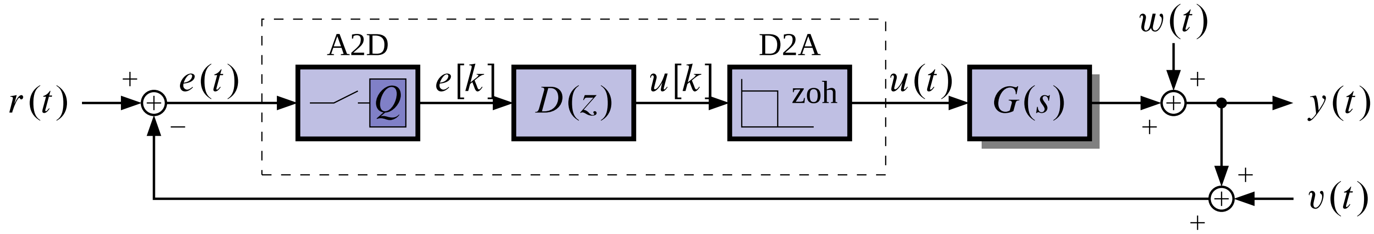
STATE-SPACE DYNAMIC SYSTEMS (DISCRETE-TIME)

3.1: The z transform

- Computer control requires analog-to-digital (A2D) and digital-to-analog (D2A) conversion.
- Everything shown in the dashed box below represents the internals of a computer or embedded system in a digital control scenario.



- The A2D comprises a sampler and quantizer; the D2A performs a zero-order-hold operation.
- The controller is designed in discrete time, where the z transform is used as a design tool rather than the Laplace transform.

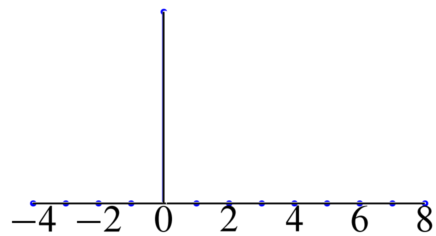
DEFINITION: The z transform is defined for a signal $x(kT)$ to be

$$X(z) = x(0) + x(T)z^{-1} + x(2T)z^{-2} + x(3T)z^{-3} + \cdots$$
$$= \sum_{k=0}^{\infty} x(kT)z^{-k}$$
$$\left(\text{or, } X(z) = \sum_{k=0}^{\infty} x[k]z^{-k}.\right)$$

- As with the Laplace-transform variable "s," the z-transform variable "z" is a complex number.
- I assume that you have studied the z transform before, and so include only a short review in these notes.

EXAMPLE: Define the digital impulse (pulse) function to be

$$\delta[k] = \begin{cases} 1, & k = 0; \\ 0, & \text{otherwise.} \end{cases}$$



- Note: This is very different from an analog impulse (*e.g.*, $\delta[0]$ is defined), but plays a similar role.

$$\begin{aligned} \text{so, let } x[k] &= \delta[k] \\ X(z) &= \sum_{k=0}^{\infty} x[k]z^{-k} \\ &= x[0] + x[1]z^{-1} + x[2]z^{-2} + \cdots \\ &= x[0] = 1. \end{aligned}$$

- This sum converges regardless of the value of z, so ROC $= \{|z| > 0\}$.

Selected important properties of the z transform

- Some select properties of the z transform may be useful.

LINEARITY: The z transform is linear.

- If $x[k] \Longleftrightarrow X(z)$ and $v[k] \Longleftrightarrow V(z)$, then

$$ax[k] + bv[k] \Longleftrightarrow aX(z) + bV(z).$$

RIGHT SHIFT IN TIME (DELAY): When shifting a signal in time,

$$x[k-1] \Longleftrightarrow z^{-1}X(z) + x[-1]$$

$$x[k-2] \Longleftrightarrow z^{-2}X(z) + x[-2] + z^{-1}x[-1]$$

$$\vdots$$

$$x[k-q] \Longleftrightarrow z^{-q}X(z) + x[-q] + z^{-1}x[-q+1] + \cdots + z^{-q+1}x[-1].$$

CONVOLUTION: Discrete-time convolution is analogous to continuous-time convolution:

$$(x * v)[k] = \sum_{i=0}^{k} x[i]v[k-i] \qquad k \ge 0$$

$$\text{or,} \quad = \sum_{i=0}^{k} v[i]x[k-i]. \qquad k \ge 0.$$

- Then, assuming that $x[k]$ and $v[k]$ are zero for negative k,

$$(x * v)[k] \Longleftrightarrow X(z)V(z).$$

INITIAL-VALUE THEOREM: If $x[k] \Longleftrightarrow X(z)$, $x[0] = \lim_{z\to\infty} X(z)$.

FINAL-VALUE THEOREM: If a finite final value exists (all poles of $X(z)$ strictly inside the unit circle except perhaps for a single pole on the unit circle at $z = 1$),

$$\lim_{k\to\infty} x[k] = \lim_{z\to 1}(z-1)X(z).$$

- The table on the next page lists z transforms for commonly encountered sampled time-domain signals:

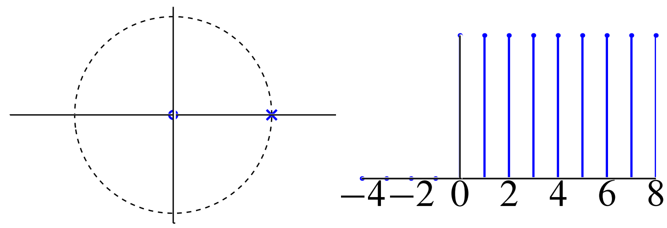
Example table showing z transform of sampled time-domain signals:

Time function $e(t)$	Laplace transform $E(s)$	z-Transform $E(z)$
$1(t)$	$\frac{1}{s}$	$\frac{z}{z-1}$
$t1(t)$	$\frac{1}{s^2}$	$\frac{Tz}{(z-1)^2}$
$\frac{t^2}{2}1(t)$	$\frac{1}{s^3}$	$\frac{T^2z(z+1)}{2(z-1)^3}$
$t^{k-1}1(t)$	$\frac{(k-1)!}{s^k}$	$\lim_{a\to 0}(-1)^{k-1}\frac{\partial^{k-1}}{\partial a^{k-1}}\left[\frac{z}{z-e^{-aT}}\right]$
$e^{-at}1(t)$	$\frac{1}{s+a}$	$\frac{z}{z-e^{-aT}}$
$te^{-at}1(t)$	$\frac{1}{(s+a)^2}$	$\frac{Tze^{-aT}}{(z-e^{-aT})^2}$
$t^ke^{-at}1(t)$	$\frac{(k-1)!}{(s+a)^k}$	$(-1)^k\frac{\partial^k}{\partial a^k}\left[\frac{z}{z-e^{-aT}}\right]$
$1-e^{-at}$	$\frac{a}{s(s+a)}$	$\frac{z(1-e^{-aT})}{(z-1)(z-e^{-aT})}$
$t-\frac{1-e^{-aT}}{a}$	$\frac{a}{s^2(s+a)}$	$\frac{z[(aT-1+e^{-aT})z+(a-e^{-aT}-aTe^{-aT})]}{a(z-1)^2(z-e^{-aT})}$
$1-(1+at)e^{-at}$	$\frac{a^2}{s(s+a)^2}$	$\frac{z}{z-1}-\frac{z}{z-e^{-aT}}-\frac{aTe^{-aT}z}{(z-e^{-aT})^2}$
$e^{-aT}-e^{-bT}$	$\frac{b-a}{(s+a)(s+b)}$	$\frac{(e^{-aT}-e^{-bT})z}{(z-e^{-aT})(z-e^{-bT})}$
$\sin(at)$	$\frac{a}{s^2+a^2}$	$\frac{z\sin(aT)}{z^2-2z\cos(aT)+1}$
$\cos(at)$	$\frac{s}{s^2+a^2}$	$\frac{z(z-\cos(aT))}{z^2-2z\cos(aT)+1}$
$e^{-at}\sin(bt)$	$\frac{b}{(s+a)^2+b^2}$	$\frac{ze^{-aT}\sin(bT)}{z^2-2ze^{-aT}\cos(bT)+e^{-2aT}}$
$e^{-at}\cos(bt)$	$\frac{s+a}{(s+a)^2+b^2}$	$\frac{z^2-ze^{-aT}\cos(bT)}{z^2-2ze^{-aT}\cos(bT)+e^{-2aT}}$
$1-e^{-at}\left(\cos(bt)+\frac{a}{b}\sin(bt)\right)$	$\frac{a^2+b^2}{s[(s+a)^2+b^2]}$	$\frac{z(Az+B)}{(z-1)(z^2-2ze^{-aT}\cos(bT)+e^{-2aT})}$ $A=1-e^{-aT}\left(\cos(bT)+\frac{a}{b}\sin(bT)\right)$ $B=e^{-2aT}+e^{-aT}\left(\frac{a}{b}\sin(bT)-\cos(bT)\right)$
$\frac{1}{ab}+\frac{e^{-at}}{a(a-b)}+\frac{e^{-bt}}{b(b-a)}$	$\frac{1}{s(s+a)(s+b)}$	$\frac{(Az+B)z}{(z-e^{-aT})(z-e^{-bT})(z-1)}$ $A=\frac{b(1-e^{-aT})-a(1-e^{-bT})}{ab(b-a)}$ $B=\frac{ae^{-aT}(1-e^{-bT})-be^{-bT}(1-e^{-aT})}{ab(b-a)}$

3.2: Working with the z transform

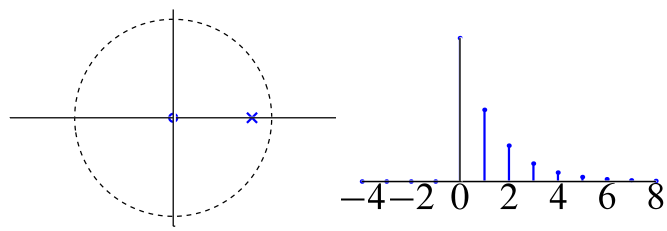
- The following shows some correspondence between the z-plane and some discrete-time unit-pulse-response signals.

Unit step:



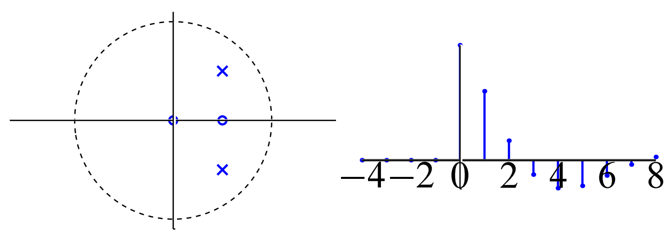
- $h[k] = 1[k]$.
- $H(z) = \dfrac{z}{z-1}, \quad |z| > 1.$

Exponential (geometric):



- $h[k] = a^k 1[k], \quad |a| < 1.$
- $H(z) = \dfrac{z}{z-a}, \quad |z| > |a|.$

General cosinusoid:



- $h[k] = a^k \cos[\omega k] 1[k], \; |a| < 1.$
- $H(z) = \dfrac{z(z - a\cos\omega)}{z^2 - 2a(\cos\omega)z + a^2}$ for $|z| > |a|$.
- The radius to the two poles is a; the angle to the poles is ω.
- The zero (not at the origin) has the same real part as the two poles.
  - If $\omega = 0$, $H(z) = \dfrac{z}{z-a} \ldots$ geometric!
  - If $\omega = 0$, $a = 1$, $H(z) = \dfrac{z}{z-1} \ldots$ step!
- Pole *radius* a is the geometric factor, determines settling time.

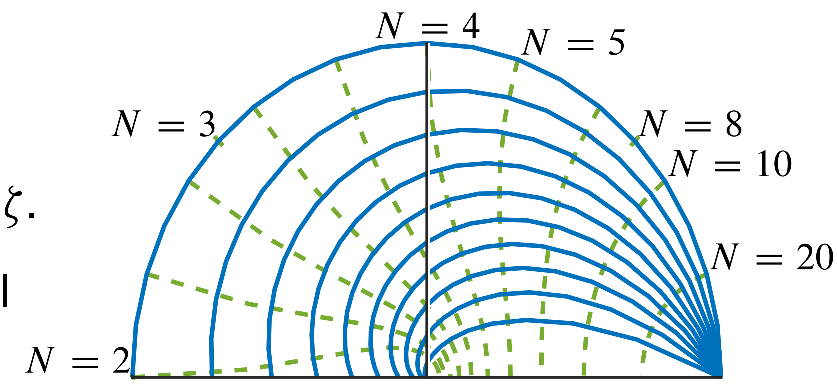
1. $|a| = 0$, finite-duration response. *e.g.*, $\delta[k-N] \Longleftrightarrow z^{-N}$.
2. $|a| > 1$, growing signal that will not decay.
3. $|a| = 1$, signal with constant amplitude; either step or cosine.
4. $|a| < 1$, decaying signal. Small a = fast decay (see below).

a	0.9	0.8	0.6	0.4
$\approx$ duration N	43	21	9	5

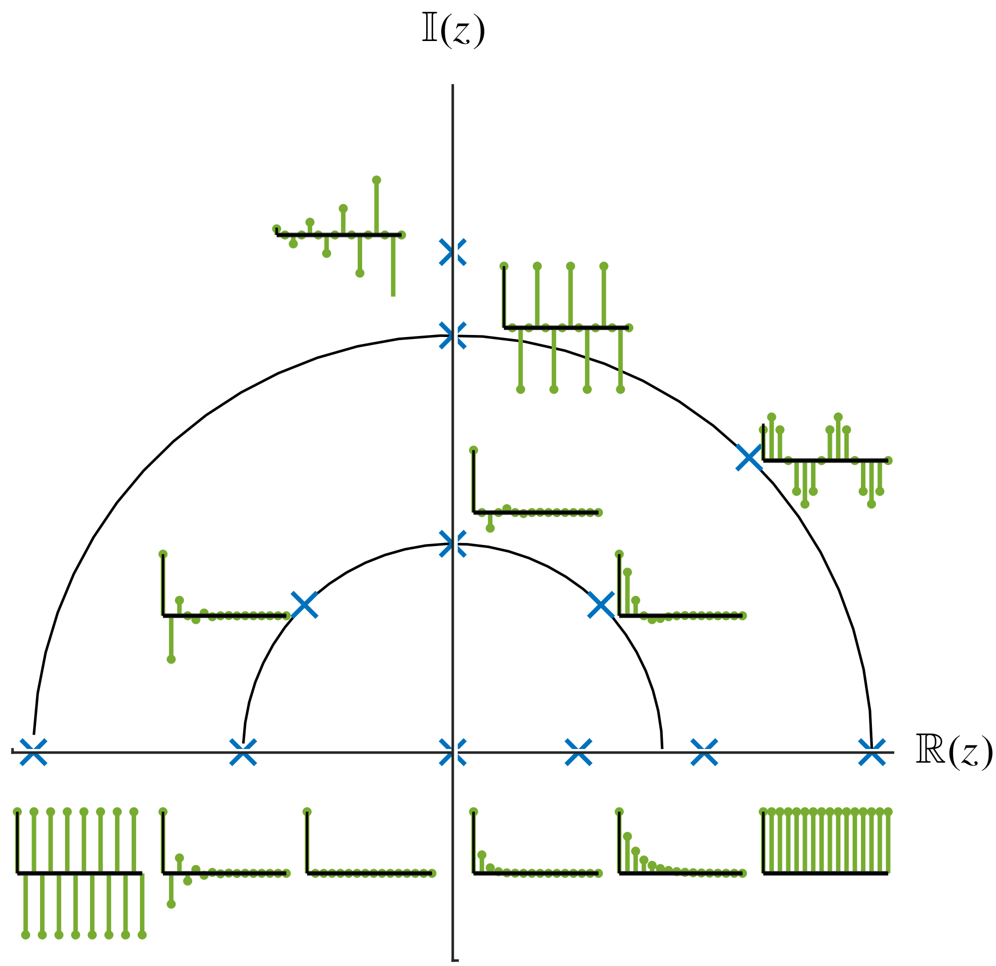
- Pole *angle* ω determines number of samples per oscillation.
  - That is, if we require $\cos[\omega k] = \cos[\omega(k+N)]$, then

$$N = \left.\frac{2\pi}{\omega}\right|_{\text{rad}} = \left.\frac{360}{\omega}\right|_{\text{deg}}.$$

- Solid: cst. damping ratio ζ.
- Dashed: constant natural frequency ω_n.



- Plot to right shows discrete-time unit-pulse responses versus pole locations.



Discrete Impulse Responses versus Pole Locations

Correspondence with continuous-time signals

- Let $g(t) = e^{-at}\cos(bt)1(t)$.
- Suppose

$$\left.\begin{aligned} a &= 0.3567/T \\ b &= \frac{\pi/4}{T} \end{aligned}\right\} \quad T = \text{sampling period.}$$

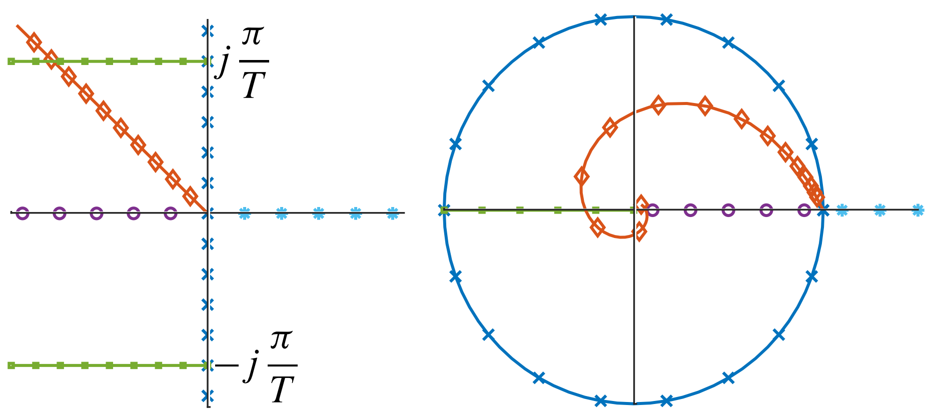
- Then,

$$g[k] = g(kT) = \left(e^{-0.3567}\right)^k \cos\left(\frac{\pi k}{4}\right)1[k]$$
$$= 0.7^k \cos\left(\frac{\pi k}{4}\right)1[k].$$

(This is the cosinusoid example used in the earlier example).

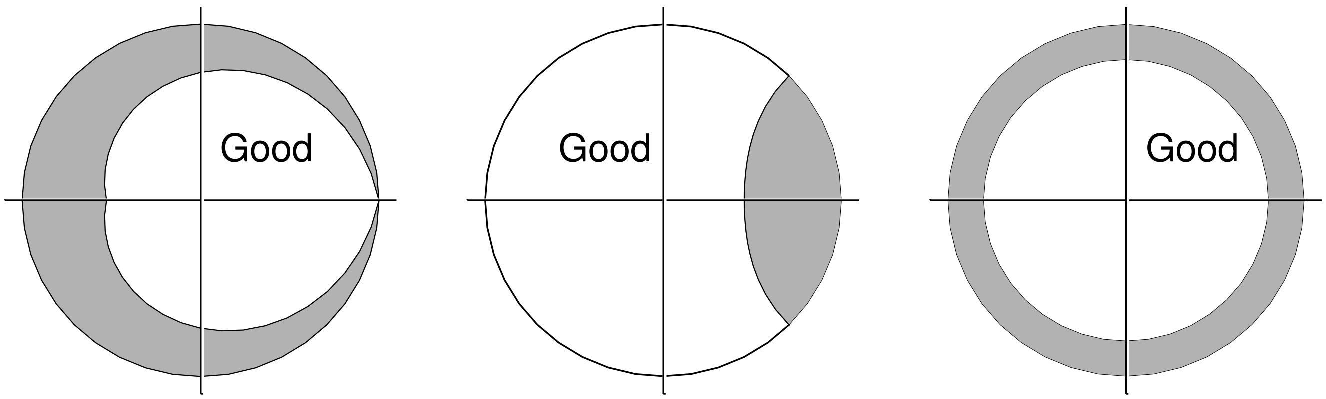
- $G(s)$ has poles at $s_{1,2} = -a + jb$ and $-a - jb$.
- $G(z)$ has poles at radius e^{-aT} angle $\omega = \pm bT$ or at $e^{-aT \pm jbT}$.
  - So, $z_{1,2} = e^{s_1T}$ and e^{s_2T}.
- In general, *poles* convert between the s-plane and z-plane via $z = e^{sT}$.

EXAMPLE: Some corresponding pole locations:



s-plane z-plane

- $j\omega$-axis maps to unit circle.
- Constant damping ratio ζ maps to strange spiral.
- When considering system response to a step input for controls purposes, the following diagrams may be helpful:



Damping ζ Frequency ω_n Settling time

- Higher-order systems:
  - Pole moving toward $z = 1$, system slows down.
  - Zero moving toward $z = 1$, overshoot.
  - Pole and zero moving close to each other cancel.

3.3: Discrete-time state-space form

- Discrete-time systems can also be represented in state-space form.

$$x[k+1] = A_d x[k] + B_d u[k]$$

$$y[k] = C_d x[k] + D_d u[k].$$

- The subscript "d" is used here to emphasize that, in general, the "A", "B", "C" and "D" matrices are *different* for discrete-time and continuous-time systems, even if the underlying plant is the same.
- I will usually drop the "d" and expect you to interpret the system from its context.

Formulating from transfer functions

- Discrete-time dynamics are represented as difference equations. *e.g.,*

$$y[k+3]+a_1y[k+2]+a_2y[k+1]+a_3y[k] = b_1u[k+2]+b_2u[k+1]+b_3u[k]$$

$$y[k]+a_1y[k-1]+a_2y[k-2]+a_3y[k-3] = b_1u[k-1]+b_2u[k-2]+b_3u[k-3].$$

- This particular example has transfer function

$$G(z) = \frac{b_1z^2 + b_2z + b_3}{z^3 + a_1z^2 + a_2z + a_3} = \frac{Y(z)}{U(z)}.$$

- This transfer function may be converted to state-space in a very similar way to continuous-time systems.
- First, consider the poles:

$$G_p(z) = \frac{1}{z^3 + a_1z^2 + a_2z + a_3} = \frac{V(z)}{U(z)}$$

$$\Rightarrow \quad v[k+3] + a_1v[k+2] + a_2v[k+1] + a_3v[k] = u[k].$$

- Choose current and advanced versions of $v[k]$ as state.

$$x[k] = \begin{bmatrix} v[k+2] & v[k+1] & v[k] \end{bmatrix}^T.$$

Then

$$\begin{aligned} x[k+1] &= \begin{bmatrix} v[k+3] \\ v[k+2] \\ v[k+1] \end{bmatrix} \\ &= \begin{bmatrix} -a_1 & -a_2 & -a_3 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} v[k+2] \\ v[k+1] \\ v[k] \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} u[k]. \end{aligned}$$

- We now add zeros.

$$G(z) = \frac{b_1 z^2 + b_2 z + b_3}{z^3 + a_1 z^2 + a_2 z + a_3} = \frac{Y(z)}{U(z)}.$$

Break up transfer function into two parts. $\dfrac{V(z)}{U(z)}$ contains all of the poles of $\dfrac{Y(z)}{U(z)}$. Then,

$$Y(z) = \left[b_1 z^2 + b_2 z + b_3\right] V(z).$$

Or,

$$y[k] = b_1 v[k+2] + b_2 v[k+1] + b_3 v[k].$$

Then

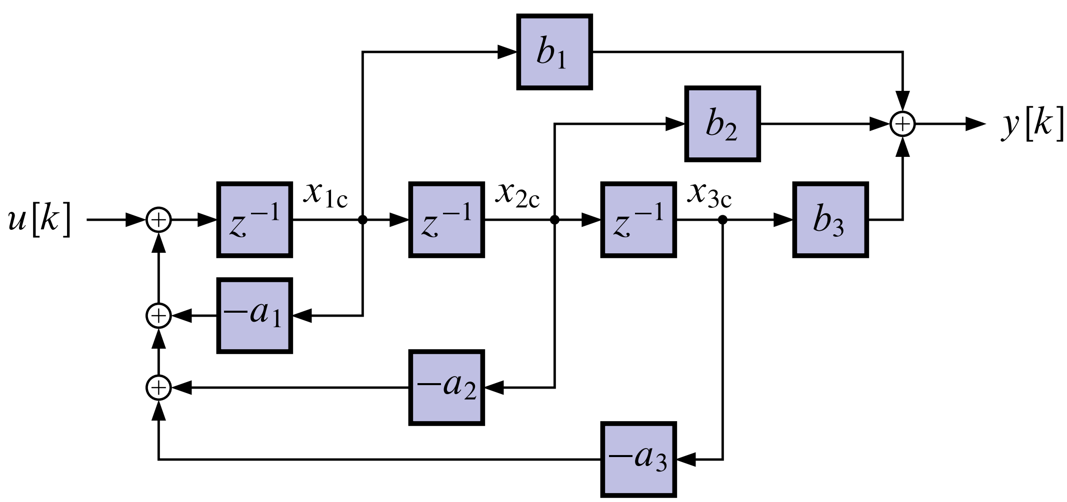
$$x[k+1] = \begin{bmatrix} -a_1 & -a_2 & -a_3 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} v[k+2] \\ v[k+1] \\ v[k] \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} u[k]$$

$$y[k] = \begin{bmatrix} b_1 & b_2 & b_3 \end{bmatrix} x[k] + \begin{bmatrix} 0 \end{bmatrix} u[k].$$

- Many discrete-time transfer functions are not strictly proper. Solve by polynomial long division, and setting D equal to the quotient.
- In MATLAB, `[A,B,C,D]=tf2ss(num,den)` converts a transfer function form to state-space form (same syntax as for cts.-time, but `num` and `den` must have same order, maybe zero padded).
- As with continuous-time systems, we have a lot of freedom when making state-space models (*i.e.*, in choosing components of $x[k]$).

Canonical forms

- In discrete-time we have the same canonical forms: Controller, observer, controllability, observability, modal, and Jordan.
- They are derived in the same way, as demonstrated above for the controller form.
- A block diagram for controller form is:



Time (dynamic) response

Homogeneous part

- First, consider the scalar case

$$x[k+1] = ax[k], \quad x[0].$$

- Take z-transform. $X(z) = (z-a)^{-1}zx[0]$.
- Inverse z-transform. $x[k] = a^k x[0]$.
- Similarly, the full solution (vector case) is

$$x[k] = A^k x[0].$$

Aside: Nilpotent systems

- A is nilpotent if some power of n exists such that

$$A^n = 0.$$

- A does not just decay to zero, it is exactly zero!
- This might be a desirable control design! (Why?) You might imagine that all the eigenvalues of A must be zero for this to work.

Forced solution

- The full solution is:

$$x[k] = A^k x[0] + \underbrace{\sum_{j=0}^{k-1} A^{k-1-j} Bu[j]}_{\text{convolution}}.$$

- This can be proved by induction from the equation

$$x[k+1] = Ax[k] + Bu[k], \quad x[0]$$

- Clearly, if $y[k] = Cx[k] + Du[k]$,

$$y[k] = \underbrace{CA^k x[0]}_{\text{initial resp.}} + \underbrace{\sum_{j=0}^{k-1} CA^{k-1-j} Bu[k]}_{\text{convolution}} + \underbrace{Du[k]}_{\text{feedthrough}}.$$

3.4: More on discrete-time state-space models

State-space to transfer function

- Start with the state equations

$$\begin{aligned} x[k+1] &= Ax[k] + Bu[k] \\ y[k] &= Cx[k] + Du[k]. \end{aligned}$$

- z-transform

$$\begin{aligned} zX(z) - zx[0] &= AX(z) + BU(z) \\ Y(z) &= CX(z) + DU(z) \end{aligned}$$

or

$$\begin{aligned} (zI - A)X(z) &= BU(z) + zx[0] \\ X(z) &= (zI - A)^{-1}BU(z) + (zI - A)^{-1}zx[0] \end{aligned}$$

and

$$Y(z) = \underbrace{[C(zI - A)^{-1}B + D]}_{\text{transfer function of system}} U(z) + \underbrace{C(zI - A)^{-1}zx[0]}_{\text{response to initial conditions}}.$$

- So,

$$\frac{Y(z)}{U(z)} = C(zI - A)^{-1}B + D$$

- Same form as for continuous-time systems.
- Poles of system are roots of $\det[zI - A] = 0$.

Transformation

- State-space representations are not unique. Selection of state x are quite arbitrary.

- Analyze the transformation of

$$x[k+1] = Ax[k] + Bu[k]$$
$$y[k] = Cx[k] + Du[k]$$

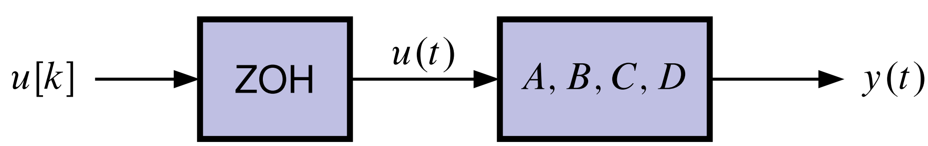
Let $x[k] = Tw[k]$, where T is an invertible (similarity) transformation matrix.

$$w[k+1] = \underbrace{T^{-1}AT}_{\bar{A}} w[k] + \underbrace{T^{-1}B}_{\bar{B}} u[k]$$
$$y[k] = \underbrace{CT}_{\bar{C}} w[k] + \underbrace{D}_{\bar{D}} u[k]$$
$$\text{so,}\quad w[k+1] = \bar{A}w[k] + \bar{B}u[k]$$
$$y[k] = \bar{C}w[k] + \bar{D}u[k].$$

- Same as for continuous-time.

Converting plant dynamics to discrete time

- Combine the dynamics of the zero-order hold and the plant.



- The continuous-time dynamics of the plant are:

$$\dot{x}(t) = Ax(t) + Bu(t)$$
$$y(t) = Cx(t) + Du(t).$$

- Evaluate $x(t)$ at discrete times. Recall

$$x(t) = \int_0^t e^{A(t-\tau)}Bu(\tau)\,\mathrm{d}\tau$$

$$x[k+1] = x((k+1)T) = \int_0^{(k+1)T} e^{A((k+1)T-\tau)} Bu(\tau)\,\mathrm{d}\tau$$

- With malice aforethought, break up the integral into two pieces. The first piece will become A_d times $x(kT)$. The second part will become B_d times $u(kT)$.

$$\begin{aligned} &= \int_0^{kT} e^{A((k+1)T-\tau)} Bu(\tau)\,\mathrm{d}\tau + \int_{kT}^{(k+1)T} e^{A((k+1)T-\tau)} Bu(\tau)\,\mathrm{d}\tau \\ &= \int_0^{kT} e^{AT} e^{A(kT-\tau)} Bu(\tau)\,\mathrm{d}\tau + \int_{kT}^{(k+1)T} e^{A((k+1)T-\tau)} Bu(\tau)\,\mathrm{d}\tau \\ &= e^{AT} x(kT) + \int_{kT}^{(k+1)T} e^{A((k+1)T-\tau)} Bu(\tau)\,\mathrm{d}\tau. \end{aligned}$$

In the remaining integral, note that $u(\tau)$ is constant from kT to $(k+1)T$, and equal to $u(kT)$; let $\sigma = (k+1)T - \tau$; $\tau = (k+1)T - \sigma$; $\mathrm{d}\tau = -\mathrm{d}\sigma$.

$$x((k+1)T) = e^{AT} x(kT) + \left[\int_0^T e^{A\sigma} B\,\mathrm{d}\sigma\right] u(kT)$$

$$\text{or, } x[k+1] = e^{AT} x[k] + \left[\int_0^T e^{A\sigma} B\,\mathrm{d}\sigma\right] u[k].$$

- So, we have a discrete-time state-space representation from the continuous-time representation.

$$x[k+1] = A_d x[k] + B_d u[k]$$

where $A_d = e^{AT}$, $B_d = \displaystyle\int_0^T e^{A\sigma} B\,\mathrm{d}\sigma$.

- Similarly,

$$y[k] = Cx[k] + Du[k].$$

That is, $C_d = C$; $D_d = D$.

Calculating A_d, B_d, C_d and D_d

- C_d and D_d require no calculation since $C_d = C$ and $D_d = D$.
- A_d is calculated via the *matrix* exponential $A_d = e^{AT}$. This is different from taking the exponential of each element in AT.
- If MATLAB is handy, you can type in `Ad=expm(A*T)`
- If MATLAB is not handy, then you need to work a little harder. Recall from the previous set of notes that $e^{At} = \mathcal{L}^{-1}[(sI - A)^{-1}]$. So,

$$e^{AT} = \mathcal{L}^{-1}[(sI - A)^{-1}]\Big|_{t=T},$$

  which is probably the "easiest" way to work it out by hand. Or,

$$e^{AT} = I + AT + \frac{A^2T^2}{2!} + \frac{A^3T^3}{3!} + \cdots$$

  which is a convergent series so may be approximated with only a few terms.

- Now we focus on computing B_d. Recall that

$$\begin{aligned} B_d &= \int_0^T e^{A\sigma} B \,\mathrm{d}\sigma \\ &= \int_0^T \left(I + A\sigma + A^2\frac{\sigma^2}{2} + \ldots \right) B \,\mathrm{d}\sigma \\ &= \left(IT + A\frac{T^2}{2!} + A^2\frac{T^3}{3!} + \ldots \right) B \\ &= A^{-1}(e^{AT} - I)B \\ &= A^{-1}(A_d - I)B, \end{aligned}$$

  if A^{-1} exists.

- So, calculating B_d is easy once we have already calculated A_d.
- Also, in MATLAB, `[Ad,Bd]=c2d(A,B,T)`

3.5: Linear time-varying and nonlinear discrete-time systems

Linear time-varying discrete-time systems

- A linear time-varying system can be written as

$$x[k+1] = A[k]x[k] + B[k]u[k]$$

$$y[k] = C[k]x[k] + D[k]u[k].$$

- Analysis is somewhat easier than for the continuous-time counterpart.
- Consider first the homogeneous case

$$x[k+1] = A[k]x[k], \qquad x[k_0] = x_0, \qquad k \geq 0.$$

- The solution to this is

$$x[k] = \Phi[k, k_0]x_0$$

where

$$\Phi[k, k_0] = \begin{cases} I, & k = k_0 \\ A[k-1]A[k-2]\cdots A[k_0], & k > k_0. \end{cases}$$

- We see that the state-transition matrix can be computed readily, and does not involve any difficult integrals.
- Some properties of the state-transition matrix include:

$$\Phi[k+1, k_0] = A[k]\Phi[k, k_0]$$

$$\Phi[k_0, k_0] = I$$

$$\Phi[k, s]\Phi[s, \tau] = \Phi[k, \tau].$$

- The solution to the nonhomogeneous case can be shown to be

$$x[k] = \Phi[k, k_0]x_0 + \sum_{\tau=k_0}^{k-1} \Phi[k, \tau+1]B[\tau]u[\tau]$$

$$y[k] = C[k]\Phi[k,k_0]x_0 + \sum_{\tau=k_0}^{k-1} C[k]\Phi[k,\tau+1]B[\tau]u[\tau] + D[k]u[k].$$

Nonlinear discrete-time systems

- The approach to working with nonlinear discrete-time systems is similar to that used for nonlinear continuous-time systems.
  - We can linearize around an equilibrium point or around a solution trajectory.
- We define a nonlinear state-space form as

$$\begin{aligned} x[k+1] &= f(x[k],u[k]) \\ y[k] &= g(x[k],u[k]). \end{aligned}$$

Linearizing around an equilibrium point

- We can linearize around an equilibrium point if there exists an equilibrium constant solution

$$u[k] = u^{\text{eq}}, \qquad x[k] = x^{\text{eq}}, \qquad y[k] = y^{\text{eq}}.$$

- Then, let the actual input signal and initial state be written as

$$u[k] = u^{\text{eq}} + \delta u[k], \qquad x[k_0] = x^{\text{eq}} + \delta x^{\text{eq}}[k_0]$$

  for *small* $\delta u[k]$.
- Then, we can write actual state and output as

$$x[k] = x^{\text{eq}} + \delta x[k], \qquad y[k] = y^{\text{eq}} + \delta y[k].$$

- Following the same process used in the prior chapter, the linearized perturbation system is

$$\delta x[k+1] = A\delta x[k] + B\delta u[k]$$

$$\delta y[k] = C\delta x[k] + D\delta u[k],$$

where the linearized state-space matrices are

$$A = \left(\frac{\mathrm{d}f(x^{\text{eq}}, u^{\text{eq}})}{\mathrm{d}x}\right); \qquad B = \left(\frac{\mathrm{d}f(x^{\text{eq}}, u^{\text{eq}})}{\mathrm{d}u}\right)$$

$$C = \left(\frac{\mathrm{d}g(x^{\text{eq}}, u^{\text{eq}})}{\mathrm{d}x}\right); \qquad D = \left(\frac{\mathrm{d}g(x^{\text{eq}}, u^{\text{eq}})}{\mathrm{d}u}\right)$$

and the overall state and output can be computed as

$$x[k] = x^{\text{eq}} + \delta x[k] \qquad \text{and} \qquad y[k] = y^{\text{eq}} + \delta y[k].$$

Linearizing around a solution trajectory

- Alternately, suppose it is known that

$$u^{\text{sol}}[k], \qquad x^{\text{sol}}[k], \qquad y^{\text{sol}}[k]$$

form a time-varying solution to the nonlinear dynamics of the system.

- This means that

$$x^{\text{sol}}[k+1] = f(x^{\text{sol}}[k], u^{\text{sol}}[k])$$

$$y^{\text{sol}}[k] = g(x^{\text{sol}}[k], u^{\text{sol}}[k]).$$

- Then, let general input, state, and output be

$$u[k] = u^{\text{sol}}[k] + \delta u[k]$$

$$x[k] = x^{\text{sol}}[k] + \delta x[k]$$

$$y[k] = y^{\text{sol}}[k] + \delta y[k].$$

- Proceeding as before, we find the perturbation system

$$\delta x[k+1] \approx \underbrace{\left(\frac{\mathrm{d}f(x^{\text{sol}}, u^{\text{sol}})}{\mathrm{d}x[k]}\right)}_{A[k]} \delta x[k] + \underbrace{\left(\frac{\mathrm{d}f(x^{\text{sol}}, u^{\text{sol}})}{\mathrm{d}u[k]}\right)}_{B[k]} \delta u[k]$$

$$\delta y[k] \approx \underbrace{\left(\frac{\mathrm{d}g(x^{\text{sol}}, u^{\text{sol}})}{\mathrm{d}x[k]}\right)}_{C[k]} \delta x[k] + \underbrace{\left(\frac{\mathrm{d}g(x^{\text{sol}}, u^{\text{sol}})}{\mathrm{d}u[k]}\right)}_{D[k]} \delta u[k].$$

- The overall state and output can be computed as

$$x[k] = x^{\text{sol}} + \delta x[k] \qquad \text{and} \qquad y[k] = y^{\text{sol}} + \delta y[k].$$

- Notice that even if the nonlinear system is time-invariant, the linearized system will be time-varying, in general.

Where to from here?

- We have now seen how to model both continuous-time and discrete-time systems in state-space form.
- There are many commonalities between the two, which we will take advantage of in the remainder of the course.
- The main difference is in interpreting the result. For example,
  - Eigenvalues of the continuous-time A matrix correspond to s-plane locations, and should be in the left-half s-plane for stability;
  - Eigenvalues of the discrete-time A matrix correspond to z-plane locations, and should be in the unit circle for stability.
- In fact, stability is a very important concept, which must be evaluated and ensured before we talk more about control.
- This will be our next topic.