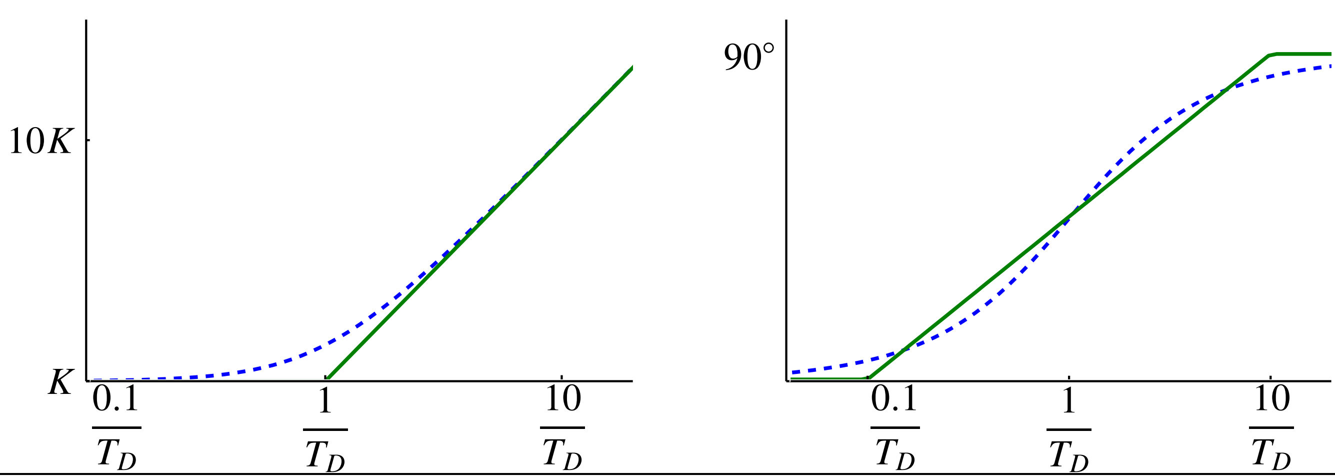
FREQUENCY-RESPONSE DESIGN

9.1: PD and lead compensation networks

- The frequency-response methods we have seen so far largely tell us about stability and stability margins of a closed-loop system based on open-loop response.
- Now, we look at frequency-response based design methods which primarily aim at improving stability margins.
- Also, given the relationship between PM and performance, we have some idea of transient response as well.
- Start thinking of Bode-magnitude and Bode-phase plots as "LEGO" to make the frequency response we want.

PD compensation

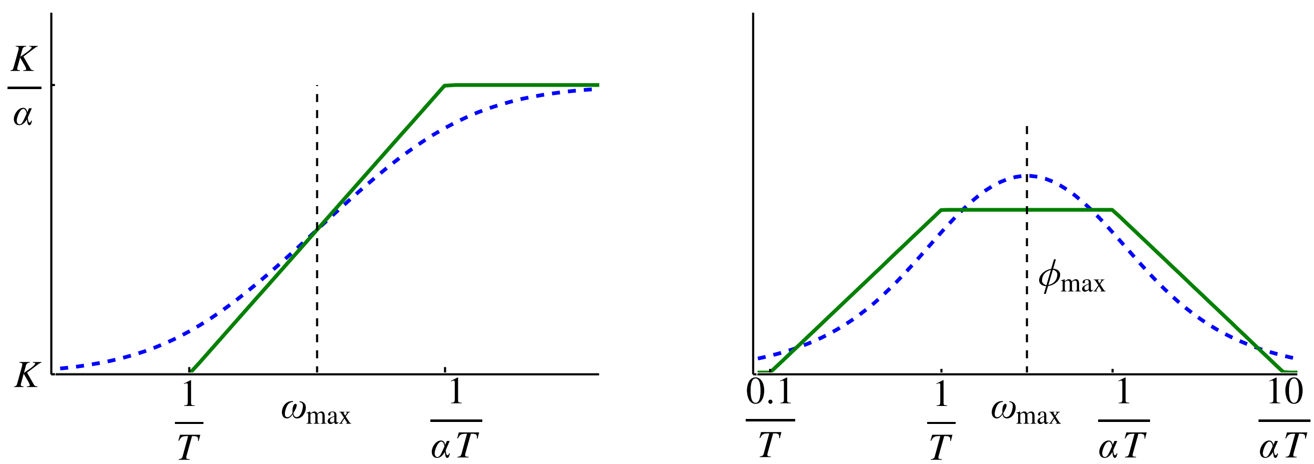
- Compensator $D(s) = K(1 + T_D s)$.
- We have seen from root-locus that this has a stabilizing effect.
- Magnitude and phase effect:



- PD controller increases phase for frequencies over $\omega \approx 0.1/T_D$. So, locate $1/T_D <$ crossover so that phase margin at crossover is better.
- Problem: Magnitude response continues to increase as frequency increases. This amplifies high-frequency sensor noise.

Lead compensation

- Compensator $D(s) = K\left[\dfrac{Ts+1}{\alpha Ts+1}\right], \qquad \alpha < 1.$
- Approximate PD control for frequencies up to $\omega = \dfrac{1}{\alpha T}$.
- Magnitude and phase effect:



- The phase contributed at frequency ω is

$$\phi = \tan^{-1}(T\omega) - \tan^{-1}(\alpha T\omega).$$

- To find where the maximum phase contribution is, we take the derivative of ϕ with respect to ω and set it to zero

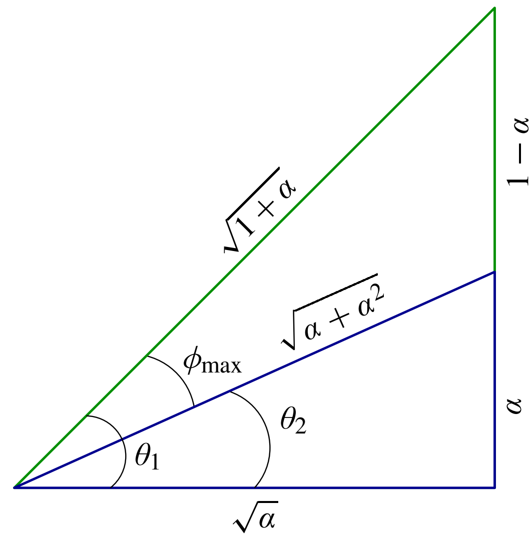
$$\begin{aligned}\frac{d\phi}{d\omega} &= \frac{d\tan^{-1}(T\omega)}{d\omega} - \frac{d\tan^{-1}(\alpha T\omega)}{d\omega} \\ &= \frac{T}{T^2\omega^2+1} - \frac{\alpha T}{\alpha^2T^2\omega^2+1} = 0\end{aligned}$$

$$\alpha^2 T^3\omega^2 + T = \alpha T^3\omega^2 + \alpha T$$
$$T^3\omega^2\alpha(\alpha - 1) = T(\alpha - 1)$$
$$\omega_{\max} = \frac{1}{T\sqrt{\alpha}}.$$

- For this frequency, the amount of phase added is

$$\phi_{\max} = \tan^{-1}(T\omega_{\max}) - \tan^{-1}(\alpha T\omega_{\max})$$
$$= \underbrace{\tan^{-1}\left(\frac{1}{\sqrt{\alpha}}\right)}_{\theta_1} - \underbrace{\tan^{-1}\left(\frac{\alpha}{\sqrt{\alpha}}\right)}_{\theta_2}$$

To get a simpler expression, consider the triangles to the right.



- We can solve for $\phi_{\max}$ using the law of cosines. This gives:

$$(1-\alpha)^2 = (1+\alpha) + (\alpha+\alpha^2) - 2\sqrt{1+\alpha}\sqrt{\alpha+\alpha^2}\cos(\phi_{\max})$$
$$1 - 2\alpha + \alpha^2 = 1 + 2\alpha + \alpha^2 - 2\sqrt{1+\alpha}\sqrt{\alpha+\alpha^2}\cos(\phi_{\max})$$
$$\cos(\phi_{\max}) = \frac{4\alpha}{2\sqrt{1+\alpha}\sqrt{\alpha+\alpha^2}} = \frac{2\sqrt{\alpha}}{(1+\alpha)}.$$

- Now, we remember that $\cos^2(\theta) + \sin^2(\theta) = 1$, so

$$\sin^2(\phi_{\max}) = 1 - \frac{4\alpha}{(1+\alpha)^2}$$
$$= \frac{(1+\alpha)^2 - 4\alpha}{(1+\alpha)^2}$$
$$= \frac{(1-\alpha)^2}{(1+\alpha)^2}.$$

- So, we conclude that

$$\phi_{\max} = \sin^{-1}\left(\frac{1-\alpha}{1+\alpha}\right).$$

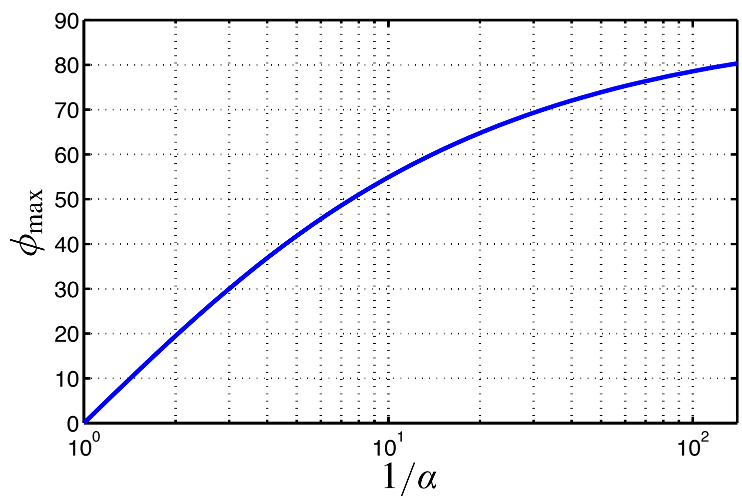
- We can invert this to find α to add a certain phase $\phi_{\max}$

$$\alpha = \frac{1-\sin(\phi_{\max})}{1+\sin(\phi_{\max})}.$$

- We can show that $\omega_{\max}$ occurs mid-way between $\dfrac{1}{T}$ and $\dfrac{1}{\alpha T}$ on a log scale

$$\log(\omega_{\max}) = \log\left(\frac{\frac{1}{\sqrt{T}}}{\sqrt{\alpha T}}\right) = \log\left(\frac{1}{\sqrt{T}}\right) + \log\left(\frac{1}{\sqrt{\alpha T}}\right)$$
$$= \frac{1}{2}\left[\log\left(\frac{1}{T}\right) + \log\left(\frac{1}{\alpha T}\right)\right].$$

- How much can we improve phase with a lead network?



- If we need more phase improvement, we can use a double-lead compensator:

$$D(s) = K\left[\frac{Ts+1}{\alpha Ts+1}\right]^2.$$

- Need to compromise between good phase margin and good sensor noise rejection at high frequency.

9.2: Design method #1 for lead controllers

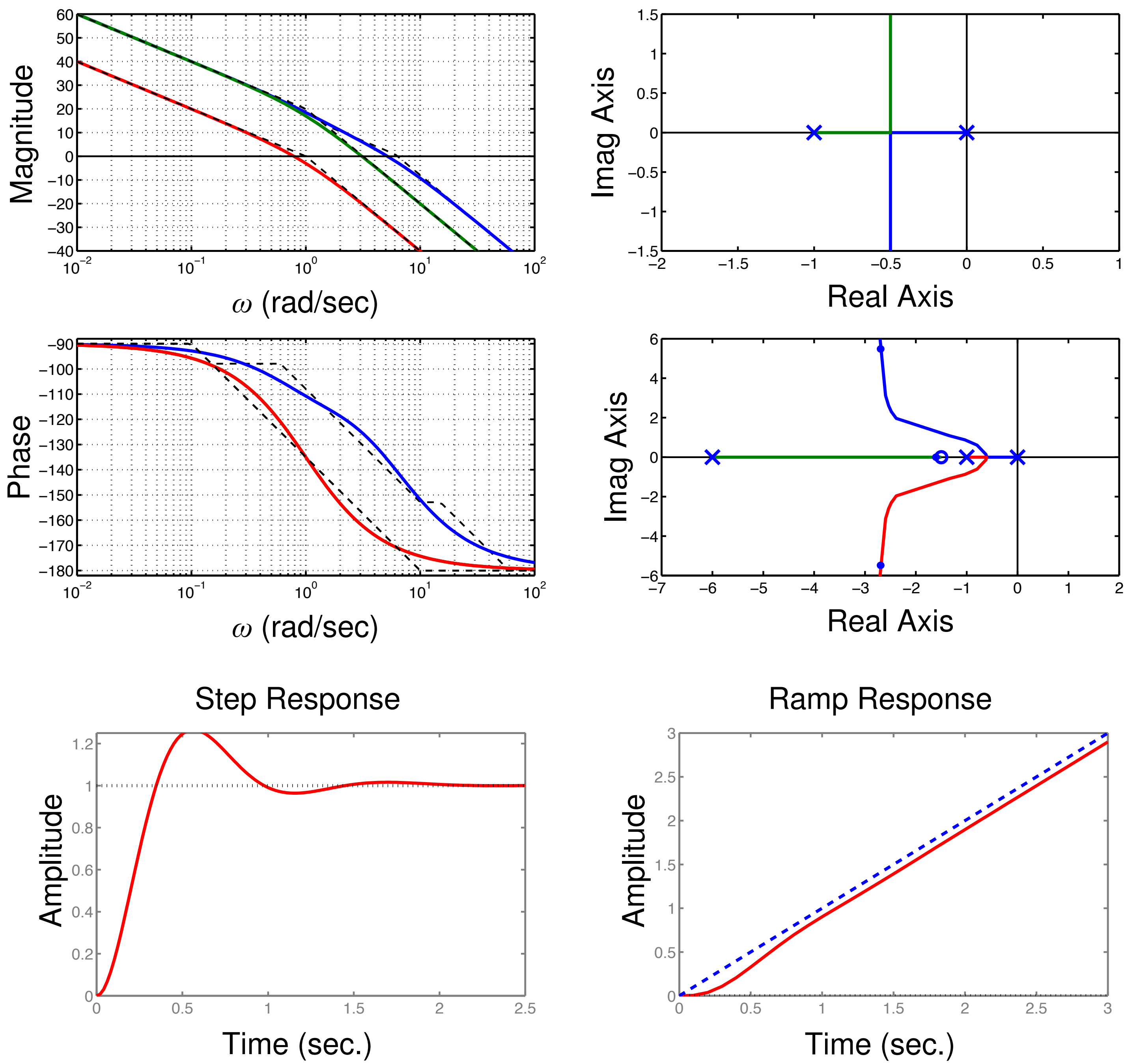
- Design specification include:
  - Desired steady-state error.
  - Desired phase margin.
- Compute steady-state error of compensated system. Set equal to desired steady-state error by computing K.
- Evaluate the PM of the uncompensated system using the value of K from above. $\omega_{\max}$ initially set to crossover frequency.
- Add a small amount of PM (5° to 12°) to the needed phase lead. (Lead network moves crossover point, so need to design conservatively): Gives $\phi_{\max}$.
- Determine $\alpha = \dfrac{1-\sin(\phi_{\max})}{1+\sin(\phi_{\max})}$.
- Determine $T = \dfrac{1}{\omega_{\max}\sqrt{\alpha}}$.
- Iterate if necessary (choose a slightly different $\omega_{\max}$ or $\phi_{\max}$).

EXAMPLE: Plant $G(s) = \dfrac{1}{s(s+1)}$

- Specification 1: Want steady-state error for ramp input $<$ 0.1.
- Specification 2: Want overshoot $M_p < 25\%$.
- Start by computing steady-state error

$$
\begin{aligned}
e_{ss} &= \lim_{s\to 0} s\left[\frac{1}{1+D(s)G(s)}\right]R(s)\\
&= \lim_{s\to 0}\left[\frac{1}{s+D(s)\frac{1}{(s+1)}}\right] = \frac{1}{D(0)}.
\end{aligned}
$$

- $D(s) = K\left[\dfrac{Ts+1}{\alpha Ts+1}\right]$ so $D(0) = K$.
- So, $\dfrac{1}{D(0)} = \dfrac{1}{K} = 0.1 \quad \ldots \quad K = 10.$
- In Bode plot below, red line is original plant. Green line is plant with added gain of $K = 10$, but no lead compensator.
- We see that the new crossover is at 3 rad s^{-1}, so $\omega_{\max} = 3$ rad s^{-1}.
- Overshoot specification $M_p < 25\%$ gives $PM > 45°$.
- Evaluating (red line) Bode diagram of $KG(s)$ at crossover, we see we have a phase margin of $18°$. Need about $27°$ more to meet spec.
- Note, addition of pole and zero from lead network will shift crossover frequency somewhat. To be safe, design for added phase of $37°$ instead of $27°$.
- $\alpha = \dfrac{1-\sin(37°)}{1+\sin(37°)} = 0.25. \qquad T = \dfrac{1}{\sqrt{0.25}(3)} = 0.667.$
- So, $D(s) = 10\left[\dfrac{2s/3+1}{2s/12+1}\right]$.
- In plots below, the blue line is the Bode plot of the compensated system $KD(s)G(s)$.
  - Also, uncompensated root locus is plotted top-right; compensated root locus is plotted bottom-right.
- We find that we just missed specifications: $PM = 44°$. Iterate if desired to meet specs.



- Summary of design example.

  1. Determine dc-gain so that steady-state errors meet spec.
  2. Select α and T to achieve an acceptable PM at crossover.

9.3: Design method #2 for lead controllers

- Design specifications
  - Desired closed-loop bandwidth requirements (rather than e_{ss}).
  - Desired PM requirements.
- Choose open-loop crossover frequency ω_c to be half the desired closed-loop bandwidth.
- Evaluate $K_G = |G(j\omega_c)|$ and $\phi_G = \angle G(j\omega_c)$.
- Compute phase lead required $\phi_{\max} = PM - \phi_G - 180$.
- Compute $\alpha = \dfrac{1 - \sin(\phi_{\max})}{1 + \sin(\phi_{\max})}$.
- Compute $T = \dfrac{1}{\sqrt{\alpha}\omega_c}$.
- Compensation is

$$D(s) = K\left[\frac{Ts+1}{\alpha Ts+1}\right]$$

$$|D(j\omega_c)| = K\sqrt{\frac{\frac{1}{\alpha}+1}{\alpha+1}} = K\frac{\sqrt{\alpha}}{\sqrt{\alpha}}\sqrt{\frac{\frac{1}{\alpha}+1}{\alpha+1}} = \frac{K}{\sqrt{\alpha}}.$$

- Open-loop gain at ω_c is designed to be 1:

$$|D(j\omega_c)||G(j\omega_c)| = 1$$

$$\frac{K}{\sqrt{\alpha}}|G(j\omega_c)| = 1$$

$$K = \frac{\sqrt{\alpha}}{K_G}.$$

- Our design is now complete.

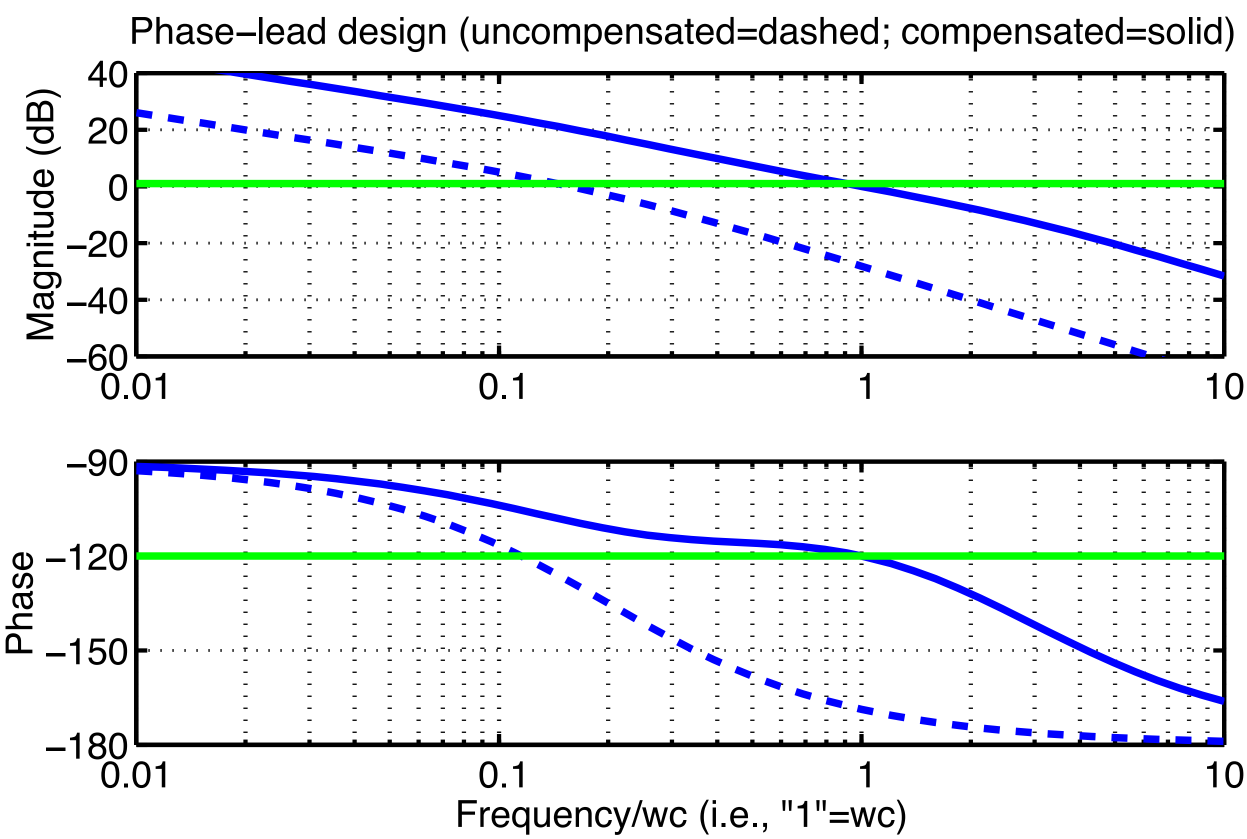
EXAMPLE: Desired gain crossover frequency of 10 radians per second. Desired PM of 60°. $G(s) = \dfrac{1}{s(s+1)}$.

- $K_G = \left|\dfrac{1}{j10(j10+1)}\right| = 0.01, \qquad \phi_G = -174.3°.$
- $\phi_{\max} = 60° + 174.3° - 180° = 54.3°.$
- $\alpha = \dfrac{1-\sin(\phi_{\max})}{1+\sin(\phi_{\max})} = 0.1037.$
- $T = \dfrac{1}{\sqrt{\alpha}10} = 0.3105.$
- $K_D = \dfrac{\sqrt{\alpha}}{K_G} = 32.36.$
- Matlab code to automate this procedure:

```
% [K,T,alpha]=bod_lead(np,dp,wc,PM)
% Computes the lead compensation of the plant np/dp to have
% phase margin PM (in degrees) at crossover frequency wc (in radians/sec).
% e.g., [K,T,alpha]=bod_lead([1],[1 1 0],5,60);
function [K,T,alpha]=bod_lead(np,dp,wc,PM)

% first compute the plant response at wc.
[magc,phc]=bode(np,dp,wc);

% now, compute the needed phase lead at wc and convert to radians
% for use with "sine"
phir=(-180+PM-phc)*pi/180;

if abs(phir)pi/2,
  fprintf('A simple phase lead/lag cannot change the phase by more\n');
  fprintf('than +/- 90 degrees.\n');
  error('Aborting.');
end;

% Compute alpha, T, and K for compensator
alpha=(1-sin(phir))/(1+sin(phir));
T=1/(wc*sqrt(alpha)); K=sqrt(alpha)/magc;
```

```
% compute the new open-loop system by convolving the plant polynomials
% with the compensator polynomials.
nol=conv(np,K*[T 1]); dol=conv(dp,[alpha*T 1]);

% check the solution by plotting the Bode plot for the new open-loop
% polynomials. Include the frequency w=1
% to get the full resonance response to show the gain margin. Also, plot
% the uncompensated Bode response.
w=logspace(-2,1)*wc; w(34)=wc; clf;
[mag1,ph1]=bode(np,dp,w); [mag2,ph2]=bode(nol,dol,w);
subplot(211);
semilogx(w/wc,20*log10(mag1),'--'); hold on;
semilogx(w/wc,20*log10(mag2)); grid; plot(w/wc,0*w+1,'g-');
ylabel('Magnitude (dB)');
title('Phase-lead design (uncompensated=dashed; compensated=solid)');

subplot(212);
semilogx(w/wc,ph1,'--'); hold on; semilogx(w/wc,ph2); grid;
plot(w/wc,0*w-180+PM,'g-');
ylabel('Phase'); xlabel('Frequency/wc (i.e., "1"=wc)');
```

■ Matlab results:

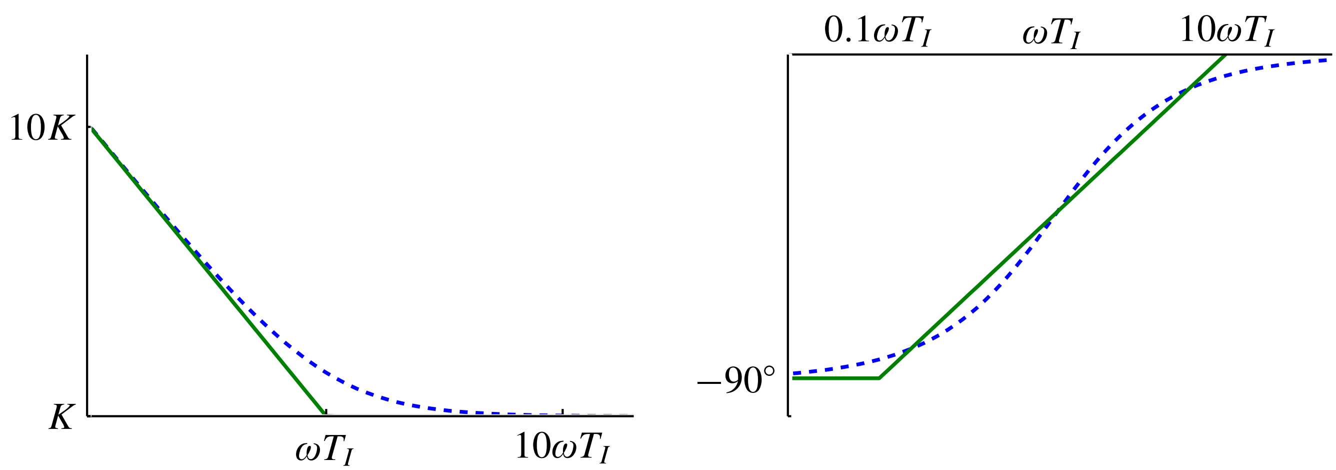


9.4: PI and lag compensation

PI compensation

- In many problems it is important to keep bandwidth low, and also reduce steady-state error.
- PI compensation used here.

$$D(s) = K\left[1 + \frac{1}{T_I s}\right].$$



- Infinite gain at zero frequency
  - Reduces steady-state error to step, ramp, etc.
  - But also has integrator “anti-windup” problems.
- Adds phase below breakpoint.
  - We want to keep breakpoint frequency very low to keep from destabilizing system.

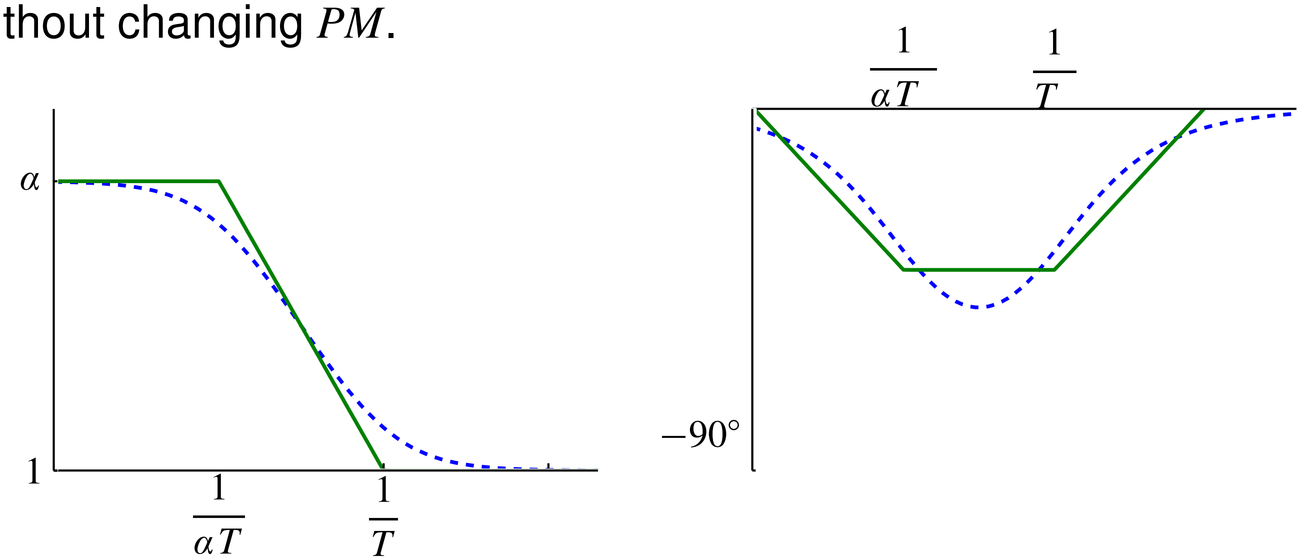
$$\frac{1}{T_I} \ll \omega_c.$$

Lag compensation

- Approximates PI, but without integrator overflow.

$$D(s) = \alpha\left[\frac{Ts+1}{\alpha Ts+1}\right] \qquad \alpha > 1.$$

- Primary objective of lag is to add $20\log_{10}\alpha$ dB gain to *low* frequencies without changing PM.



- Steady-state response improves with little effect on transient response.

Typical process

- Assumption is that we need to modify (increase) the dc-gain of the loop transfer function.
  - If we apply only a gain, then ω_c typically increases and the phase margin decreases. *NOT GOOD.*
  - Instead, use lag compensation to lower high-frequency gain.
- Assume plant has gain K (adjustable), or that we insert a gain K into the system. Adjust the open-loop gain K to meet phase margin requirements (plus about $5°$ slop factor) at crossover without additional compensation.
- Draw Bode diagram of system using the gain K from above. Evaluate low-frequency gain.
- Determine α to meet low-frequency gain requirements.

- Choose one corner frequency $\omega = \dfrac{1}{T}$ (the zero) to be about one decade below crossover frequency. (This way, the phase added by the lag compensator will minimally affect PM. The phase added at crossover will be about $5°$, hence our previous slop factor).
- The other corner frequency is at $\omega = \dfrac{1}{\alpha T}$.
- Iterate design to meet spec.

EXAMPLE:

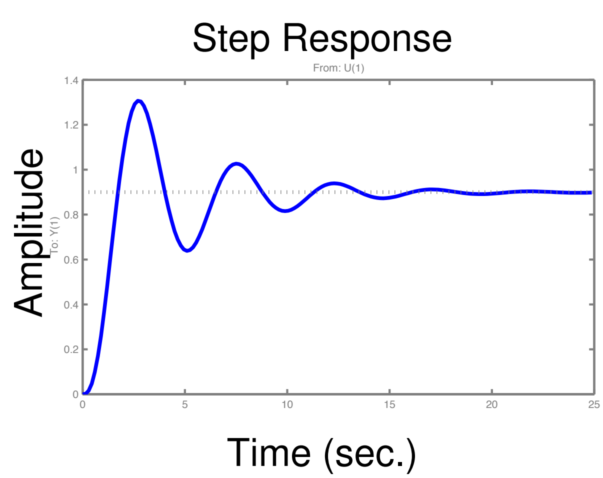
$$G(s) = \frac{K}{\left(\frac{1}{0.5}s+1\right)(s+1)\left(\frac{1}{2}s+1\right)}.$$

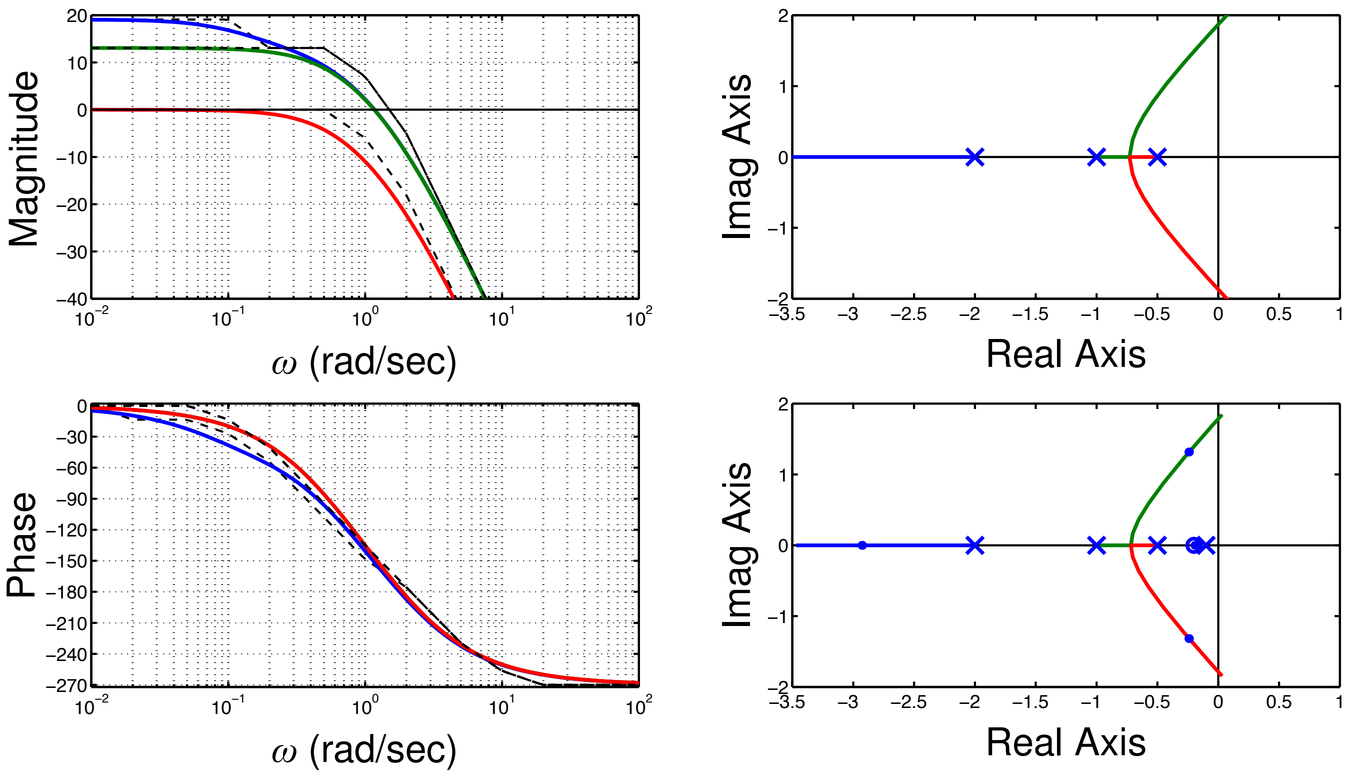
- Bode plot for plant is red line, below. Uncompensated root locus is top-right plot.
- We want to design compensator for $PM \geq 25°$, $K_p = 9$.

1. Set $K = 4.5$ for $PM = 30°$. This gives crossover at $\omega_c \approx 1.2$ rads/sec. (Green line on Bode plot.)
2. Low-frequency gain now about 13 dB or $10^{(13/20)} = 4.5$.
3. Should be raised by a factor of 2 to get $K_p = 9$. So, $\alpha = 2$.
4. Choose corner frequency at $\omega = 0.2$ rads/sec. $\dfrac{1}{T} = 0.2$, or $T = 5$.
5. We then know other corner frequency $\omega = \dfrac{1}{\alpha T} = \dfrac{1}{10}$.

- Altogether, we now have the compensator

$$D(s) = 2\left[\frac{5s+1}{10s+1}\right].$$

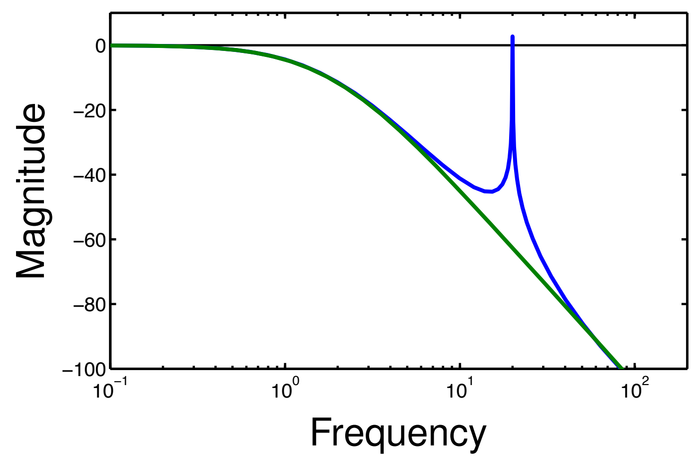




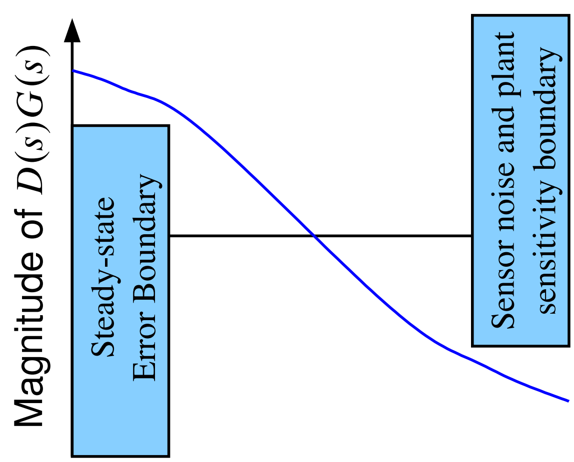
- Please understand that the “recipes” in these notes are not meant to be exhaustive!
- And, we often require both lead and lag to meet specs.
- Many other approaches to control design using frequency-response methods exist.
- In fact, once you are comfortable with Bode plots, you can add poles and zeros like LEGOs to build whatever frequency response you might like (within the limitations of Bode’s gain-phase theorem).
- Our next section gives some guidance on this.

9.5: Design based on sensitivity

- Goal: Develop conditions on Bode plot of loop transfer function $D(s)G(s)$ that will ensure good performance with respect to sensitivity, steady-state errors and sensor noise.
  - Steady-state performance = lower bound on system low-freq. gain.
  - Sensor noise = upper bound on high-frequency gain.



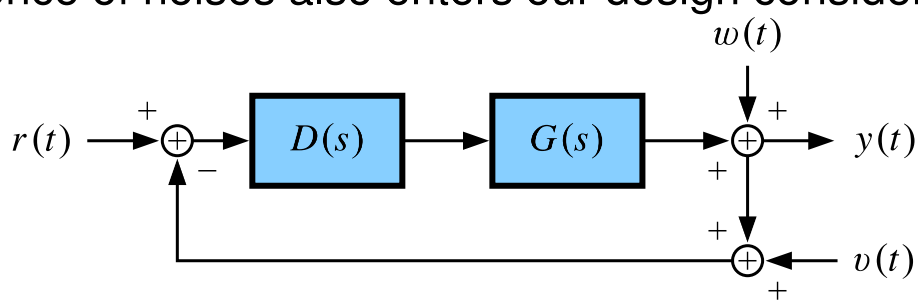
- Consider also the risk of an unmodeled system resonance:
- Magnitude may go over 1. Can cause instability. Must ensure that high-frequency gain is low so magnitude does not go over 1.



- Together, these help us design a desired loop frequency response:
  - Magnitude must be high at low frequencies, and
  - Low at high frequencies.

Sensitivity functions

- The presence of noises also enters our design considerations.



$$Y(s) = W(s) + G(s)D(s)[R(s) - V(s) - Y(s)]$$

$$[1 + G(s)D(s)]\,Y(s) = W(s) + G(s)D(s)[R(s) - V(s)]$$

$$\text{or, } Y(s) = \frac{1}{1 + G(s)D(s)}W(s) + \frac{G(s)D(s)}{1 + G(s)D(s)}[R(s) - V(s)].$$

- Tracking error $\stackrel{\triangle}{=} R(s) - Y(s)$

$$E(s) = R(s) - \frac{1}{1 + G(s)D(s)}W(s) - \frac{G(s)D(s)}{1 + G(s)D(s)}[R(s) - V(s)]$$

$$= \frac{1}{1 + G(s)D(s)}[R(s) - W(s)] + \frac{1}{1 + G(s)D(s)}G(s)D(s)V(s)$$

- Define the "sensitivity function" $S(s)$ to be

$$S(s) \stackrel{\triangle}{=} \frac{1}{1 + G(s)D(s)}$$

which is the transfer function from $r(t)$ to $e(t)$ and from $w(t)$ to $-e(t)$.

- The "complementary sensitivity function" $T(s) = 1 - S(s)$

$$1 - S(s) = \frac{G(s)D(s)}{1 + G(s)D(s)} = T(s)$$

which is the transfer function from $r(t)$ to $y(t)$.

- If $V = 0$, then

$$Y(s) = S(s)W(s) + T(s)R(s)$$

and

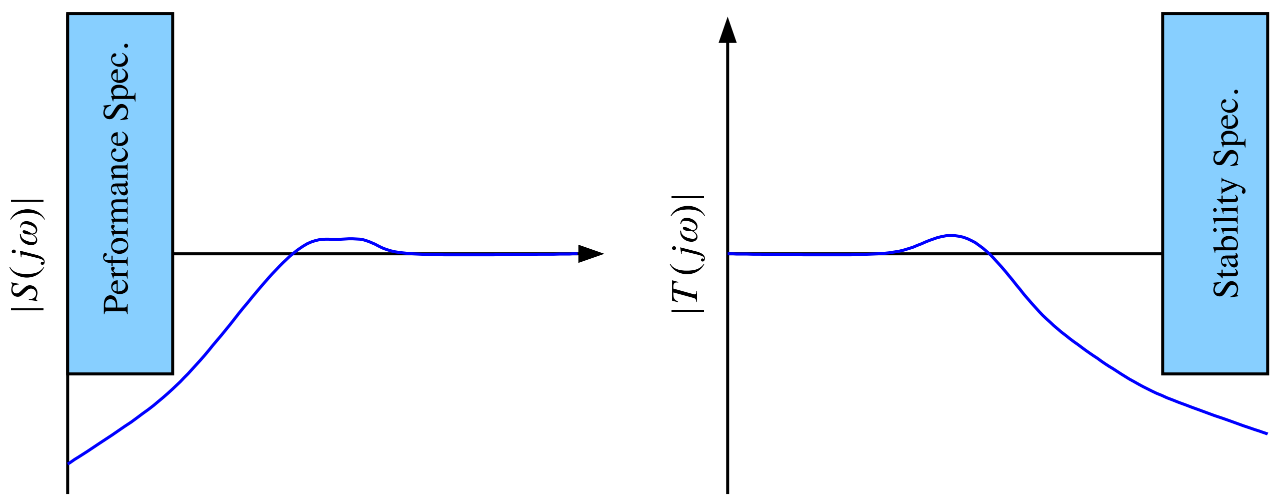
$$E(s) = S(s)[R(s) - W(s)].$$

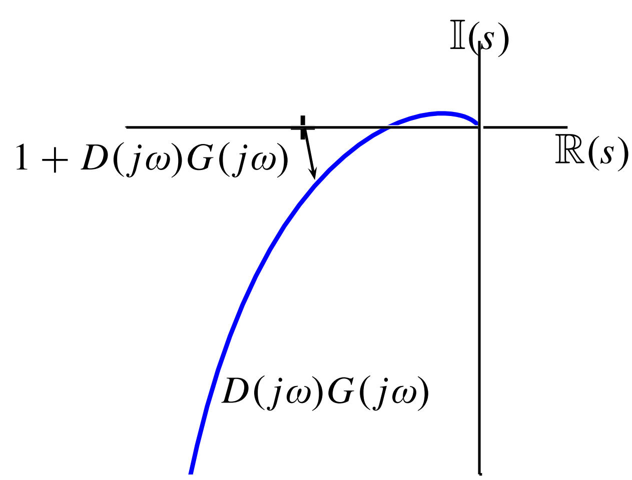
- The sensitivity function here is related to the one we saw several weeks ago

$$S_G^T = \frac{\partial T}{\partial G} \cdot \frac{G}{T} = \frac{1 + D(s)G(s) - D(s)G(s)}{[1 + D(s)G(s)]^2} \cdot \frac{G(s)[1 + D(s)G(s)]}{G(s)}$$

$$= \frac{1}{1 + D(s)G(s)} = S(s).$$

- So $S(s)$ is the sensitivity of the transfer function to plant perturbations.
- Recall that $T(s) + S(s) = 1$, regardless of $D(s)$ and $G(s)$.
- We would like $T(s) = 1$. Then $S(s) = 0$; Disturbance is cancelled, design is insensitive to plant perturbation, steady-state error $\approx$ 0.
- *BUT*, for physical plants, $G(s) \to 0$ for high frequencies (which forces $S(s) \to 1$).
- Furthermore, the transfer function between $V(s)$ and $E(s)$ is $T(s)$. To reduce high frequency noise effects, $T(s) \to 0$ as frequency increases, and $S(s) \approx 1$.
- Typical sensitivity and complementary sensitivity (closed-loop transfer) functions are:





- Another view of sensitivity:
- So, $1 + D(j\omega)G(j\omega)$ is the distance between the Nyquist curve to the -1 point, and $S(j\omega) = \dfrac{1}{1 + D(j\omega)G(j\omega)}$.

- A large value of $|S(j\omega)|$ indicates a nearly unstable Nyquist plot.
- The maximum value of $|S|$ is a more accurate measure of stability than PM or GM. So, we want $\max|S(j\omega)|$ small. How small?
- Note: $E(j\omega) = S(j\omega)R(j\omega)$

$$|E(j\omega)| = |S(j\omega)R(j\omega)| \le |S(j\omega)||R(j\omega)|$$

put a frequency-based error bound

$$|E(j\omega)| \le |S(j\omega)||R(j\omega)| \le e_b$$

- Let $W_1(\omega) = R(j\omega)/e_b$. Then,

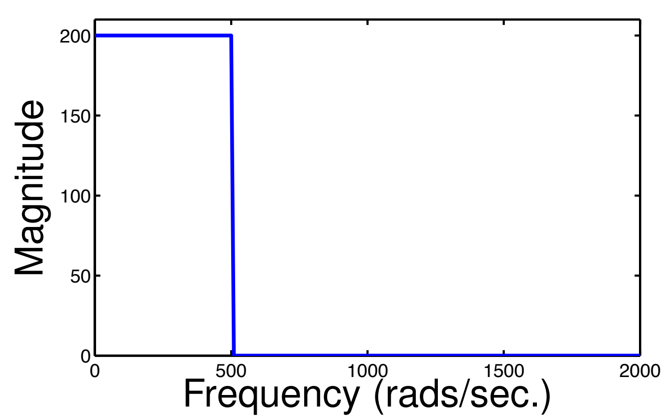
$$|S(j\omega)| \le \frac{1}{W_1(\omega)}.$$

EXAMPLE: A unity-feedback system is to have an error less than 0.005 for all unity-amplitude sinusoids below 500 rads/sec. Draw $|W_1(j\omega)|$ for this design.

- Spectrum of $R(j\omega)$ is unity for $0 \le \omega \le 500$.
- Since $e_b = 0.005$,

$$W_1(\omega) = \frac{1}{0.005} = 200$$

for this range.



- We can translate this requirement into a loop-gain requirement. When errors are small, loop gain is high, so $|S(j\omega)| \approx \dfrac{1}{|D(j\omega)G(j\omega)|}$ and

$$\frac{1}{|D(j\omega)G(j\omega)|} \le \frac{1}{W_1(\omega)}$$

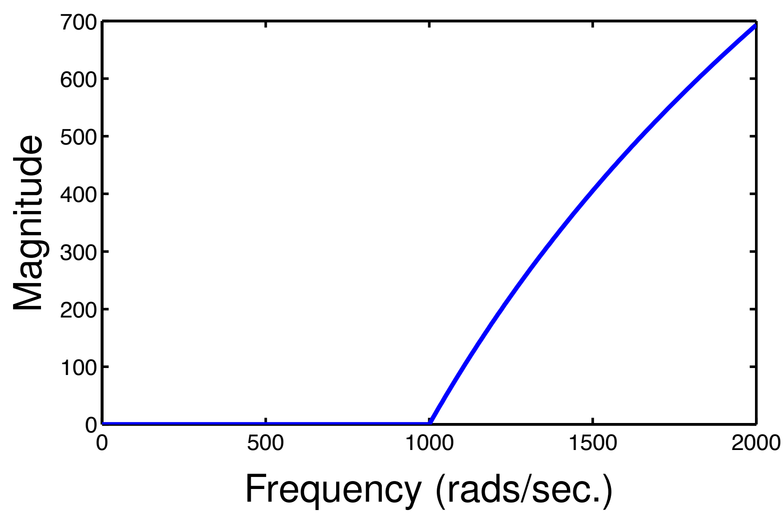
or, $|D(j\omega)G(j\omega)| \ge W_1(\omega)$

9.6: Robustness

- Typically there is some uncertainty in the plant transfer function. We want our design to be robustly stable, and to robustly give good performance (often called H_∞ design).
- Uncertainty often expressed as multiplicative

$$G(j\omega) = G_n(j\omega)[1 + W_2(j\omega)\Delta(j\omega)]$$

- $W_2(\omega)$ is a function of frequency expressing uncertainty, or size of possible error in transfer function as a function of frequency.
- $W_2(\omega)$ is almost always small at low frequencies.
- $W_2(\omega)$ increases at high frequencies as unmodeled structural flexibility is common.
- "Typical W_2"



- $\Delta(j\omega)$ expresses uncertainty in phase. The only restriction is

$$|\Delta(j\omega)| \le 1.$$

Design

- Assume design for nominal plant $G_n(s)$ is stable. Thus, $1 + D(j\omega)G_n(j\omega) \ne 0 \,\forall\, \omega$.

- For robust stability,

$$1 + D(j\omega)G(j\omega) \neq 0\,\forall\,\omega$$

$$1 + D(j\omega)G_n(j\omega)[1 + W_2(\omega)\Delta(j\omega)] \neq 0$$

$$\underbrace{\frac{1 + D(j\omega)G_n(j\omega)}{1 + D(j\omega)G_n(j\omega)}}_{\neq 0 \text{ by assumption}} + \frac{D(j\omega)G_n(j\omega)}{1 + D(j\omega)G_n(j\omega)}W_2(\omega)\Delta(j\omega) \neq 0$$

$$\text{recall, } T(j\omega) = \frac{D(j\omega)G_n(j\omega)}{1 + D(j\omega)G_n(j\omega)},$$

$$[1 + T(j\omega)W_2(\omega)\Delta(j\omega)] \neq 0$$

$$\text{so, } |T(j\omega)W_2(\omega)\Delta(j\omega)| < 1$$

$$\text{or, } |T(j\omega)|W_2(\omega) < 1.$$

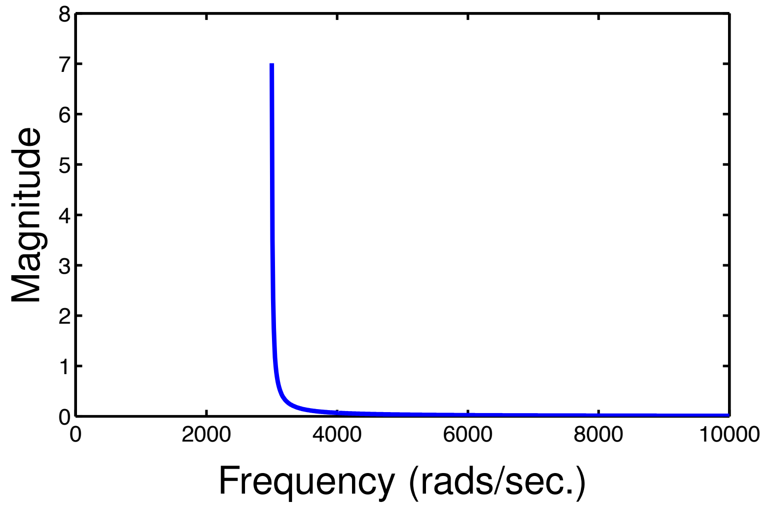
- For high freqencies $D(j\omega)G_n(j\omega)$ is typically small, so $T(j\omega) \approx D(j\omega)G_n(j\omega)$. Thus

$$|D(j\omega)G_n(j\omega)|W_2(\omega) < 1$$

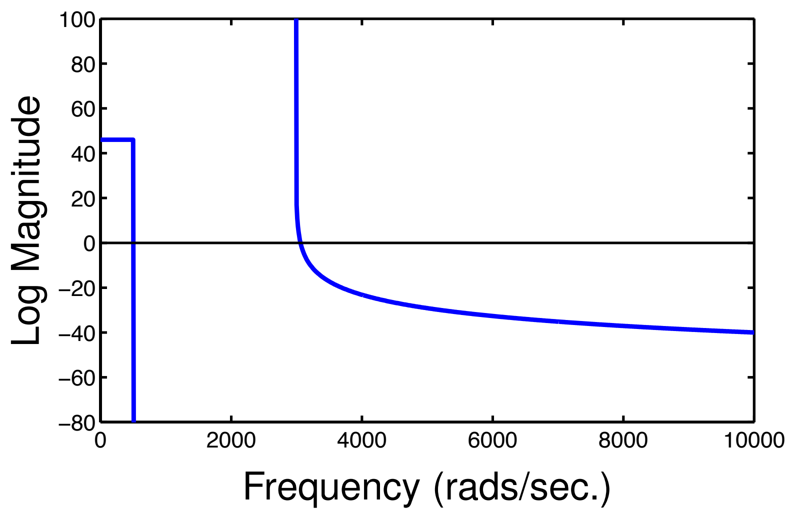
$$|D(j\omega)G_n(j\omega)| < \frac{1}{W_2(\omega)}.$$

EXAMPLE: The uncertainty in a plant model is described by a function $W_2(\omega)$ which is zero until $\omega = 3000$ rads/sec, and increases linearly from there to a value of 100 at $\omega = 10,000$ rads/sec. It remains constant at 100 for higher frequencies. Plot constraint on $D(j\omega)G_n(j\omega)$.

- Where $W_2(\omega) = 0$, there is no constraint on the magnitude of the loop gain. Above $\omega = 3000$, $1/W_2(\omega)$ is a hyperbola from ∞ to 0.01 at $10,000$.



- Combining $W_1(\omega)$ and $W_2(\omega)$ requirements,



- Limitation: Crossover needs to be with slope ≈ -1. So, cannot make constraints too strict, or design will be unstable.