SYSTEM MODELING IN THE TIME DOMAIN

2.1: What is a model? Why do we need one?

- We use the term model to refer to a set of mathematical equations used to represent a physical system, relating the system's output signal to its input signal.
- A model is required in order to:
  1. Understand system behavior (analysis).
  2. Design a controller (synthesis).

KEY POINT: It is necessary to understand how the system works naturally in order to know how to be able to change how it works using a feedback controller.

- Developing a reasonable mathematical model is the most important part of the entire analysis.
- It is also often the most difficult, amounting to $\approx 80\,\%$–$90\,\%$ of the effort in designing a controller.
- There are two basic approaches to modeling:
  1. Analytic system modeling—we focus on these methods.
  2. Empirical system identification. (In practice, there is always an empirical component to system modeling: cf. ECE5560.)
- It is important to realize that no model is ever exact! Inaccuracies arise because of

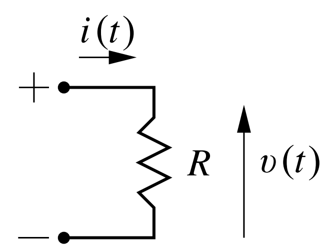
1. Unknown parameter values, or
2. Unmodeled dynamics (to make simpler model).

- There is always a tradeoff between simplicity and accuracy:
  - It's often possible to improve the accuracy of a mathematical model by increasing its complexity
  - Simplification often means ignoring some inherent physical properties
    - *e.g.*, ignore nonlinearities in linear, lumped-parameter models
  - In general, it's desirable to start with a simplified model to get a "general feel," increasing complexity only if the controlled system does not meet performance requirements.
    - Simplifications often ignore some high-frequency behaviors, which requires that the controllers must operate with slower transient-response requirements in order to be robust.

KEY POINT: "All models are wrong, but some are useful"

(George E. P. Box, statistician.)

EXAMPLE: Consider a 1 Ω, 2 W resistor.



- Ohm's Law (model) says: $v(t) = i(t) \cdot R$.
- Apply 1 V. What happens?
  - 1 A of current is predicted to flow.
  - Power dissipated $= V^2/R = 1$ W.
- Model should be accurate.

- Now apply 10 V.
  - 10 A of current is predicted to flow.
  - Power dissipated $= V^2/R =$ 100 W!

  ➤ Model will no longer be accurate.

  ➤ True behavior depends on input signal level—nonlinear.

  ➤ Model is accurate only in certain range of input-signal values.

- Ohm's law is definitely useful, but it is "wrong" in the sense that it applies only under certain conditions, and even then is an averaged version of what is truly happening at the microscopic scale.

LTI systems

- This example shows that it is important to know the properties of your model, as well as the model itself.
- I claimed that the resistor exhibited "nonlinear" behavior, in some sense.
- In the next sections, we look at two critical properties of systems:
  - Whether the system is "linear" (or not),
  - Whether the system is "time invariant" (or not).
- This course teaches methods to control linear-time-invariant (LTI) systems.
- Again, none exist! But, many are "close enough" for the techniques developed here to work very well.
- Also, we'll look at ways of linearizing equations in Topic 2.7.

2.2: System properties of linearity and time invariance

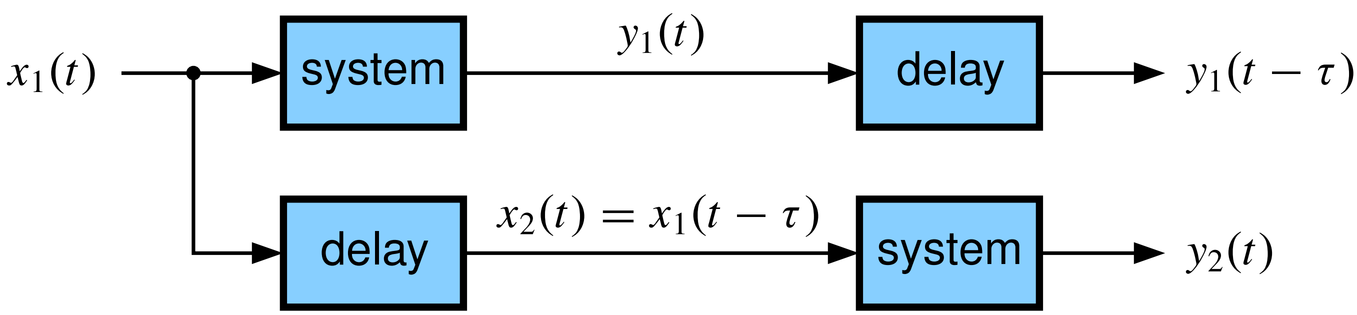
TIME INVARIANT: The first system property that we look at is that of time invariance.

- A system is either time-varying or time-invariant, not both.
- A time-invariant system does not change its fundamental behavior over different periods of time. Its parameter values are constant.
- A time invariant system satisfies the property (for any $x(t)$, τ)

$$x(t-\tau) \mapsto y(t-\tau)$$

when $x(t) \mapsto y(t)$.

- We can test a system for this property using ideas from the figure.



- A time-invariant system will have $y_2(t) = y_1(t-\tau)$ for all $x_1(t)$ and τ.

TEST: To test for time-invariance, we must

- Input $x_1(t)$ to the system and measure the output $y_1(t)$.
- Input $x_2(t) = x_1(t-\tau)$ to the system and measure $y_2(t)$.
- If $y_2(t) = y_1(t-\tau)$ for all possible delays τ and signals $x_1(t)$, then the system is time-invariant.

EXAMPLE: For example, consider a square-law system $y(t) = (x(t))^2$.

- Input $x_1(t)$ to the system and measure $y_1(t)$: $y_1(t) = (x_1(t))^2$.
- Input $x_2(t)$ to the system and measure $y_2(t)$: $y_2(t) = (x_2(t))^2$.
- But, $x_2(t) = x_1(t-\tau)$, so $y_2(t) = (x_1(t-\tau))^2 = y_1(t-\tau)$.
- Since this relationship holds for all τ and all $x_1(t)$, the square-law system *is* time-invariant.

EXAMPLE: Let us examine a "delay operator." (The delay operator is a fundamental building-block of digital-signal-processing systems and digital control systems).

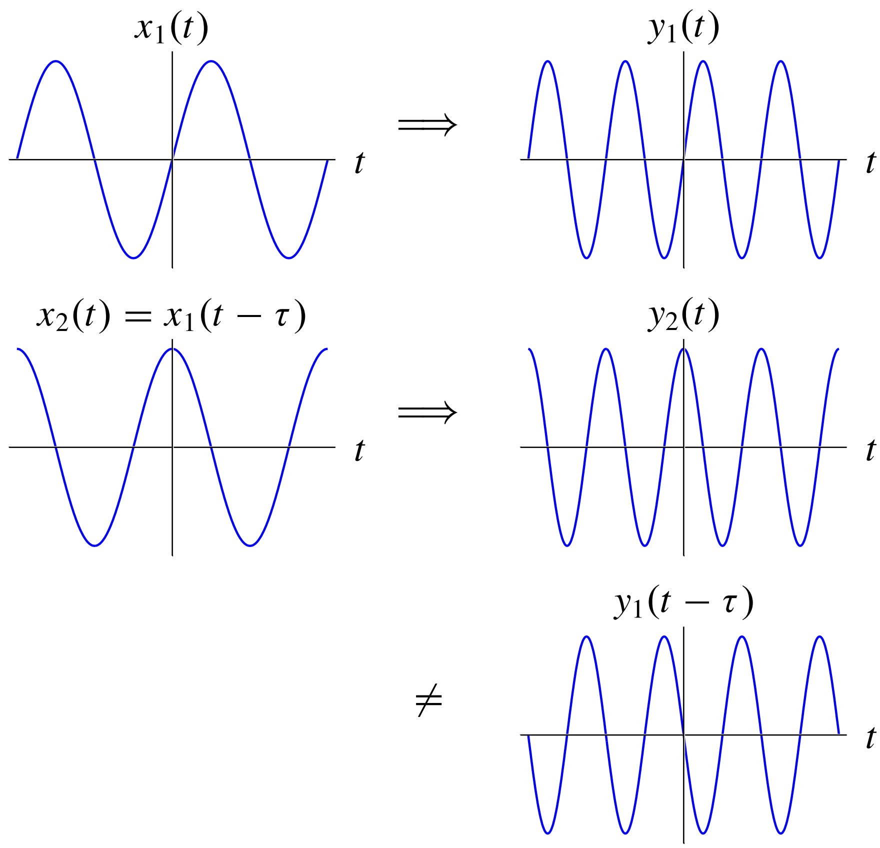
- The output of a delay is equal to the input, but shifted a constant amount λ seconds: $y(t) = x(t-\lambda), \quad \lambda \geq 0$.
- Input $x_1(t)$ to the system and measure $y_1(t)$: $y_1(t) = x_1(t-\lambda)$.
- Input $x_2(t)$ to the system and measure $y_2(t)$: $y_2(t) = x_2(t-\lambda)$.
- But, $x_2(t) = x_1(t-\tau)$, so $y_2(t) = x_1(t-\tau-\lambda) = y_1(t-\tau)$.
- Since this relationship holds for all τ and all $x_1(t)$, the delay operator *is* time-invariant.

EXAMPLE: Let us examine a "time compressor" whose output is equal to its input, but "squashed" in time:$y(t) = x(kt)$.

- Input $x_1(t)$ to the system and measure $y_1(t)$: $y_1(t) = x_1(kt)$.
- Input $x_2(t)$ to the system and measure $y_2(t)$: $y_2(t) = x_2(kt)$.
- But, $x_2(t) = x_1(t-\tau)$, so

$$y_2(t) = x_1(kt-\tau) \neq x_1(kt-k\tau) = y_1(t-\tau).$$

- Therefore, the compressor is time-varying.



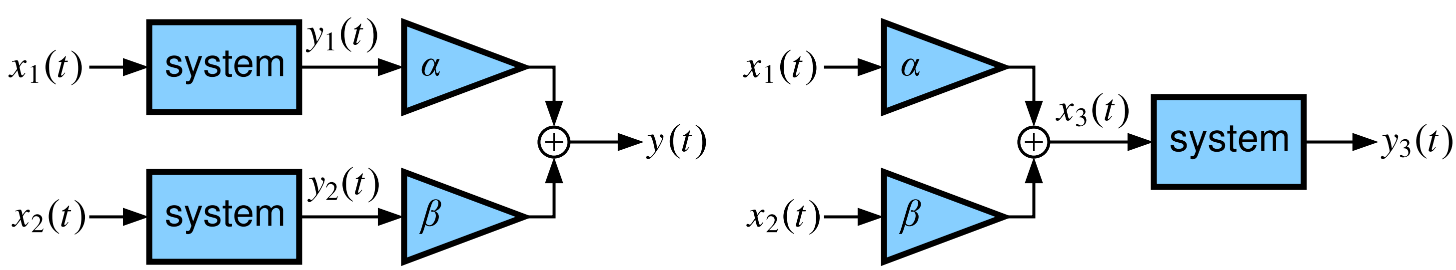
LINEAR: The second property that we look at is linearity.

- For linear systems, if $x_1(t) \mapsto y_1(t)$ and $x_2(t) \mapsto y_2(t)$, then

$$x_3(t) = \alpha x_1(t) + \beta x_2(t) \mapsto y_3(t) = \alpha y_1(t) + \beta y_2(t),$$

  for any such $x_1(t)$, $x_2(t)$, α, β.

- We can test a system for this property using ideas from the figure.



TEST: To test for linearity, we must

- Input $x_1(t)$ to the system and measure the output $y_1(t)$.
- Input $x_2(t)$ to the system and measure $y_2(t)$.
- Input $x_3(t) = \alpha x_1(t) + \beta x_2(t)$ to the system and measure $y_3(t)$.
- If $y_3(t) = \alpha y_1(t) + \beta y_2(t)$ for all possible α and β values, and $x_1(t)$ and $x_2(t)$, then the system is linear.

EXAMPLE: Is the following system, described by the differential equation $\dot{y}(t) + ty(t) = x(t)$, linear?[1]

- Input $x_1(t)$ and output is $y_1(t)$: $\dot{y}_1(t) + ty_1(t) = x_1(t)$.
- Input $x_2(t)$ and output is $y_2(t)$: $\dot{y}_2(t) + ty_2(t) = x_2(t)$.
- Input $x_3(t) = \alpha x_1(t) + \beta x_2(t)$ and measure $y_3(t)$.

$$\dot{y}_3(t) + ty_3(t) = x_3(t);$$

but, $x_3(t) = \alpha x_1(t) + \beta x_2(t)$, so

$$\begin{aligned}
\dot{y}_3(t) + ty_3(t) &= \alpha x_1(t) + \beta x_2(t) \\
&= \alpha\left(\dot{y}_1(t) + ty_1(t)\right) + \beta\left(\dot{y}_2(t) + ty_2(t)\right) \\
&= \frac{\mathrm{d}}{\mathrm{d}t}\left(\alpha y_1(t) + \beta y_2(t)\right) + t\left(\alpha y_1(t) + \beta y_2(t)\right).
\end{aligned}$$

By examining both sides of this equation, we realize that $y_3(t) = \alpha y_1(t) + \beta y_2(t)$. Therefore, the system is linear.

[1] Note, the "dot" decoration on a variable indicates a time derivative. That is, $\dot{y}(t) = \mathrm{d}y(t)/\mathrm{d}(t)$, and so forth.

EXAMPLE: Trying this on the square-law system,

- Input $x_1(t)$ and output is $y_1(t)$: $y_1(t) = (x_1(t))^2$.
- Input $x_2(t)$ and output is $y_2(t)$: $y_2(t) = (x_2(t))^2$.
- Input $x_3(t) = \alpha x_1(t) + \beta x_2(t)$ and measure $y_3(t)$.

$$\begin{aligned} y_3(t) &= (x_3(t))^2 \\ &= (\alpha x_1(t) + \beta x_2(t))^2 \\ &= \alpha^2 (x_1(t))^2 + 2\alpha\beta x_1(t) x_2(t) + \beta^2 (x_2(t))^2 \\ &\neq \alpha\,(x_1(t))^2 + \beta\,(x_2(t))^2 . \end{aligned}$$

- So, the square-law system is *not* linear (it is a *nonlinear system*).

KEY POINT: If a system is LTI, then it has an impulse response*.* This entirely characterizes the system's dynamics. The Laplace transform of the impulse response is the transfer function*.* Working with the transfer function eliminates the need to mess around with trying to solve complicated differential equations.

2.3: Dynamics of mechanical systems (translational)

- We now begin to review some basic physics as a refresher to developing models of dynamic systems.
- We'll focus on mechanical and electrical systems, but will mention some others too.

Translational motion

- Newton's second law, applied to translational motion, states:

$$\sum F = ma.$$

- That is, the vector sum of forces = mass of object times inertial acceleration.
- "Free-body diagrams" are a tool to apply this law.

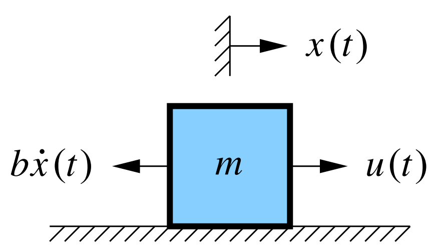
EXAMPLE: Cruise control model.

- Write the equations of motion for the speed and forward motion of a car assuming that the engine imparts a forward force of $u(t)$.
  1. Assume rotational inertia of wheels is negligible.
  2. Assume that friction is proportional to car's speed (viscous friction).

$$\sum F = ma$$

$$u(t) - b\dot{x}(t) = m\ddot{x}(t)$$

or, $$\ddot{x}(t) + \frac{b}{m}\dot{x}(t) = \frac{u(t)}{m}$$



- If the variable of interest is speed ($v(t) = \dot{x}(t)$), not position,

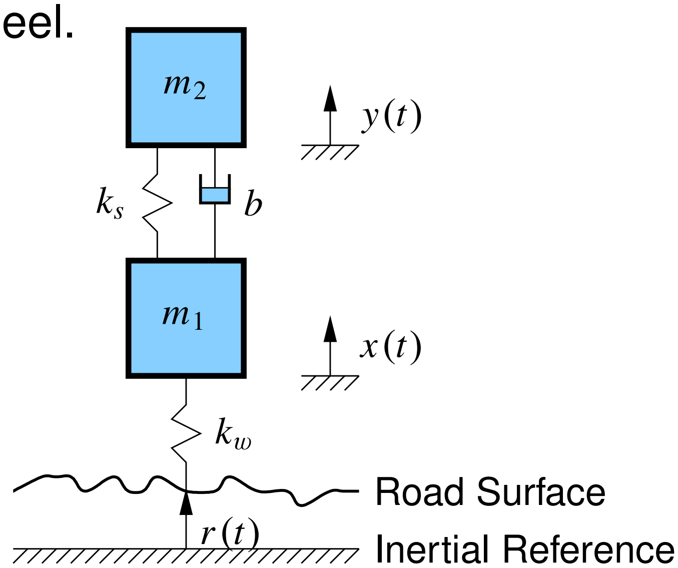
$$\dot{v}(t) + \frac{b}{m}v(t) = \frac{u(t)}{m}$$

- Notice that the differential equation has "output variables" on the left of "$=$", and "input variables" on the right.

IMPORTANT POINT: All of our models of dynamical systems will be differential equations involving the input (*e.g.*, $u(t)$) and its derivatives and the output (*e.g.*, $y(t)$) and its derivatives. No other signals (intermediate variables) are allowed in our solutions.

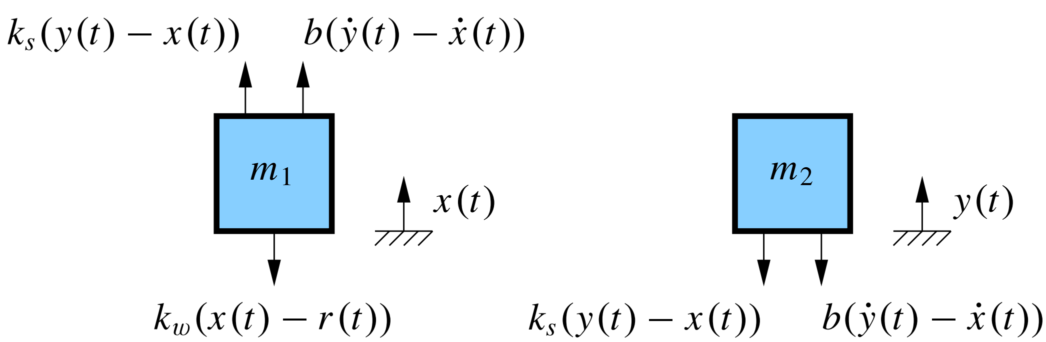
EXAMPLE: Car suspension.

Each wheel in a car suspension system has a tire, shock absorber and spring. Write the one-dimensional (vertical) equations of motion for the car body and wheel.



- "Quarter-car model"

- Free-body diagram:



- The force from the spring is proportional to its stretch. The force from the shock absorber is proportional to the rate-of-change of its stretch.

$$\sum F = ma$$

$$b\left(\dot{y}(t) - \dot{x}(t)\right) + k_s\left(y(t) - x(t)\right) - k_w\left(x(t) - r(t)\right) = m_1\ddot{x}(t)$$

$$-k_s\left(y(t) - x(t)\right) - b\left(\dot{y}(t) - \dot{x}(t)\right) = m_2\ddot{y}(t)$$

- Re-arrange:

$$\ddot{x}(t) + \frac{b}{m_1}(\dot{x}(t) - \dot{y}(t)) + \frac{k_s}{m_1}(x(t) - y(t)) + \frac{k_w}{m_1}x(t) = \frac{k_w}{m_1}r(t)$$

$$\ddot{y}(t) + \frac{b}{m_2}(\dot{y}(t) - \dot{x}(t)) + \frac{k_s}{m_2}(y(t) - x(t)) = 0$$

<u>*Implementation in Simulink*</u>

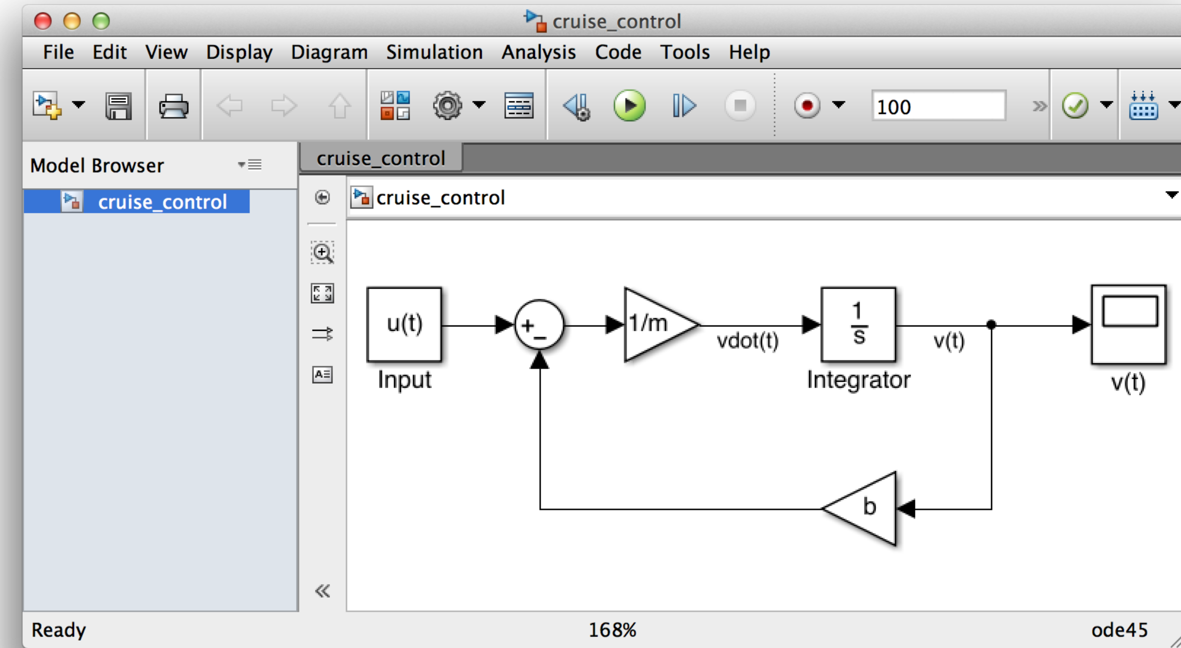
- Simulink is a component of MATLAB that is very useful for simulating dynamic systems using a block-diagram approach.
- Consider the cruise-control model, where we wish to control vehicle velocity:

$$\dot{v}(t) + \frac{b}{m}v(t) = \frac{u(t)}{m}.$$

- To implement this model in Simulink, we re-write the equation to have only the highest derivative term on the left-hand-side

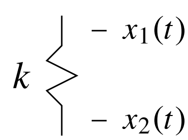
$$\dot{v}(t) = \frac{u(t)}{m} - \frac{b}{m}v(t).$$

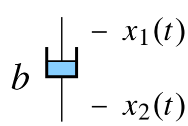
- We wire up a diagram like:



- If the appropriate parameters are entered in the MATLAB workspace for m and b, then this will simulate the car's dynamics.

Important components for mechanical-translational systems:

1. Mass $\boxed{m}$
2. Spring 

$f(t) = k(x_1(t) - x_2(t))$
3. Damper 

$f(t) = b(\dot{x}_1(t) - \dot{x}_2(t))$

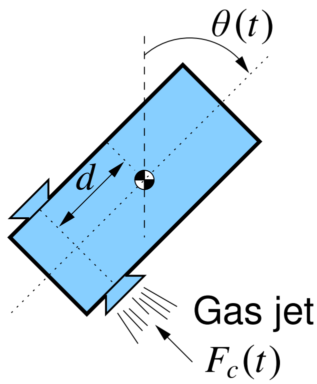
2.4: Dynamics of mechanical systems (rotational)

- Newton's second law, applied to rotational motion, states:

$$\sum M = J\alpha \quad \text{or} \quad I\alpha.$$

- That is, the vector sum of moments = moment of inertia times angular acceleration. ("moment"="torque").

EXAMPLE: Satellites require attitude control so that sensors, antennas, etc., are properly pointed. Let's consider one axis of rotation.

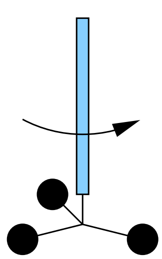


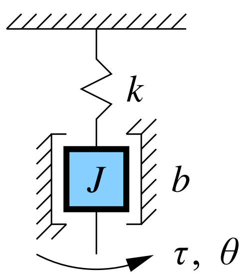
moment $= F_c(t) \cdot d$, so,

$$F_c(t)d = J\ddot{\theta}(t)$$
$$\ddot{\theta}(t) = \frac{F_c(t)d}{J}$$

Note: Output of system $\theta(t)$ integrates torques twice—"double-integrator plant."

EXAMPLE: A torsional pendulum is used, for example, in clocks enclosed in glass domes. A similar device is the read-write head on a hard-disk drive.





k: "Springiness" of suspension wire.

b: Viscous friction.

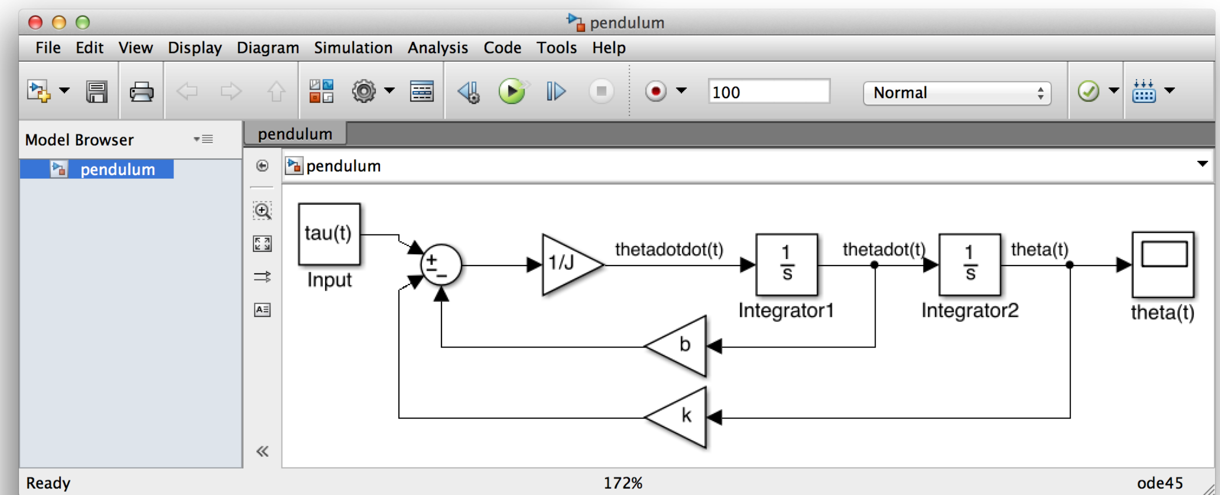
$$\sum M = J\ddot{\theta}(t)$$
$$J\ddot{\theta}(t) = \tau(t) - b\dot{\theta}(t) - k\theta(t)$$
$$\ddot{\theta}(t) + \frac{b}{J}\dot{\theta}(t) + \frac{k}{J}\theta(t) = \frac{\tau(t)}{J}$$

Implementation in Simulink

- The same basic principle applies to implementing this system in Simulink as well, except now we have a second-derivative term.
- No problem! Again, we re-write the equation to have only the highest derivative term on the left-hand-side

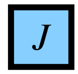
$$\ddot{\theta}(t) = \frac{\tau(t)}{J} - \frac{b}{J}\dot{\theta}(t) - \frac{k}{J}\theta(t).$$

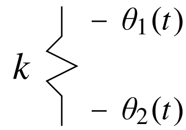
- We wire up a diagram like:



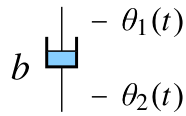
- If the appropriate parameters are entered in the MATLAB workspace for J, b, and k, then this will simulate the pendulum's dynamics.

Important components for mechanical rotational systems:

1. Inertia 

2. Spring 

$\tau(t) = k(\theta_1(t) - \theta_2(t))$

3. Damper 

$\tau(t) = b(\dot{\theta}_1(t) - \dot{\theta}_2(t))$

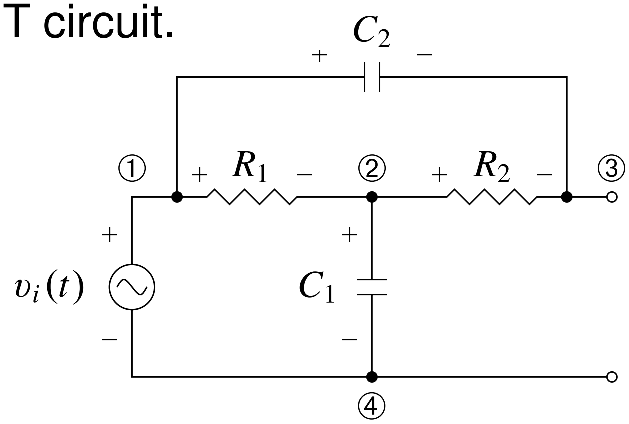
Summary of Developing Models for Rigid Bodies:

1. Assign variables such as $x(t)$ and $\theta(t)$ that are both necessary and sufficient to describe and arbitrary position of the object.
2. Draw a free-body diagram of each component, and indicate all forces acting on each body and the accelerations of the center of mass with respect to an inertial reference.
3. Apply Newton's laws: $\sum F = ma, \quad \sum M = J\alpha.$
4. Combine the equations to eliminate internal forces.
5. The final form must be in terms of ONLY the input to the system and its derivatives, and the output of the system and its derivatives.

2.5: Dynamics of electrical circuits

- Kirchhoff's Law's:
  - Current Law (KCL): The algebraic sum of currents entering a node equals the algebraic sum of currents leaving the node.
  - Voltage Law (KVL): The algebraic sum of all voltages taken around a closed path in a circuit is zero.
- "Node analysis" is a tool to apply these laws. (*i.e.*, select one node as reference (*e.g.*, ground) and assume all other voltages are unknown. Write equations for the unknowns using KCL. KVL must be used for voltage sources.)

EXAMPLE: Bridged-T circuit.



- Select reference = ④.
  - KVL at ①: $v_①(t) = v_i(t)$.
  - KCL at ②: $\dfrac{v_①(t) - v_②(t)}{R_1} - \dfrac{v_②(t) - v_③(t)}{R_2} - C_1\dot{v}_②(t) = 0.$
  - KCL at ③: $\dfrac{v_②(t) - v_③(t)}{R_2} + C_2(\dot{v}_①(t) - \dot{v}_③(t)) = 0.$

$$v_②(t) - v_③(t) + R_2C_2(\dot{v}_①(t) - \dot{v}_③(t)) = 0$$

$$v_③(t) + R_2C_2(\dot{v}_③(t) - \dot{v}_①(t)) = v_②(t)$$

$$\frac{v_{①}(t) - [v_{③}(t) + R_2C_2(\dot{v}_{③}(t) - \dot{v}_{①}(t))]}{R_1} -$$

$$\frac{[v_{③}(t) + R_2C_2(\dot{v}_{③}(t) - \dot{v}_{①}(t))] - v_{③}(t)}{R_2} -$$

$$C_1\,[\dot{v}_{③}(t) + R_2C_2(\ddot{v}_{③}(t) - \ddot{v}_{①}(t))] = 0$$

$$R_2\,(v_{①}(t) - [v_{③}(t) + R_2C_2(\dot{v}_{③}(t) - \dot{v}_{①}(t))]) -$$

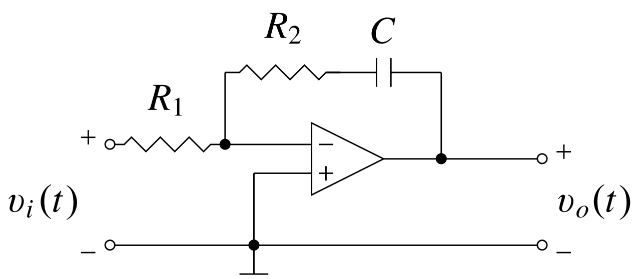
$$R_1\,([v_{③}(t) + R_2C_2(\dot{v}_{③}(t) - \dot{v}_{①}(t))] - v_{③}(t)) -$$

$$R_1R_2C_1\,[\dot{v}_{③}(t) + R_2C_2(\ddot{v}_{③}(t) - \ddot{v}_{①}(t))] = 0.$$

$$\left(R_1R_2^2C_1C_2\right)\ddot{v}_{③}(t) + \left(R_2^2C_2 + R_1R_2C_2 + R_1R_2C_1\right)\dot{v}_{③}(t) + (R_2)\,v_{③}(t)$$

$$= (R_1R_2^2C_1C_2)\ddot{v}_{①}(t) + (R_2^2C_2 + R_1R_2C_2)\dot{v}_{①}(t) + (R_2)v_{①}(t)$$

EXAMPLE: Op-amp circuit.



$$i(t) = \frac{v_i(t)}{R_1}$$

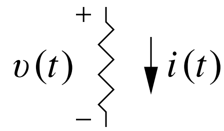
$$v_o(t) = -R_2 i(t) - v_c(t).$$

$$\begin{aligned}\frac{\mathrm{d}v_o(t)}{\mathrm{d}t} &= -R_2\frac{\mathrm{d}i(t)}{\mathrm{d}t} - \frac{\mathrm{d}v_c(t)}{\mathrm{d}t}\\ &= -\frac{R_2}{R_1}\frac{\mathrm{d}v_i(t)}{\mathrm{d}t} - \frac{i(t)}{C}\\ &= -\frac{R_2}{R_1}\frac{\mathrm{d}v_i(t)}{\mathrm{d}t} - \frac{1}{R_1C}v_i(t)\\ R_1C\dot{v}_o(t) &= -R_2C\dot{v}_i(t) - v_i(t)\end{aligned}$$

(as $C \to \infty$, we get an inverting amplifier.)

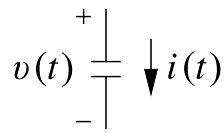
Important components for electrical systems:

1. Resistor



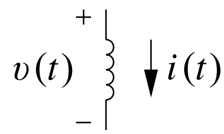
$v(t) = Ri(t)$

2. Capacitor



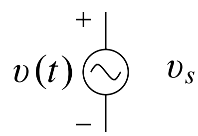
$i(t) = C\dfrac{\mathrm{d}\,v(t)}{\mathrm{d}t}$

3. Inductor



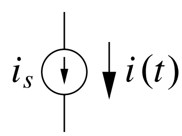
$v(t) = L\dfrac{\mathrm{d}\,i(t)}{\mathrm{d}t}$

4. Voltage source



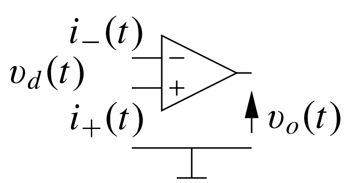
$v(t) = v_s$

5. Current source



$i(t) = i_s$

6. Operational Amplifier



$v_d(t) = 0$

$i_-(t) = i_+(t) = 0$

$v_o(t) = A_o(v_+(t) - v_-(t))$

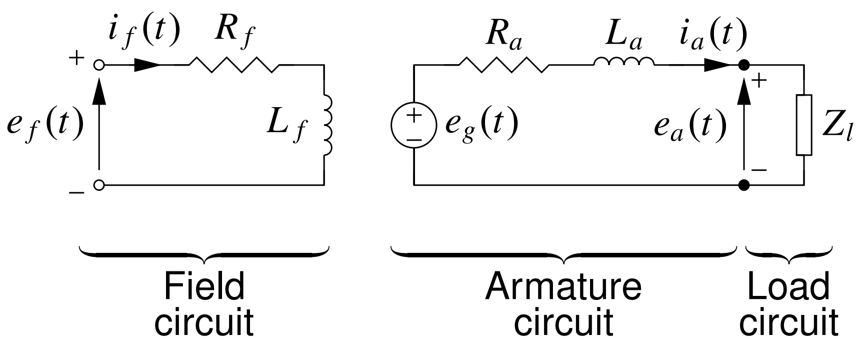
as $A_o \to \infty$

2.6: Dynamics of electro-mechanical systems (etc.)

- These are systems that convert energy from electrical to mechanical, or vice versa.

EXAMPLE: DC generator.

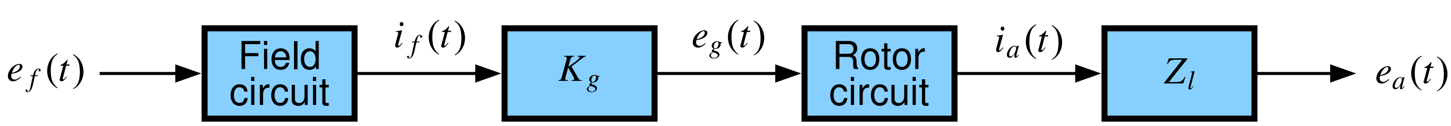
- Assume generator is driven at constant speed.
- Generator has field windings (input), and rotor/armature windings (output).



- $e_f(t) = R_f i_f(t) + L_f \dfrac{\mathrm{d}i_f(t)}{\mathrm{d}t}$ — $e_f(t)$ is input, $i_f(t)$ is output.
- $e_g(t) = K\phi \dfrac{\mathrm{d}\theta(t)}{\mathrm{d}t}$ — K depends on generator structure.

  $= K_g i_f(t).$ — $\mathrm{d}\theta(t)/\mathrm{d}t$ = angular velocity = cst.

  ϕ = flux, proportional to $i_f(t)$.
- $e_g(t) = R_a i_a(t) + L_a \dfrac{\mathrm{d}i_a(t)}{\mathrm{d}t} + Z_l i_a(t).$ — $e_g(t)$ is input, $i_a(t)$ is output.
- $e_a(t) = Z_l i_a(t).$ — $i_a(t)$ is input, $e_a(t)$ is output.

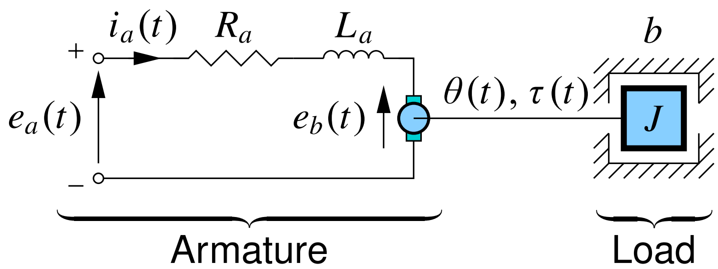


This is a preview of a block diagram used to simplify our understanding of the system dynamics.

EXAMPLE: DC motor (servo-motor).

- Directly generates rotational motion.

- Indirectly generates translational motion.



- Mechanical resistance of load is translated into an electrical "resistance" called the back e.m.f.
- $e_b(t) = K_e \dfrac{\mathrm{d}\theta(t)}{\mathrm{d}t}$ $\qquad K_e = K\phi$, as with generator.
- $e_a(t) = R_a i_a(t) + L_a \dfrac{\mathrm{d}i_a(t)}{\mathrm{d}t} + e_b(t)$
- $\tau(t) = K_\tau i_a(t)$ $\qquad K_\tau = K_1\phi$
- Combining these equations of motion, recall Newton:

$$\sum M = J\alpha$$

$$J\ddot{\theta}(t) = \tau(t) - b\dot{\theta}(t)$$

$$= K_\tau i_a(t) - b\dot{\theta}(t)$$

- Assume (FOR NOW ONLY) electrical response is faster than mechanical. $L_a \approx 0$.

$$J\ddot{\theta}(t) = K_\tau\left(\frac{e_a(t) - e_b(t)}{R_a}\right) - b\dot{\theta}(t)$$

$$J\ddot{\theta}(t) + \underbrace{\left(b + \frac{K_\tau K_e}{R_a}\right)}_{\text{back e.m.f. indistinguishable from friction!}}\dot{\theta}(t) = \frac{K_\tau}{R_a}e_a(t)$$

Dynamics of heat flow/ dynamics of fluid flow

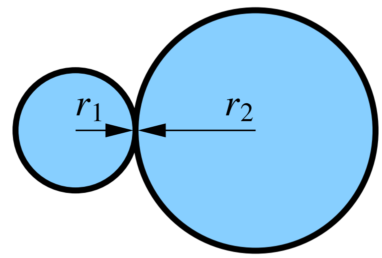
- These two subjects will not be covered here. Refer to texts on thermodynamics or fluid-dynamics.

Transformers and gears

- Ideally, both of these devices simply scale their input value.

$$\text{Transformer}: \quad \frac{N_1}{N_2} = \frac{e_1}{e_2} = \frac{i_2}{i_1}$$

$$\text{Gears}: \quad \frac{r_1}{r_2} = \frac{\theta_2}{\theta_1} = \frac{\tau_1}{\tau_2}$$



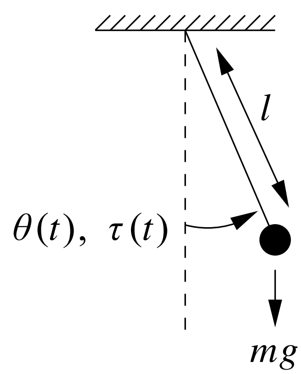
System identification (SYS ID)

- When we generate models of system dynamics, we are performing “system identifications.”
- When we use known properties from physics and knowledge of the system’s structure (as we have done here) we are performing “white box system ID.”
- If the system is very complex, or if the physics are not well understood, we need to use input/output data to generate a system model: “black-box system ID.”
- A topic for the whole course! (ECE5560)

2.7: Linearization and analogous systems

- We will study how to control linear systems.
- Linear systems are rare.
- We can "linearize" a non-linear system—the controller designed for linearized model will work on the true nonlinear system (but not as well as a controller designed directly for the non-linear system.)

EXAMPLE: *NONLINEAR* rotational pendulum.



Moment of inertia: $J = ml^2$.

$$\sum M = J\ddot{\theta}(t)$$

$$J\ddot{\theta}(t) = \tau(t) - mgl\sin(\theta(t))$$

$$\ddot{\theta}(t) + \frac{g}{l}\underbrace{\sin(\theta(t))}_{\text{Nonlinear!}} = \frac{\tau(t)}{ml^2}$$

- *If motion is "small,"* $\sin(\theta(t)) \approx \theta(t)$.

$$\ddot{\theta}(t) + \frac{g}{l}\theta(t) = \frac{\tau(t)}{ml^2} \qquad \text{Linear.}$$

This is a preview of linearization.

KEY POINT: We can convert any differential equation into a first-order vector differential equation:

$$\dot{\vec{x}} = f(\vec{x}, u); \qquad \vec{x} = \text{vector},\ u = \text{input.}$$

Iff the system is linear, this will be of the form:

$$\dot{\vec{x}} = A\vec{x} + Bu; \qquad A \text{ and } B \text{ are constant matrices.}$$

EXAMPLE: Torsional pendulum revisited (pg. 2–13)

$$\ddot{\theta}(t) + \frac{b}{J}\dot{\theta}(t) + \frac{k}{J}\theta(t) = \frac{\tau(t)}{J}$$

$$\text{let } \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} \theta(t) \\ \dot\theta(t) \end{bmatrix}$$

$$\dot x_2(t) + \frac{b}{J}x_2(t) + \frac{k}{J}x_1(t) = \frac{\tau(t)}{J}$$

$$\begin{bmatrix} \dot x_1(t) \\ \dot x_2(t) \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & 1 \\ -k/J & -b/J \end{bmatrix}}_{A} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \underbrace{\begin{bmatrix} 0 \\ 1/J \end{bmatrix}}_{B} \tau(t).$$

- So, our model of the torsional pendulum is linear.

EXAMPLE: Rotational pendulum revisited (pg. 2–22)

$$\ddot\theta(t) + \frac{g}{l}\sin(\theta(t)) = \frac{\tau(t)}{ml^2}$$

$$\text{let } \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} \theta(t) \\ \dot\theta(t) \end{bmatrix}$$

$$\begin{bmatrix} \dot x_1(t) \\ \dot x_2(t) \end{bmatrix} = \begin{bmatrix} x_2(t) \\ -\dfrac{g}{l}\sin(x_1(t)) \end{bmatrix} + \begin{bmatrix} 0 \\ \dfrac{1}{ml^2} \end{bmatrix} \tau(t).$$

- Not linear because we cannot make a constant A matrix.

Small-signal linearization

- Uses a Taylor-series expansion of the differential equation around some operating condition. (Equilibrium value where $\dot x_0 = 0 = f(x_0, u_0)$).

$$\text{let } x = x_0 + \delta x \qquad x_0 = \text{operating state}$$

$$u = u_0 + \delta u \qquad u_0 = \text{nominal control value.}$$

$$\dot{x} = f(x, u).$$

- Taylor-series expansion:

$$\dot{x} = \dot{x}_0 + \delta\dot{x} \approx f(x_0, u_0) + A\delta x + B\delta u \qquad \text{plus higher-order terms}$$

- Subtract out equilibrium (nominal) solution;

$$\delta\dot{x} = A\delta x + B\delta u,$$

  which is linear. This is exactly how we linearized the rotational pendulum before, with $\tau_0 = 0;\ \theta_0 = 0.$

$$\begin{aligned}\sin(\theta) &= \theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \cdots \\ &\approx \theta.\end{aligned}$$

Feedback linearization (computed torque)

- For rotational pendulum, $ml^2\ddot{\theta}(t) + mgl\sin(\theta(t)) = \tau(t)$.
  - COMPUTE: $\tau(t) = mgl\sin(\theta(t)) + u(t)$.
  - THEN: $ml^2\ddot{\theta}(t) = u(t)$, no matter how large $\theta(t)$ becomes!
  - Sometimes used in robotics and airplane flight control, but very computationally intensive.

Analogous systems

- The linearized differential equations of many very different physical systems appear identical.
- One would suppose they behave in similar ways (dynamic response) and can be controlled with similar controllers.

Mechanical Translational	$m\ddot{x}(t) + b\dot{x}(t) + kx(t) = u(t)$
Mechanical Rotational	$J\ddot{\theta}(t) + b\dot{\theta}(t) + k\theta(t) = \tau(t)$
Satellite	$J\ddot{\theta}(t) = f(t) \cdot d$
DC Motor (for $L_a = 0$)	$J\ddot{\theta}(t) + \left(b + \dfrac{k_\tau k_e}{R_a}\right)\dot{\theta}(t) = \dfrac{k_\tau}{R_a}e_a(t)$
Generator	$(L_a L_f)\ddot{e}_a(t) + (L_f(R_a + R_l) + L_a R_f)\dot{e}_a(t) +$ $R_f(R_a + R_l) = (k_g R_l)e_f(t)$

- These are all of the form

$$a_2\ddot{x}(t) + a_1\dot{x}(t) + a_0x(t) = b_2\ddot{u}(t) + b_1\dot{u}(t) + b_0u(t)$$

  which is called a second-order form.

- Therefore, we have seen very specific examples of a very general class of system. If we learn how to control the general class, we can apply this knowledge to specific systems.