $1 \times 1$ .

## Lyapunov stability in discrete time

- The following five conditions are equivalent for LTI systems
  1. The system is asymptotically stable.
  2. The system is exponentially stable.
  3. All the eigenvalues of  $A$  have magnitude strictly smaller than 1.
  4. For every symmetric positive-definite matrix  $Q$ , there exists a unique solution  $P$  to the following discrete-time Lyapunov equation (aka Stein equation)

$$A^T P A - P = -Q.$$

Furthermore,  $P$  is positive-definite.

5. There exists a symmetric positive-definite matrix  $P$  for which the following Lyapunov matrix inequality holds

$$A^T P A - P < 0.$$

- These conditions can be proven in a similar way to their continuous-time counterparts.
- Now, however, we use the energy function

$$v[k] = x^T[k] P x[k], \quad k > k_0$$

which evolves according to

$$v[k + 1] = x^T[k + 1] P x[k + 1] = x^T[k] A^T P A x[k].$$

- The discrete-time Lyapunov equation guarantees that

$$v[k + 1] = x^T[k](P - Q)x[k] = v[k] - x^T[k]Qx[k], \quad k \geq k_0.$$

- From this, we conclude that  $v[k]$  is nonincreasing, and with some work that it decreases to zero exponentially quickly.

### Solving discrete-time Lyapunov equations

- In discrete time, a Lyapunov equation takes on the form

$$W - AWA^T = BB^T.$$

- This may be converted to a standard Lyapunov equation. Let

$$A_c = (A + I)^{-1}(A - I)$$

and

$$C_c = -2(A + I)^{-1}BB^T(A^T + I)^{-1}.$$

- Expand the continuous-time Lyapunov equation  $A_c W + WA_c^T = C_c$  using these definitions:

$$\begin{aligned} (A + I)^{-1}(A - I)W + W(A^T - I)(A^T + I)^{-1} = \\ -2(A + I)^{-1}BB^T(A^T + I)^{-1}. \end{aligned}$$

- Replace  $(A - I)$  with  $(A + I - 2I)$  and  $(A^T - I)$  with  $(A^T + I - 2I)$ .
- Multiply through and simplify to get

$$W - (A + I)^{-1}W - W(A^T + I)^{-1} = -(A + I)^{-1}BB^T(A^T + I)^{-1}.$$

- Left multiply by  $-(A + I)$  and right multiply by  $(A^T + I)$  and simplify

$$W - AWA^T = BB^T.$$

- So, with the above definitions of  $A_c$  and  $C_c$  based on  $A$  and  $B$ , we can create a continuous-time Lyapunov equation and solve it for  $W$ .

## 4.6: Stability of locally linearized systems

- Now, consider a continuous-time homogeneous nonlinear system

$$\dot{x}(t) = f(x(t))$$

with an equilibrium point at  $x^{\text{eq}}$ . That is,  $f(x^{\text{eq}}) = 0$ .

- This gives rise to the locally linearized system

$$\delta\dot{x}(t) = A\delta x(t)$$

with  $\delta x(t) = x(t) - x^{\text{eq}}$  and  $A = \partial f(x^{\text{eq}})/\partial x$ .

- If the linearized system passes the Lyapunov stability tests, can we say anything about the stability of the original nonlinear system? Yes!
- If  $f(x(t))$  is twice differentiable and the linearized system is exponentially stable, then there exists a ball  $B \subset \mathbb{R}^n$  around  $x^{\text{eq}}$  and constants  $c, \lambda > 0$  such that for every solution  $x(t)$  to the nonlinear system that starts at  $x_0 \in B$ , we have

$$\|x(t) - x^{\text{eq}}\| \leq c e^{\lambda(t-t_0)} \|x(t_0) - x^{\text{eq}}\|, \quad t \geq t_0.$$

- We call this nonlinear system locally exponentially stable.
- To show this, because  $f(\cdot)$  is twice differentiable, we know from Taylor's theorem that

$$\begin{aligned} r(x(t)) &= f(x(t)) - (f(x^{\text{eq}}) + A\delta x(t)) \\ &= f(x(t)) - A\delta x(t) = O(\|\delta x(t)\|^2) \end{aligned}$$

which means that there exists a constant  $c$  and a ball  $\bar{B}$  around  $x^{\text{eq}}$  for which

$$\|r(x(t))\| \leq c \|\delta x(t)\|^2, \quad \forall x(t) \in \bar{B}.$$

- Since the linearized system is exponentially stable, there exists a positive-definite matrix  $P$  for which

$$A^T P + PA = -I.$$

- Inspired by our proof of the Lyapunov stability theorem, we define the scalar “energy” signal

$$v(t) = \delta x^T(t) P \delta x(t), \quad t \geq 0.$$

- We then compute its derivative along trajectories

$$\begin{aligned} \dot{v}(t) &= f^T(x(t)) P \delta x(t) + \delta x^T(t) P f(x(t)) \\ &= (A \delta x(t) + r(x(t)))^T P \delta x + \delta x^T(t) P (A \delta x(t) + r(x(t))) \\ &= \delta x^T(t) (A^T P + PA) \delta x(t) + 2 \delta x^T P r(x(t)) \\ &= -\|\delta x\|_2^2 + 2 \delta x^T(t) P r(x(t)) \\ &\leq -\|\delta x(t)\|_2^2 + 2 \|P\| \|\delta x(t)\| \|r(x)\|. \end{aligned}$$

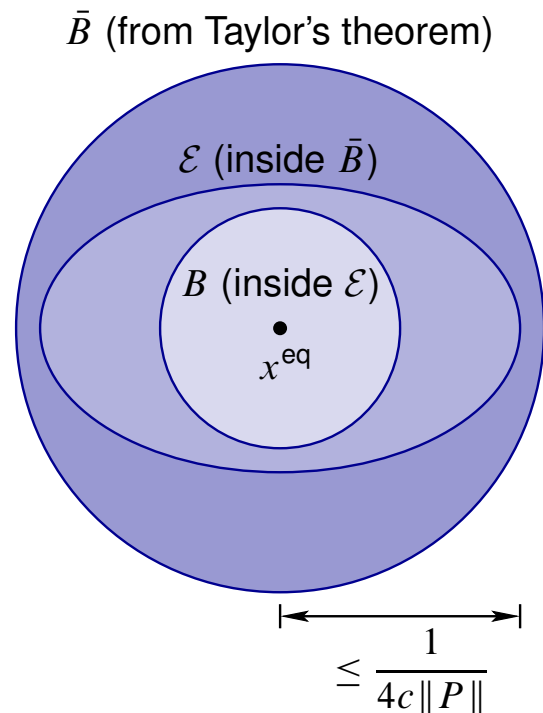
- To make the proof work, we would like to make sure that the RHS is negative. For example, it would work if we could show

$$-\|\delta x(t)\|_2^2 + 2 \|P\| \|\delta x(t)\| \|r(x)\| \leq -\frac{1}{2} \|\delta x(t)\|^2.$$

- To do so, let  $\epsilon > 0$  be sufficiently small so that the ellipsoid

$$\mathcal{E} = \{x(t) \in \mathbb{R}^n : (x(t) - x^{\text{eq}})^T P (x(t) - x^{\text{eq}}) \leq \epsilon\}$$

centered at  $x^{\text{eq}}$  satisfies the following two properties:



1.  $\mathcal{E}$  is fully contained inside the ball  $\bar{B}$ . When  $x(t)$  is inside this ellipsoid, we know  $\|r(x(t))\| \leq c\|\delta x(t)\|^2$  and so  $x(t) \in \mathcal{E}$  implies  $\dot{v}(t) \leq -\|\delta x(t)\|^2 + 2c\|P\|\|\delta x(t)\|^3 = -(1 - 2c\|P\|\|\delta x(t)\|)\|\delta x(t)\|^2$ .
2. We further shrink  $\epsilon$  so that inside the ellipsoid  $\mathcal{E}$  we have

$$1 - 2c\|P\|\|\delta x(t)\| \geq \frac{1}{2}$$

which implies that

$$\|\delta x(t)\| \leq \frac{1}{4c\|P\|}.$$

- For this choice of  $\epsilon$  we have that  $x(t) \in \mathcal{E}$  implies  $\dot{v}(t) \leq -\frac{1}{2}\|\delta x(t)\|^2$ .
- We therefore conclude that  $x(t) \in \mathcal{E}$  implies that both

$$v(t) \leq \epsilon \quad \text{and} \quad \dot{v}(t) \leq 0.$$

So,  $v(t)$  cannot increase above  $\epsilon$  and  $x(t)$  cannot exit  $\mathcal{E}$ .

- Therefore, if  $x(0)$  starts inside  $\mathcal{E}$ , it cannot exit this set.
- Moreover, since  $\dot{v}(t) \leq -\|\delta x(t)\|^2/2$  and  $\delta x^T(t)P\delta x(t) \leq \|P\|\|\delta x\|^2$ , we can further conclude that if  $x(0)$  starts within  $\mathcal{E}$ ,

$$\dot{v}(t) \leq -\frac{v(t)}{2\|P\|}$$

and therefore we can show that  $\delta x(t) = x(t) - x^{\text{eq}}$  decreases to zero exponentially quickly.

- In conclusion, if the linearized system is stable, then there exists some operating regime around  $x^{\text{eq}}$  for which the nonlinear system is also stable: any ball  $B$  inside  $\mathcal{E}$ .
- It can also be shown that if the linearized system is unstable, there exist initial conditions arbitrarily close to  $x^{\text{eq}}$  for which the trajectories do not converge as  $t \rightarrow \infty$ .

## 4.7: Input–output stability: LTV case

- We now consider a different measure of stability that expresses how the magnitude of the input affects the magnitude of the output in the absence of initial conditions.
- We consider the continuous-time LTV system

$$\dot{x}(t) = A(t)x(t) + B(t)u(t)$$

$$y(t) = C(t)x(t) + D(t)u(t).$$

- We have seen that the forced response of this system in the absence of initial conditions is

$$y(t) = \int_0^t C(t)\Phi(t, \tau)B(\tau)u(\tau) d\tau + D(t)u(t).$$

- This system is said to be (uniformly) bounded-input bounded-output (BIBO) stable if there exists a finite constant  $\gamma$  such that, for every input  $u(t)$ , its forced response satisfies

$$\sup_{t \in [0, \infty)} \|y(t)\| \leq \gamma \sup_{t \in [0, \infty)} \|u(t)\|.$$

### Time-domain conditions for BIBO stability

- The following two statements are equivalent:
  1. A LTV system is uniformly BIBO stable.
  2. Every entry of the system's  $D(t)$  is uniformly bounded and

$$\sup_{t \geq 0} \int_0^t |g_{ij}(t, \tau)| d\tau < \infty$$

for every entry  $g_{ij}(t, \tau)$  of  $C(t)\Phi(t, \tau)B(\tau)$ .



- To prove this statement, we start with

$$\|y(t)\| \leq \int_0^t \|C(t)\Phi(t, \tau)B(\tau)\| \|u(\tau)\| d\tau + \|D(t)\| \|u(t)\|.$$

- We then define

$$\mu = \sup_{t \in [0, \infty)} \|u(t)\|, \quad \delta = \sup_{t \in [0, \infty)} \|D(t)\|.$$

- Substituting, we can write

$$\|y(t)\| \leq \left( \int_0^\infty \|C(t)\Phi(t, \tau)B(\tau)\| d\tau + \delta \right) \mu.$$

- Therefore, if

$$\gamma = \left( \int_0^\infty \|C(t)\Phi(t, \tau)B(\tau)\| d\tau + \delta \right)$$

is finite, we have proven our desired result.

- To show this, we note that

$$\|C(t)\Phi(t, \tau)B(\tau)\| \leq \sum_{i,j} |g_{ij}(t, \tau)|$$

as a consequence of the triangle inequality and therefore

$$\int_0^\infty \|C(t)\Phi(t, \tau)B(\tau)\| d\tau \leq \sum_{i,j} \int_0^\infty |g_{ij}(t, \tau)| d\tau.$$

- By our assumption that

$$\sup_{t \geq 0} \int_0^t |g_{ij}(t, \tau)| d\tau < \infty,$$

we conclude that

$$\begin{aligned} \gamma &= \sup_{t \geq 0} \int_0^t \|C(t)\Phi(t, \tau)B(\tau)\| d\tau + \delta \\ &\leq \sup_{t \geq 0} \sum_{i,j} \int_0^t |g_{ij}(t, \tau)| d\tau + \delta < \infty. \end{aligned}$$

- This shows that statement 2 implies statement 1.
- To show the other direction, we show that if 2 is false then 1 must also be false. This requires two steps.
- In the first step, we consider

$$\sup_{t \geq 0} \int_0^t |g_{ij}(t, \tau)| d\tau < \infty$$

but that some entry in  $D(t)$  is unbounded.

- Choose arbitrary time  $t = T$  for  $D(t)$  to be unbounded, and choose input

$$u_T(\tau) = \begin{cases} 0, & 0 \leq \tau < T \\ e_j, & \tau \geq T \end{cases}$$

where  $e_j \in \mathbb{R}^k$  is the  $j$ th unit vector in the basis of  $\mathbb{R}^k$ .

- Then, the forced response at time  $T$  is exactly

$$y(T) = D(T)e_j.$$

- Thus, we have found an input for which

$$\sup_{t \in [0, \infty)} \|u_T(t)\| = 1$$

and

$$\sup_{t \in [0, \infty)} \|y(t)\| \geq \|y(T)\| = \|D(T)e_j\| \geq |d_{ij}(T)|,$$

where the last inequality results from the fact that the norm of the vector  $D(T)e_j$  must be larger than the absolute value of its  $i$ th entry, which is precisely  $d_{ij}(T)$ .

- As  $d_{ij}(T)$  is unbounded, we conclude that we can make  $\sup_{t \in [0, \infty)} \|y(t)\|$  arbitrarily large using inputs  $u_T(t)$  for which  $\sup_{t \in [0, \infty)} \|u_T(t)\| = 1$ .

- This is not compatible with the existence of a finite gain  $g$  that satisfies the stability criterion. Therefore,  $D(\cdot)$  must be bounded for BIBO stability.

- Now suppose that 2 is false because  $\int_0^t |g_{ij}(t, \tau)| d\tau$  is unbounded for some  $i$  and  $j$ . This will also violate BIBO stability.

- To see this, pick an arbitrary time  $T$  and a “switching” input

$$u_T(t) = \begin{cases} +e_j, & g_{ij}(t, \tau) \geq 0 \\ -e_j, & g_{ij}(t, \tau) < 0. \end{cases}$$

- For this input, the forced response at time  $T$  is given by

$$y(t) = \int_0^t C(t)\Phi(t, \tau)B(\tau)u(\tau) d\tau + D(t)u(t),$$

and its  $i$ th entry is equal to

$$\int_0^t |g_{ij}(t, \tau)| d\tau \pm d_{ij}(t).$$

- We thus have found an input for which  $\sup_{t \in [0, \infty)} \|u_T(t)\| = 1$  and

$$\sup_{t \in [0, \infty)} \|y(t)\| \geq \|y(T)\| \geq \left| \int_0^t |g_{ij}(t, \tau)| d\tau \pm d_{ij}(t) \right|.$$

- Since we have assumed that the integral of  $|g_{ij}(t, \tau)|$  is unbounded, we also now conclude that we can make  $\sup_{t \in [0, \infty)} \|y(t)\|$  arbitrarily large using inputs  $u_T(t)$  for which  $\sup_{t \in [0, \infty)} \|u_T(t)\| = 1$ .

- This is not compatible with the existence of a finite gain  $g$  that satisfies the stability criteria, which means that condition 2 must hold for the system to be BIBO stable.

## 4.8: Input–output stability: LTI case

- For a time-invariant system

$$\dot{x}(t) = Ax(t) + Bu(t)$$

$$y(t) = Cx(t) + Du(t),$$

we have that

$$C\Phi(t, \tau)B = Ce^{A(t-\tau)}B.$$

- We can therefore rewrite our prior stability criterion as finite  $D$  plus

$$\sup_{t \geq 0} \int_0^t |\bar{g}_{ij}(t - \tau)| d\tau < \infty$$

understanding that  $\bar{g}_{ij}(t - \tau)$  denotes the  $ij$ th entry of  $Ce^{A(t-\tau)}B$ .

- Making a change of variable  $\rho = t - \tau$ , we can restate

$$\sup_{t \geq 0} \int_0^t |\bar{g}_{ij}(t - \tau)| d\tau = \sup_{t \geq 0} \int_0^t |\bar{g}_{ij}(\rho)| d\rho = \int_0^\infty |\bar{g}_{ij}(\rho)| d\rho.$$

- Therefore, we can restate our prior stability observations as the equivalence between the following two statements:

1. A LTI system is uniformly BIBO stable.
2.  $D$  is finite and for every entry  $\bar{g}_{ij}(\rho)$  of  $Ce^{A\rho}B$ , we have

$$\int_0^\infty |\bar{g}_{ij}(\rho)| d\rho < \infty.$$

### Frequency-domain conditions for BIBO stability

- For LTI systems, a stability test is often easier to do in the Laplace domain.
- Taking Laplace transforms with respect to temporal variable  $\rho$ ,

$$\mathcal{L}[\bar{g}_{ij}(\rho)] = \mathcal{L}[Ce^{A\rho}B] = C(sI - A)^{-1}B.$$

- The  $ij$ th entry of this result will be a strictly proper rational polynomial

$$G_{ij}(s) = \frac{\alpha_0 s^q + \alpha_1 s^{q-1} + \cdots + \alpha_{q-1} s + \alpha_q}{(s - \lambda_1)^{m_1} (s - \lambda_2)^{m_2} \cdots (s - \lambda_k)^{m_k}}$$

where the  $\lambda$  are the unique poles and the  $m$  are their multiplicities.

- If we were to use partial-fraction expansion to invert  $G_{ij}(s)$ , we would find the intermediate result

$$G_{ij}(s) = \frac{a_{11}}{s - \lambda_1} + \frac{a_{12}}{(s - \lambda_1)^2} + \cdots + \frac{a_{1m_1}}{(s - \lambda_1)^{m_1}} + \cdots \\ \frac{a_{k1}}{s - \lambda_k} + \frac{a_{k2}}{(s - \lambda_k)^2} + \cdots + \frac{a_{km_k}}{(s - \lambda_k)^{m_k}}.$$

- The inverse Laplace transform is then given by

$$\bar{g}_{ij}(t) = \mathcal{L}^{-1}[G_{ij}(s)] \\ = a_{11}e^{\lambda_1 t} + a_{12}te^{\lambda_1 t} + \cdots + a_{1m_1}t^{m_1-1}e^{\lambda_1 t} + \cdots \\ a_{k1}e^{\lambda_k t} + a_{k2}te^{\lambda_k t} + \cdots + a_{km_k}t^{m_k-1}e^{\lambda_k t}.$$

- We therefore conclude the following

1. If for all  $G_{ij}(s)$ , all the poles have strictly negative real parts, then  $\bar{g}_{ij}(t)$  converges to zero exponentially quickly and the system is BIBO stable.

2. If at least one of the  $G_{ij}(s)$  has a pole with zero or positive real part, then  $|\bar{g}_{ij}(t)|$  does not converge to zero and the system is not BIBO stable.

- Note that adding a constant finite  $D$  to  $C(sI - A)^{-1}B$  will not change the poles, we can restate the BIBO stability relationship as

1. The system is uniformly BIBO stable.

2. Every pole of every entry of the transfer function of the system has a strictly negative real part.

### **BIBO versus Lyapunov stability**

- Systems that are exponentially stable in the sense of Lyapunov are also BIBO stable.
- However, the converse is not necessarily true. Consider the example

$$\dot{x}(t) = \begin{bmatrix} 1 & 0 \\ 0 & -2 \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t)$$

$$y(t) = \begin{bmatrix} 1 & 1 \end{bmatrix} x(t).$$

- For this system

$$e^{At} = \begin{bmatrix} e^t & 0 \\ 0 & e^{-2t} \end{bmatrix}$$

which is unbounded and therefore Lyapunov unstable.

- However

$$C e^{At} B = \begin{bmatrix} 1 & 1 \end{bmatrix} \begin{bmatrix} e^t & 0 \\ 0 & e^{-2t} \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = e^{-2t}$$

and therefore the system is BIBO stable.

- The root of this distinction between types of stability stems from the unstable part of the system being unobservable.
- We will study this more in the next chapter.

### **Discrete-time BIBO stability**

- Without going into the details, we simply state the conditions for discrete-time BIBO stability.

- In the time domain, the following two conditions are true for a discrete-time LTV or LTI system:

1. A system is uniformly BIBO stable.
2. Every entry of  $D[k]$  is uniformly bounded and

$$\sup_{k \geq 0} \sum_{\tau=0}^{k-1} |g_{ij}[k, \tau]| < \infty$$

for every entry  $g_{ij}[k, \tau]$  of  $C[k]\Phi[k, \tau]B[\tau]$ .

- For an LTI system, we also have

1. A system is uniformly BIBO stable.
2. For every entry  $\bar{g}_{ij}(\rho)$  of  $CA^\rho B$ , we have

$$\sum_{\rho=1}^{\infty} |\bar{g}_{ij}(\rho)| < \infty.$$

3. Every pole of every entry of the transfer function of the system has magnitude strictly less than 1.

### Where to from here?

- We have now seen how to determine whether a given system is stable, using two principal definitions of stability and a variety of tests.
- What if it isn't stable? Can we stabilize it? Can we control it?
- To see if we will be able to do so, we need to understand two important concepts: controllability and observability.
- We look at these next.