FREQUENCY-RESPONSE ANALYSIS

8.1: Motivation to study frequency-response methods

- Advantages and disadvantages to root-locus design approach:

ADVANTAGES:

- Good indicator of transient response.
- Explicitly shows location of closed-loop poles. ➠ Tradeoffs are clear.

DISADVANTAGES:

- Requires transfer function of plant be known.
- Difficult to infer all performance values.
- Hard to extract steady-state response (sinusoidal inputs).

- Frequency-response methods can be used to *supplement* root locus:
  - Can infer performance and stability from same plot.
  - Can use measured data when no model is available.
  - Design process is independent of system order (# poles).
  - Time delays handled correctly ($e^{-s\tau}$).
  - Graphical techniques (analysis/synthesis) are "quite simple."

What is a frequency response?

- We want to know how a linear system responds to sinusoidal input, in steady state.

- Consider system $Y(s) = G(s)U(s)$ with input $u(t) = u_0\cos(\omega t)$, so

$$U(s) = u_0\frac{s}{s^2+\omega^2}.$$

- With zero initial conditions,

$$Y(s) = u_0 G(s)\frac{s}{s^2+\omega^2}.$$

- Do a partial-fraction expansion (assume distinct roots)

$$Y(s) = \frac{\alpha_1}{s-a_1} + \frac{\alpha_2}{s-a_2} + \cdots + \frac{\alpha_n}{s-a_n} + \frac{\alpha_0}{s-j\omega} + \frac{\alpha_0^*}{s+j\omega}$$

$$y(t) = \underbrace{\alpha_1 e^{a_1 t} + \alpha_2 e^{a_2 t} + \cdots + \alpha_n e^{a_n t}}_{\text{If stable, these decay to zero.}} + \alpha_0 e^{j\omega t} + \alpha_0^* e^{-j\omega t}.$$

$$y_{ss}(t) = \alpha_0 e^{j\omega t} + \alpha_0^* e^{-j\omega t}.$$

- Let $\alpha_0 = Ae^{j\phi}$. Then,

$$\begin{aligned} y_{ss} &= Ae^{j\phi}e^{j\omega t} + Ae^{-j\phi}e^{-j\omega t} \\ &= A\left(e^{j(\omega t+\phi)} + e^{-j(\omega t+\phi)}\right) \\ &= 2A\cos(\omega t + \phi). \end{aligned}$$

We find α_0 via standard partial-fraction-expansion means:

$$\begin{aligned} \alpha_0 &= \left[(s-j\omega)Y(s)\right|_{s=j\omega} \\ &= \left[\frac{u_0 s G(s)}{(s+j\omega)}\right|_{s=j\omega} \\ &= \frac{u_0(j\omega)G(j\omega)}{(2j\omega)} = \frac{u_0 G(j\omega)}{2}. \end{aligned}$$

- Substituting into our prior result

$$y_{ss} = u_0|G(j\omega)|\cos(\omega t + \angle G(j\omega)).$$

- Important LTI-system fact: If the input to an LTI system is a sinusoid, the "steady-state" output is a sinusoid of the same frequency but different amplitude and phase.

FORESHADOWING: Transfer function at $s = j\omega$ tells us response to a sinusoid...but also about stability as $j\omega$-axis is stability boundary!

EXAMPLE: Suppose that we have a system with transfer function

$$G(s) = \frac{2}{3+s}.$$

- Then, the system's frequency response is

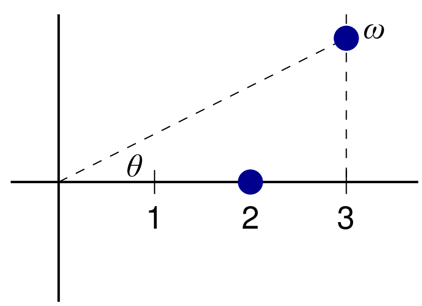
$$G(j\omega) = \left.\frac{2}{3+s}\right|_{s=j\omega} = \frac{2}{3+j\omega}.$$

- The magnitude response is

$$A(j\omega) = \left|\frac{2}{3+j\omega}\right| = \frac{|2|}{|3+j\omega|} = \frac{2}{\sqrt{(3+j\omega)(3-j\omega)}} = \frac{2}{\sqrt{9+\omega^2}}.$$

- The phase response is

$$\begin{aligned}\phi(j\omega) &= \angle\left(\frac{2}{3+j\omega}\right)\\ &= \angle\,(2) - \angle\,(3+j\omega)\\ &= 0 - \tan^{-1}(\omega/3)\,.\end{aligned}$$



- Now that we know the amplitude and phase response, we can find the amplitude gain and phase change caused by the system for any specific frequency.

- For example, if $\omega = 3\,\text{rad s}^{-1}$,

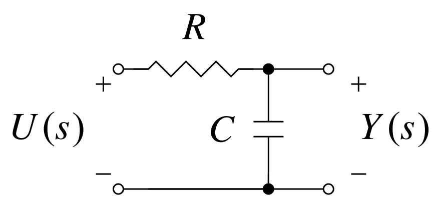
$$A(j3) = \frac{2}{\sqrt{9+9}} = \frac{\sqrt{2}}{3}$$

$$\phi(j3) = -\tan^{-1}(3/3) = -\pi/4.$$

8.2: Plotting a frequency response

- There are two common ways to plot a frequency response ➠ the magnitude and phase for all frequencies.

EXAMPLE:



$$G(s) = \frac{1}{1 + RCs}$$

- Frequency response

$$\begin{aligned} G(j\omega) &= \frac{1}{1 + j\omega RC} \qquad \text{(let } RC = 1\text{)} \\ &= \frac{1}{1 + j\omega} \\ &= \frac{1}{\sqrt{1+\omega^2}} \angle - \tan^{-1}(\omega). \end{aligned}$$

- We will need to separate magnitude and phase information from rational polynomials in $j\omega$.
  - Magnitude = magnitude of numerator / magnitude of denominator

$$\frac{\sqrt{\mathbb{R}(\text{num})^2 + \mathbb{I}(\text{num})^2}}{\sqrt{\mathbb{R}(\text{den})^2 + \mathbb{I}(\text{den})^2}}.$$

  - Phase = phase of numerator − phase of denominator

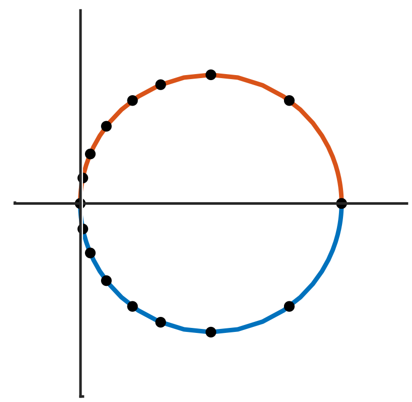
$$\tan^{-1}\left(\frac{\mathbb{I}(\text{num})}{\mathbb{R}(\text{num})}\right) - \tan^{-1}\left(\frac{\mathbb{I}(\text{den})}{\mathbb{R}(\text{den})}\right).$$

Plot method #1: Polar plot in complex plane

- Evaluate $G(j\omega)$ at each frequency for $0 \le \omega < \infty$.
- Result will be a complex number at each frequency: $a + jb$ or $Ae^{j\phi}$.

- Plot each point on the complex plane at $(a + jb)$ or $Ae^{j\phi}$ for each frequency-response value.
- Result = polar plot.
- We will later call this a "Nyquist plot".

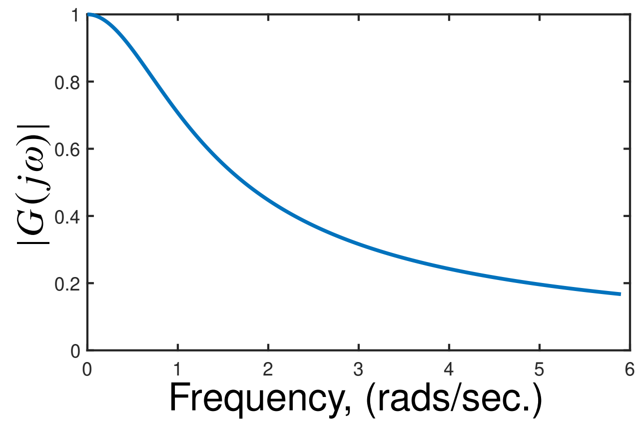
ω	$G(j\omega)$
0	$1.000\angle 0.0°$
0.5	$0.894\angle -26.6°$
1.0	$0.707\angle -45.0°$
1.5	$0.555\angle -56.3°$
2.0	$0.447\angle -63.4°$
3.0	$0.316\angle -71.6°$
5.0	$0.196\angle -78.7°$
10.0	$0.100\angle -84.3°$
∞	$0.000\angle -90.0°$

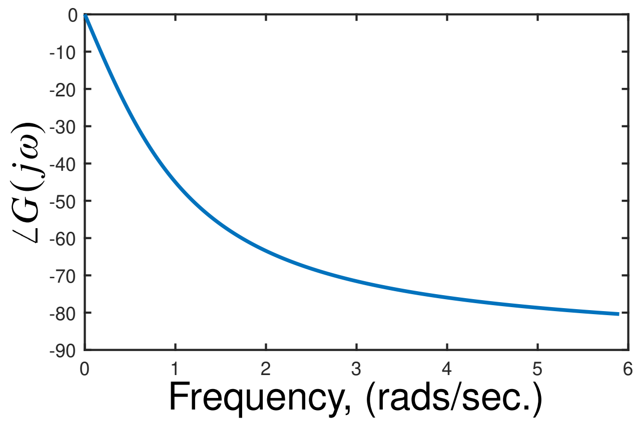


- The polar plot is parametric in ω, so it is hard to read the frequency-response for a specific frequency from the plot.
- We will see later that the polar plot will help us determine stability properties of the plant and closed-loop system.

Plot method #2: Magnitude and phase plots

- We can replot the data by separating the plots for magnitude and phase making two plots versus frequency.





- The above plots are in a natural scale, but usually a log-log plot is made ➠ This is called a "Bode plot" or "Bode diagram."

Reason for using a logarithmic scale

- Simplest way to display the frequency response of a rational-polynomial transfer function is to use a Bode Plot.
- Logarithmic $|G(j\omega)|$ versus logarithmic ω, and logarithmic $\angle G(j\omega)$ versus ω.

REASON:

$$\log_{10}\left(\frac{ab}{cd}\right) = \log_{10} a + \log_{10} b - \log_{10} c - \log_{10} d.$$

➤ The polynomial factors that contribute to the transfer function can be split up and evaluated separately.

$$G(s) = \frac{(s+1)}{(s/10+1)}$$

$$G(j\omega) = \frac{(j\omega+1)}{(j\omega/10+1)}$$

$$|G(j\omega)| = \frac{|j\omega+1|}{|j\omega/10+1|}$$

$$\log_{10}|G(j\omega)| = \log_{10}\sqrt{1+\omega^2} - \log_{10}\sqrt{1+\left(\frac{\omega}{10}\right)^2}.$$

- Consider:

$$\log_{10}\sqrt{1+\left(\frac{\omega}{\omega_n}\right)^2}.$$

- For $\omega \ll \omega_n$,

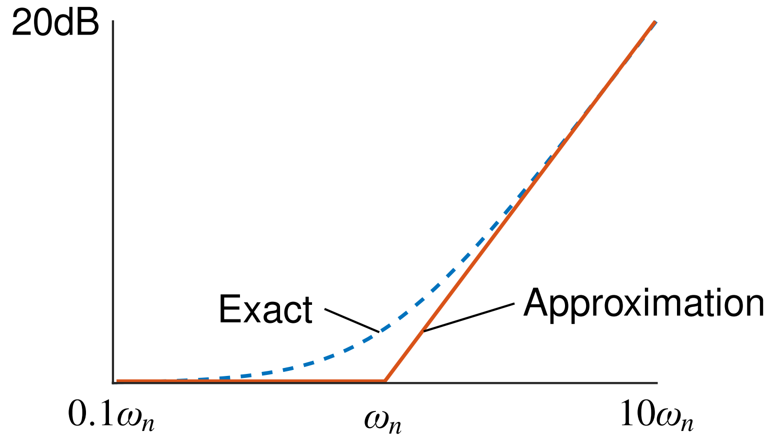
$$\log_{10}\sqrt{1+\left(\frac{\omega}{\omega_n}\right)^2} \approx \log_{10}(1) = 0.$$

- For $\omega \gg \omega_n$,

$$\log_{10}\sqrt{1+\left(\frac{\omega}{\omega_n}\right)^2} \approx \log_{10}\left(\frac{\omega}{\omega_n}\right).$$

KEY POINT: Two straight lines on a log-log plot; intersect at $\omega = \omega_n$.

- Typically plot $20\log_{10}|G(j\omega)|$; that is, in dB.



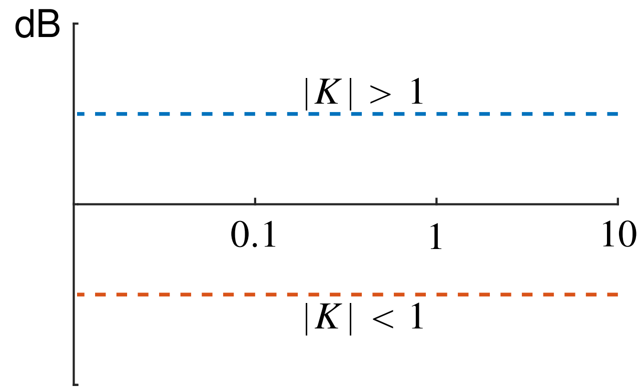
- A transfer function is made up of first-order zeros and poles, complex zeros and poles, constant gains and delays. We will see how to make straight-line (magnitude- and phase-plot) approximations for all these, and combine them to form the appropriate Bode diagram.

8.3: Bode magnitude diagrams (a)

- The $\log_{10}(\cdot)$ operator lets us break a transfer function up into pieces.
- If we know how to plot the Bode plot of each piece, then we simply add all the pieces together when we're done.

Bode magnitude: Constant gain

- dB $= 20\log_{10}|K|$.
- Not a function of frequency. Horizontal straight line. If $|K| < 1$, then negative, else positive.



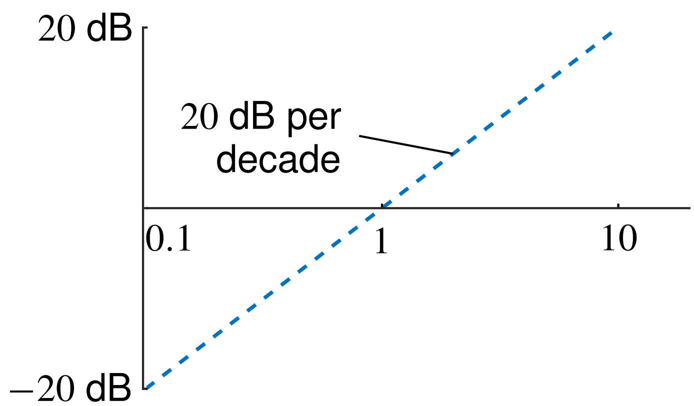
Bode magnitude: Zero or pole at origin

- For a zero at the origin,

$$G(s) = s$$

$$\text{dB} = 20\log_{10}|G(j\omega)|$$

$$= 20\log_{10}|j\omega|\ \text{dB}.$$

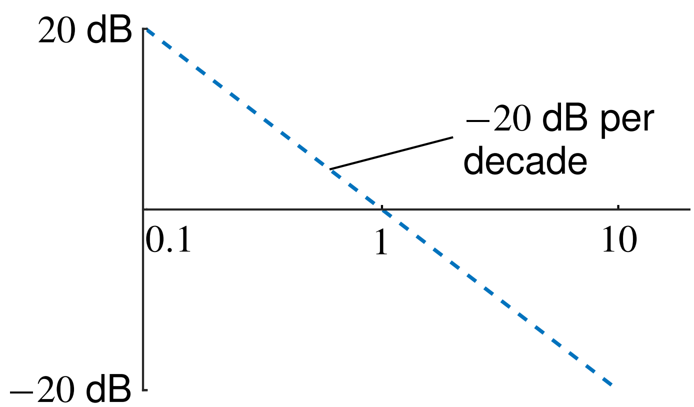


- For a pole at the origin,

$$G(s) = \frac{1}{s}$$

$$\text{dB} = 20\log_{10}|G(j\omega)|$$

$$= -20\log_{10}|j\omega|\ \text{dB}.$$



- Both are straight lines, slope $= \pm 20$ dB per decade of frequency.
  - *Line intersects ω-axis at $\omega = 1$.*
- For an nth-order pole or zero at the origin,

$$\begin{aligned} \text{dB} &= \pm 20 \log_{10} |(j\omega)^n| \\ &= \pm 20 \log_{10} \omega^n \\ &= \pm 20n \log_{10} \omega. \end{aligned}$$

  - Still straight lines.
  - Still intersect ω-axis at $\omega = 1$.
  - *But, slope* $= \pm 20$n dB *per decade.*

Bode magnitude: Zero or pole on real axis, but not at origin

- For a zero on the real axis, (LHP *or* RHP), the standard Bode form is

$$G(s) = \left(\frac{s}{\omega_n} \pm 1\right),$$

  which ensures unity dc-gain.
- If you start out with something like

$$G(s) = (s + \omega_n),$$

  then factor as

$$G(s) = \omega_n \left(\frac{s}{\omega_n} + 1\right).$$

  Draw the gain term (ω_n) separately from the zero term ($s/\omega_n + 1$).
- In general, a LHP or RHP zero has standard Bode form

$$G(s) = \left(\frac{s}{\omega_n} \pm 1\right)$$

$$G(j\omega) = \pm 1 + j\left(\frac{\omega}{\omega_n}\right)$$

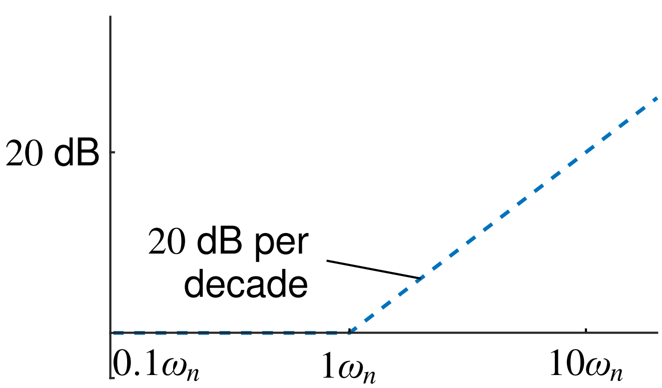
$$20\log_{10}|G(j\omega)| = 20\log_{10}\sqrt{1+\left(\frac{\omega}{\omega_n}\right)^2}$$

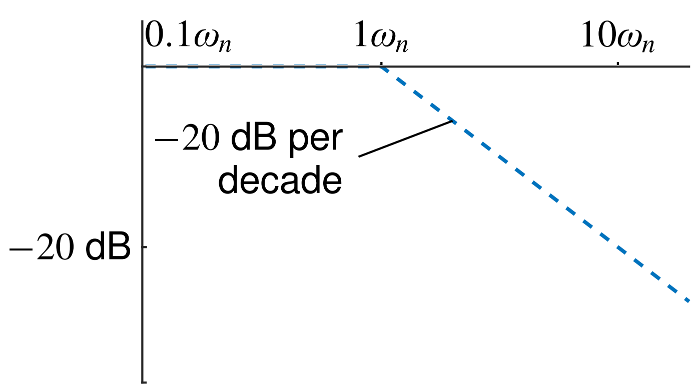
- For $\omega \ll \omega_n$, $\quad 20\log_{10}\sqrt{1+\left(\frac{\omega}{\omega_n}\right)^2} \approx 20\log_{10}\sqrt{1} = 0.$
- For $\omega \gg \omega_n$, $\quad 20\log_{10}\sqrt{1+\left(\frac{\omega}{\omega_n}\right)^2} \approx 20\log_{10}\left(\frac{\omega}{\omega_n}\right).$
- Two straight lines on a log scale which intersect at $\omega = \omega_n$.
- For a pole on the real axis, (LHP *or* RHP) standard Bode form is

$$G(s) = \left(\frac{s}{\omega_n} \pm 1\right)^{-1}$$

$$20\log_{10}|G(j\omega)| = -20\log_{10}\sqrt{1+\left(\frac{\omega}{\omega_n}\right)^2}.$$

This is the same except for a minus sign.





8.4: Bode magnitude diagrams (b)

Bode magnitude: Complex zero pair or complex pole pair

- For a complex-zero pair (LHP *or* RHP) standard Bode form is

$$\left(\frac{s}{\omega_n}\right)^2 \pm 2\zeta\left(\frac{s}{\omega_n}\right) + 1,$$

  which has unity dc-gain.

- If you start out with something like

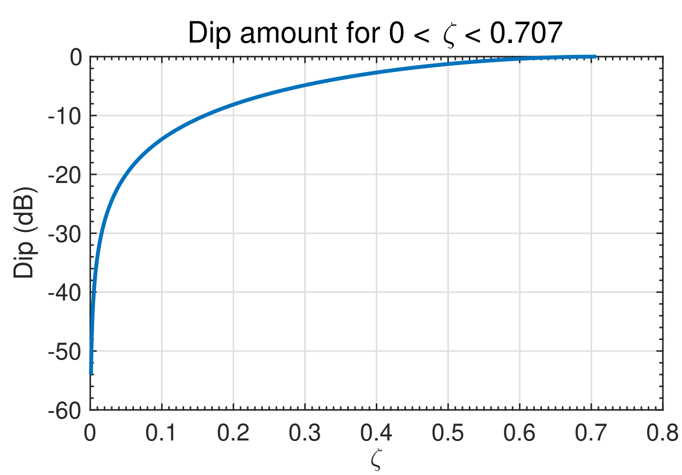
$$s^2 \pm 2\zeta\omega_n s + \omega_n^2,$$

  which we have seen before as a "standard form," the dc-gain is ω_n^2.

- Convert forms by factoring out ω_n^2

$$s^2 \pm 2\zeta\omega_n s + \omega_n^2 = \omega_n^2\left[\left(\frac{s}{\omega_n}\right)^2 \pm 2\zeta\left(\frac{s}{\omega_n}\right) + 1\right].$$

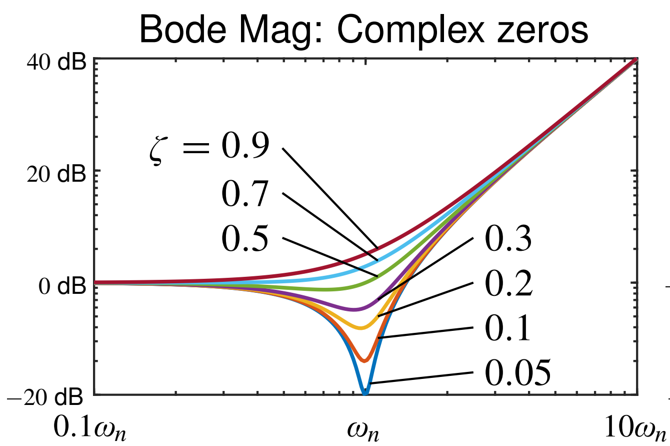
- Complex zeros do not lend themselves very well to straight-line approximation.
- If $\zeta = 1$, then this is $\left(\frac{s}{\omega_n} \pm 1\right)^2$.
- Double real zero at ω_n ➠ slope of 40 dB/decade.
- For $\zeta \neq 1$, there will be overshoot or undershoot at $\omega \approx \omega_n$.
- For other values of ζ:
  - Dip frequency: $\omega_d = \omega_n\sqrt{1 - 2\zeta^2}$
  - Value of $|H(j\omega_d)|$ is: $20\log_{10}(2\zeta\sqrt{(1-\zeta^2)})$.
  - Note: There is no dip unless $0 < \zeta < 1/\sqrt{2} \approx 0.707$.

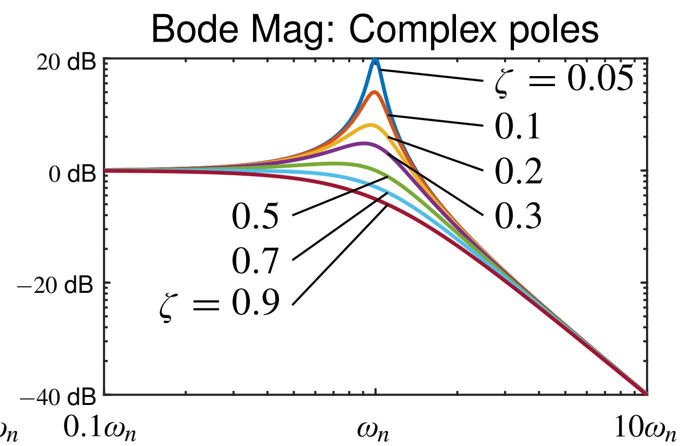


- We write complex poles (LHP *or* RHP) as

$$G(s) = \left[\left(\frac{s}{\omega_n}\right)^2 \pm 2\zeta\left(\frac{s}{\omega_n}\right) + 1\right]^{-1}.$$

  - The resonant peak frequency is $\omega_r = \omega_n\sqrt{1-2\zeta^2}$
  - Value of $|H(j\omega_r)|$ is $-20\log_{10}(2\zeta\sqrt{(1-\zeta^2)})$.
    - Same graph as for "dip" for complex-conjugate zeros.
  - Note that there is no peak unless $0 < \zeta < 1/\sqrt{2} \approx 0.707$.
  - For $\omega \ll \omega_n$, magnitude $\approx$ 0 dB.
  - For $\omega \gg \omega_n$, magnitude slope $=$ -40 dB/decade.





Bode magnitude: Time delay

- $G(s) = e^{-s\tau} \quad \ldots \quad |G(j\omega)| = 1.$
- $20\log_{10} 1 =$ 0 dB.
- Does not change magnitude response.

EXAMPLE: Sketch the Bode magnitude plot for

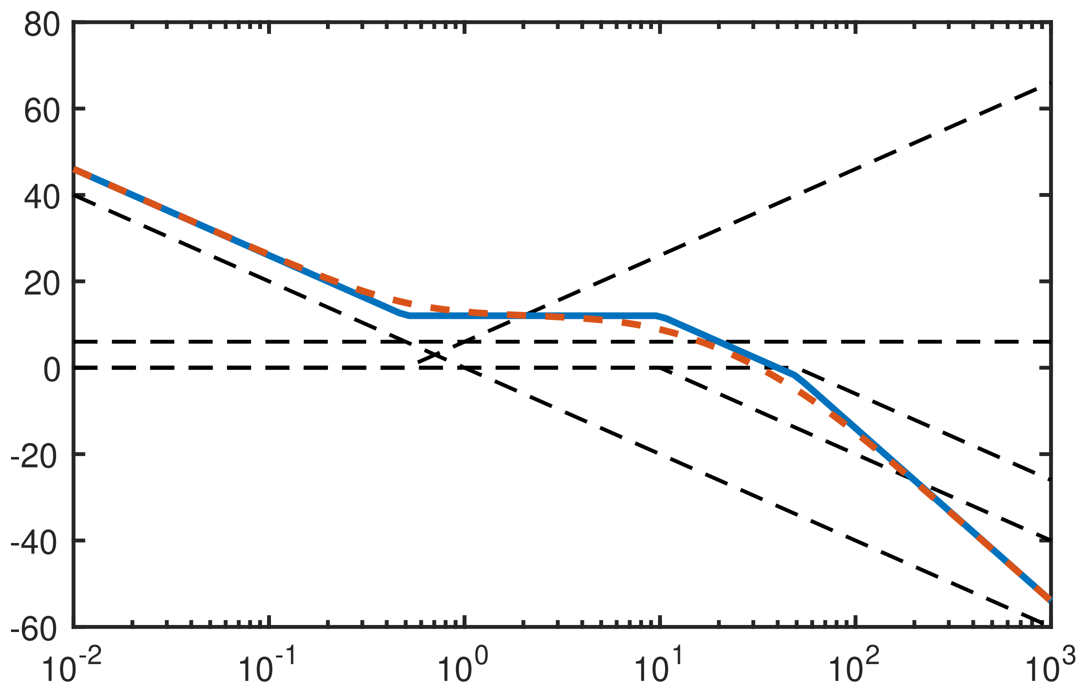
$$G(s) = \frac{2000(s+0.5)}{s(s+10)(s+50)}.$$

- The first step is to convert the terms of the transfer function into "Bode standard form".

$$G(s)=\frac{2000(s+0.5)}{s(s+10)(s+50)}=\frac{\frac{2000\cdot 0.5}{10\cdot 50}\left(\frac{s}{0.5}+1\right)}{s\left(\frac{s}{10}+1\right)\left(\frac{s}{50}+1\right)}$$

$$G(j\omega)=\frac{2\left(\frac{j\omega}{0.5}+1\right)}{j\omega\left(\frac{j\omega}{10}+1\right)\left(\frac{j\omega}{50}+1\right)}.$$

- We can see that the components of the transfer function are:
  - DC gain of $20\log_{10}2\approx 6$ dB;
  - Pole at origin;
  - One real zero not at origin, and
  - Two real poles not at origin.



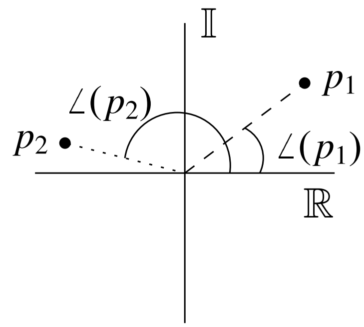
8.5: Bode phase diagrams (a)

- Bode diagrams consist of the magnitude plots we have seen so far,
- *BUT,* also phase plots. These are just as easy to draw.
- *BUT,* they differ depending on whether the dynamics are RHP or LHP.

Finding the phase of a complex number

- Plot the location of the number as a vector in the complex plane.
- Use trigonometry to find the phase.
- For numbers with positive real part,

$$\angle(\#) = \tan^{-1}\left(\frac{\mathbb{I}(\#)}{\mathbb{R}(\#)}\right).$$



- For numbers with negative real part,

$$\angle(\#) = 180^\circ - \tan^{-1}\left(\frac{\mathbb{I}(\#)}{|\mathbb{R}(\#)|}\right).$$

- If you are lucky enough to have the "atan2(y, x)" function, then

$$\angle(\#) = \text{atan2}(\mathbb{I}(\#), \mathbb{R}(\#))$$

  for *any* complex number.
- Also note,

$$\angle\left(\frac{ab}{cd}\right) = \angle(a) + \angle(b) - \angle(c) - \angle(d).$$

Finding the phase of a complex function of ω

- This is the same as finding the phase of a complex number, if specific values of ω are substituted into the function.

Bode phase: Constant gain

- $G(s) = K$.
- $\angle(K) = \begin{cases} 0^\circ, & K \geq 0; \\ -180^\circ, & K < 0. \end{cases}$.
- Constant phase of 0° or -180°.

Bode phase: Zero or pole at origin

- Zero: $G(s) = s, \quad \ldots \quad G(j\omega) = j\omega = \omega\angle 90^\circ$.
- Pole: $G(s) = \dfrac{1}{s}, \quad \ldots \quad G(j\omega) = \dfrac{1}{j\omega} = \dfrac{-j}{\omega} = \dfrac{1}{\omega}\angle - 90^\circ$.
- Constant phase of $\pm 90^\circ$.

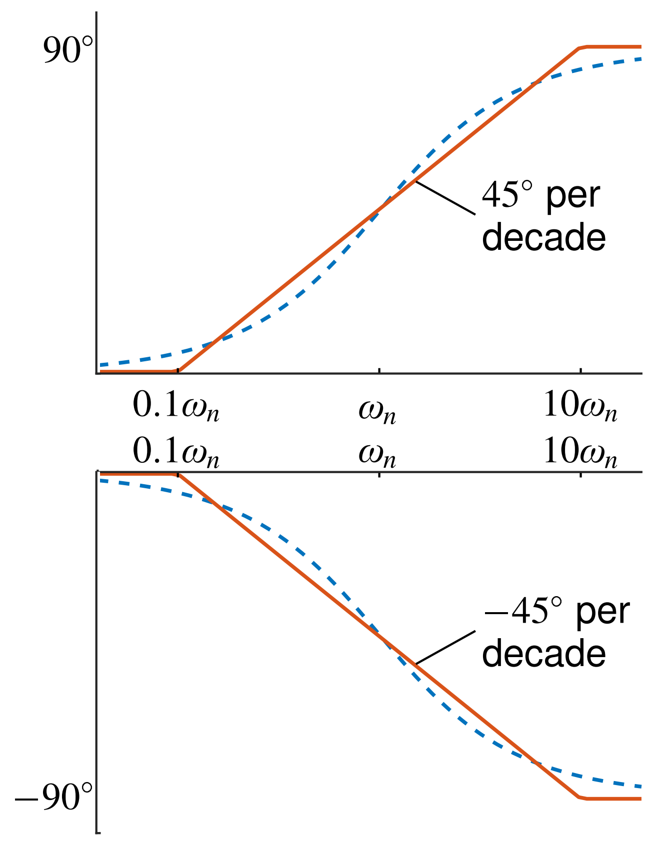
Bode phase: Real LHP zero or pole

- Zero: $G(s) = \left(\dfrac{s}{\omega_n} + 1\right)$.

$$\angle G(j\omega) = \angle\left(j\frac{\omega}{\omega_n} + 1\right) = \tan^{-1}\left(\frac{\omega}{\omega_n}\right).$$

- Pole: $G(s) = \dfrac{1}{\left(\frac{s}{\omega_n} + 1\right)}$,

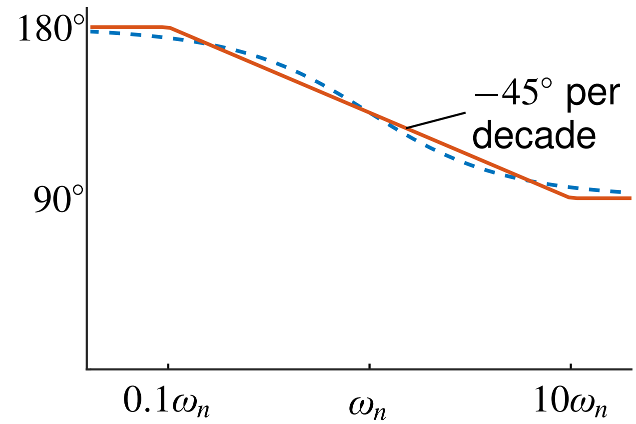
$$\angle G(j\omega) = \angle(1) - \angle\left(j\frac{\omega}{\omega_n} + 1\right) = -\tan^{-1}\left(\frac{\omega}{\omega_n}\right).$$



Bode phase: Real RHP zero or pole

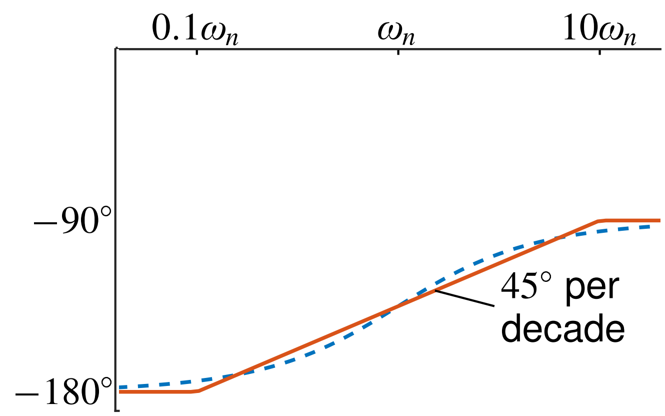
- Zero: $G(s) = \left(\dfrac{s}{\omega_n} - 1\right)$.

$$\angle G(j\omega) = \angle\left(j\frac{\omega}{\omega_n} - 1\right)$$
$$= 180^\circ - \tan^{-1}\left(\frac{\omega}{\omega_n}\right).$$



- Pole: $G(s) = \dfrac{1}{\left(\frac{s}{\omega_n} - 1\right)}$,

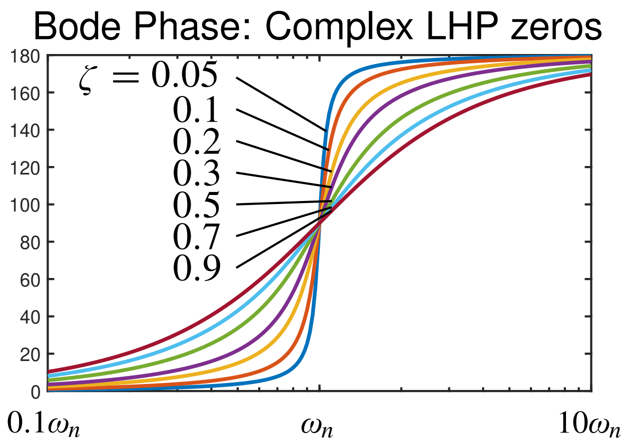
$$\angle G(j\omega) = \angle(1) - \angle\left(j\frac{\omega}{\omega_n} - 1\right)$$
$$= -\left(180^\circ - \tan^{-1}\left(\frac{\omega}{\omega_n}\right)\right)$$
$$= -180^\circ + \tan^{-1}\left(\frac{\omega}{\omega_n}\right).$$

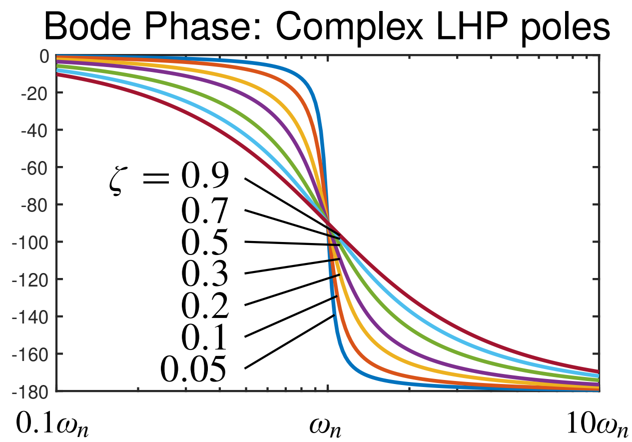


8.6: Bode phase diagrams (b)

Bode phase: Complex LHP zero pair or pole pair

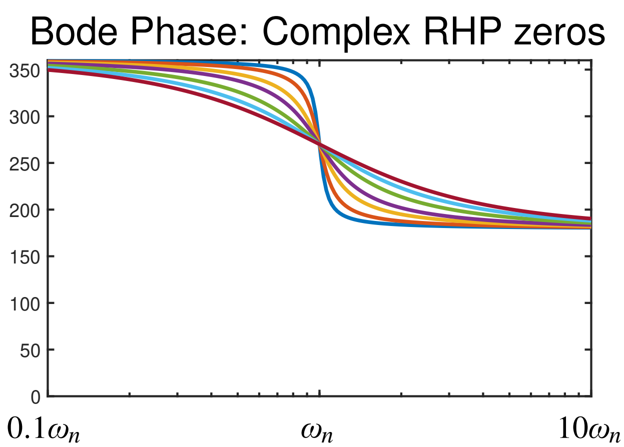
- Complex LHP zeros cause phase to go from 0° to 180°.
- Complex LHP poles cause phase to go from -180° to 0°.
- Transition happens in about $\pm\zeta$ decades, centered at ω_n.

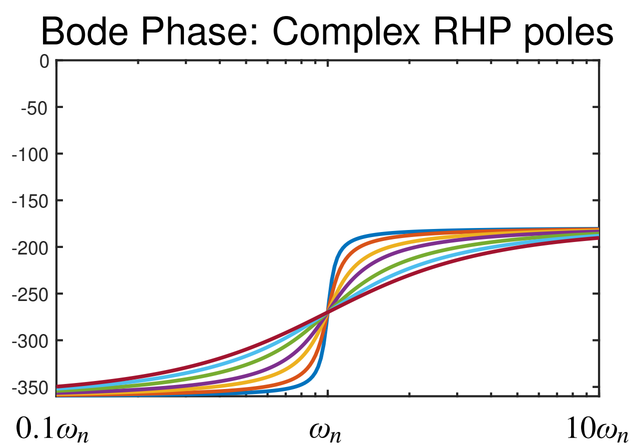




Bode phase: Complex RHP zero pair or pole pair

- Complex RHP zeros cause phase to go from 360° to 180°.
- Complex RHP poles cause phase to go from -360° to -180°.





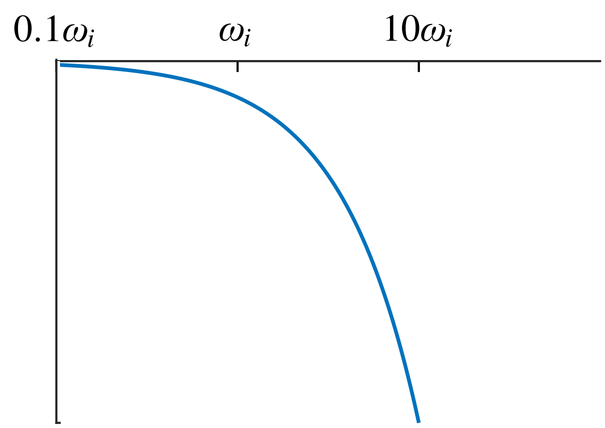
Bode phase: Time delay

- $G(s) = e^{-s\tau}$,

$$G(j\omega) = e^{-j\omega\tau} = 1\angle - \omega\tau$$

$$\angle G(j\omega) = -\omega\tau \text{ in radians.}$$

$$= -56.3\omega\tau \text{ in degrees.}$$



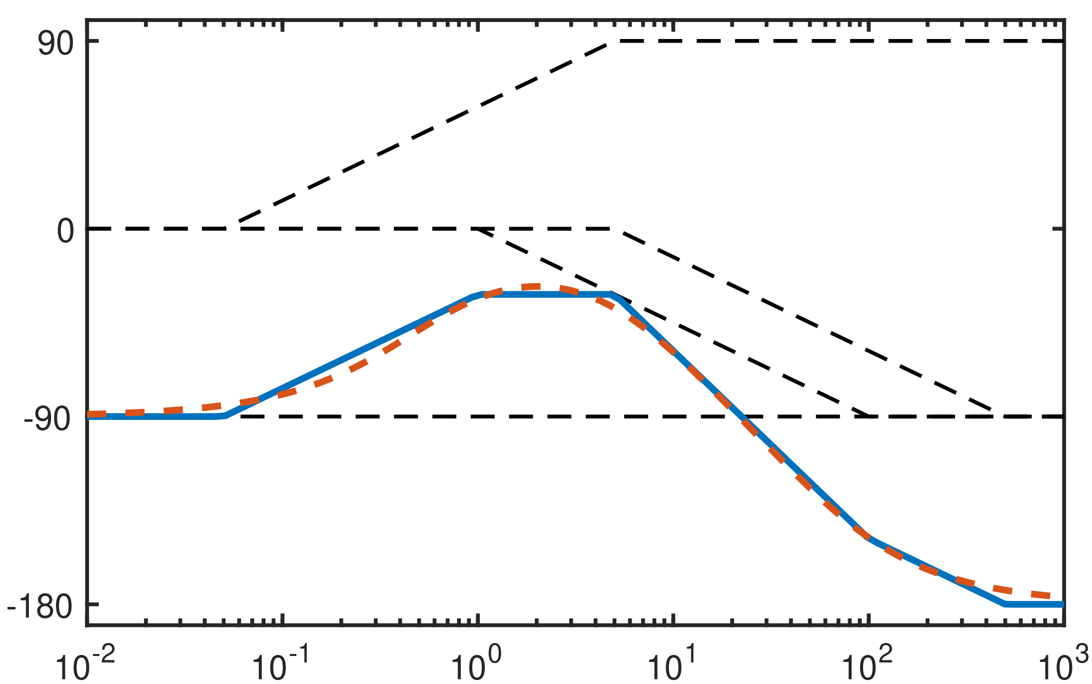
- Note: Line→curve in log scale.

EXAMPLE: Sketch the Bode phase plot for

$$G(s) = \frac{2000(s+0.5)}{s(s+10)(s+50)} \quad \text{or} \quad G(j\omega) = \frac{2\,(j\omega/0.5+1)}{j\omega\,(j\omega/10+1)\,(j\omega/50+1)},$$

where we converted to "Bode standard form" in a prior example.

- Constant: $K = +2$. Zero phase contribution.
- Pole at origin: Phase contribution of $-90°$.
- Two real LHP poles: Phase from $0°$ to $-90°$, each.
- One real LHP zero: Phase from $0°$ to $90°$.



EXAMPLE: Sketch the Bode magnitude and phase plots for

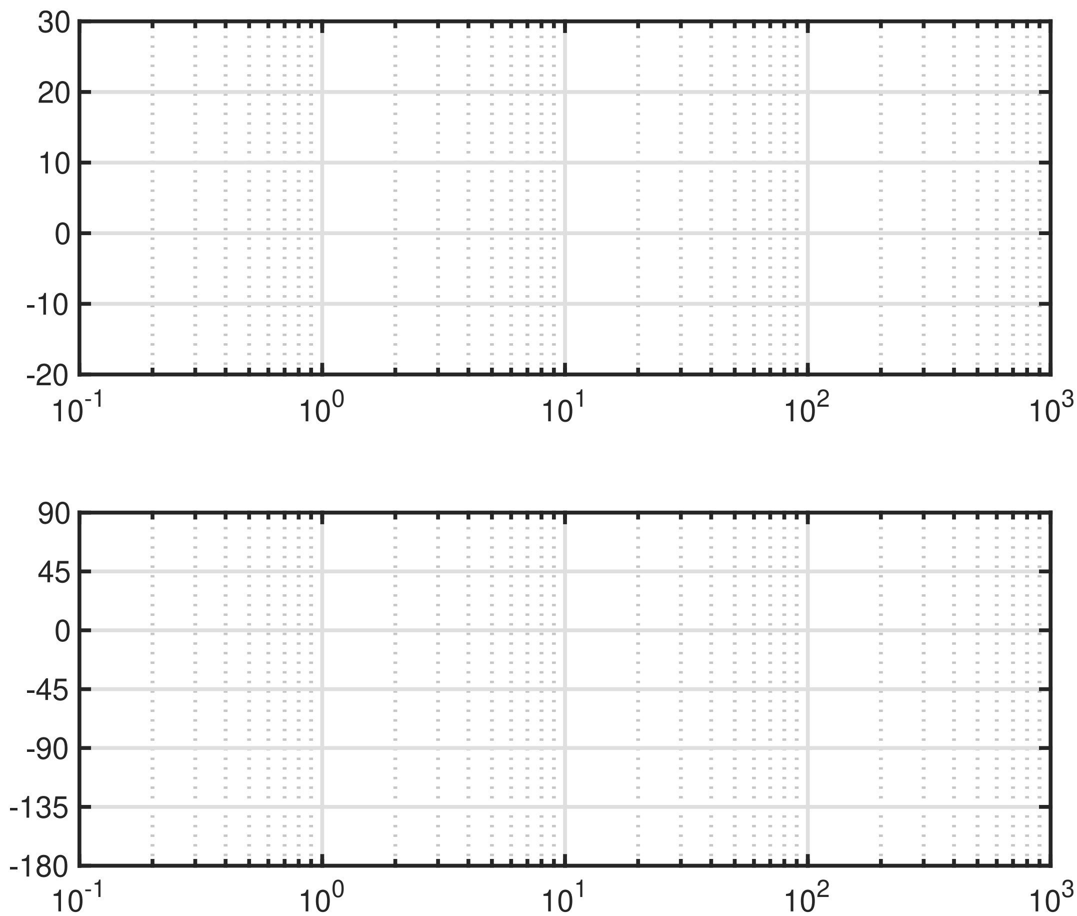
$$G(s) = \frac{1200\,(s+3)}{s\,(s+12)\,(s+50)}.$$

- First, we convert to Bode standard form, which gives

$$G(s) = \frac{1200\,(3)\left(1+\frac{s}{3}\right)}{s\,(12)\,(50)\left(1+\frac{s}{12}\right)\left(1+\frac{s}{50}\right)}$$

$$G(j\omega) = \frac{6\left(1+\frac{j\omega}{3}\right)}{j\omega\left(1+\frac{j\omega}{12}\right)\left(1+\frac{j\omega}{50}\right)}.$$

- Positive gain, one real LHP zero, one pole at origin, two real LHP poles.



8.7: Some observations based on Bode plots

Nonminimum-phase systems

- A system is called a nonminimum-phase if it has pole(s) or zero(s) in the RHP.
- Consider

$$G_1(s) = 10\,\frac{s+1}{s+10}\Bigg\}\quad \begin{matrix}\text{zero at } -1\\ \text{pole at } -10\end{matrix}\quad \Bigg\}\ \begin{matrix}\text{minimum}\\ \text{phase}\end{matrix}$$

$$G_2(s) = 10\,\frac{s-1}{s+10}\Bigg\}\quad \begin{matrix}\text{zero at } +1\\ \text{pole at } -10\end{matrix}\quad \Bigg\}\ \begin{matrix}\text{nonminimum}\\ \text{phase}\end{matrix}$$

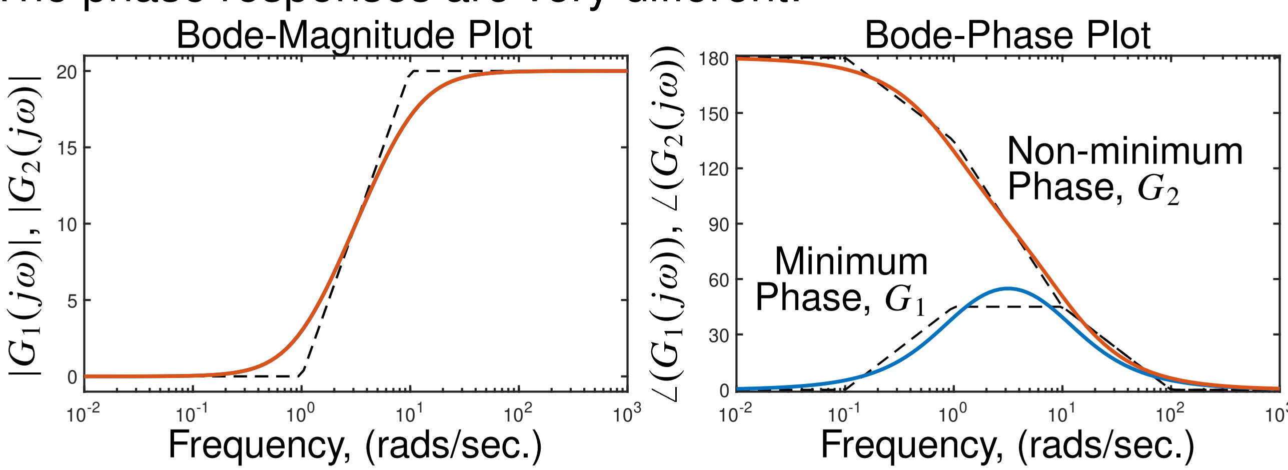
- The magnitude responses of these two systems are:

$$|G_1(j\omega)| = 10\frac{|j\omega+1|}{|j\omega+10|} = 10\frac{\sqrt{\omega^2+1}}{\sqrt{\omega^2+100}}$$

$$|G_2(j\omega)| = 10\frac{|j\omega-1|}{|j\omega+10|} = 10\frac{\sqrt{\omega^2+1}}{\sqrt{\omega^2+100}}$$

which are the same!

- The phase responses are very different:



- Note that the change in phase of G_1 is much smaller than change of phase in G_2. Hence G_1 is "minimum phase" and G_2 is "nonminimum-phase"

- Non-minimum phase usually associated with delay.

$$G_2(s) = G_1(s)\underbrace{\frac{s-1}{s+1}}_{\text{Delay}}$$

- Note: $\dfrac{s-1}{s+1}$ is very similar to a first-order Padé approximation to a delay. It is the same when evaluated at $s = j\omega$.
- Consider using feedback to control a nonminimum-phase system. What do the root-locus plotting techniques tell us?
- Consequently, nonminimum-phase systems are harder to design controllers for; step response often tends to "go the wrong way," at least initially.

Steady-state errors from Bode magnitude plot

- Recall our discussion of steady-state errors to step/ramp/parabolic inputs versus "system type" (summarized on pg. 4–24)
- Consider a *unity-feedback* system.
- If the open-loop plant transfer function has N poles at $s = 0$ then the system is "type N"
  - K_p is error constant for type 0.
  - K_v is error constant for type 1.
  - K_a is error constant for type 2...
- For a unity-feedback system, $K_p = \lim_{s\to 0} G(s)$.
  - *At low frequency*, a type 0 system will have $G(s) \approx K_p$.
  - We can read this off the Bode-magnitude plot directly!

- Horizontal y-intercept at low frequency = K_p.

$$\Rightarrow \quad e_{ss} = \frac{1}{1+K_p} \quad \text{for step input.}$$

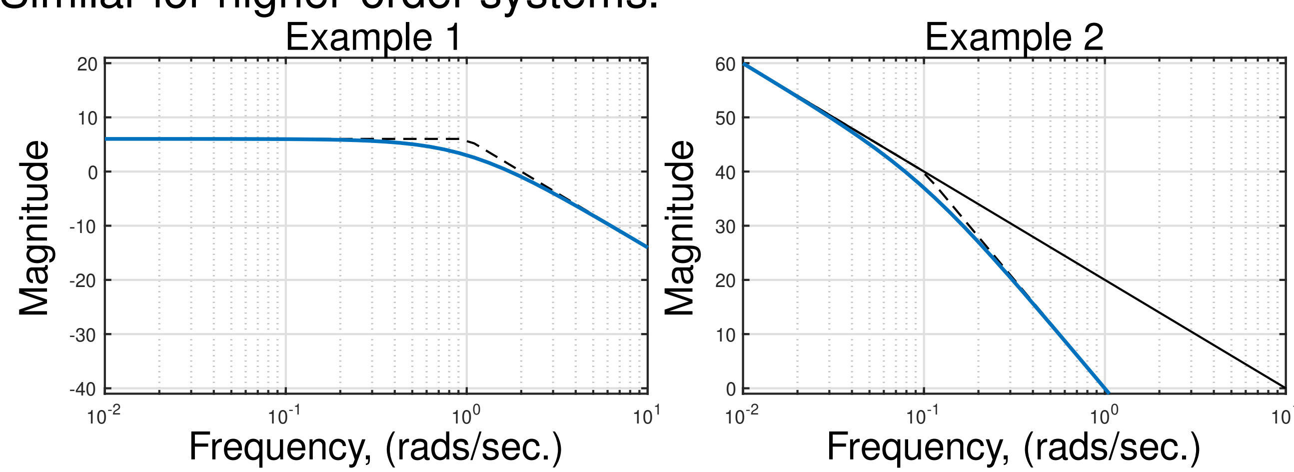
- $K_v = \lim_{s\to 0} sG(s)$, and is nonzero for a type 1 system.
  - *At low frequency*, a type 1 system will have $G(s) \approx \dfrac{K_v}{s}$.
  - *At low frequency*, $|G(j\omega)| \approx \dfrac{K_v}{\omega}$. Slope of -20 dB/decade.
  - Use the above approximation to extend the low-frequency asymptote to $\omega = 1$. The asymptote (*NOT THE ORIGINAL* $|G(j\omega)|$) evaluated at $\omega = 1$ is K_v.

$$\Rightarrow \quad e_{ss} = \frac{1}{K_v} \quad \text{for ramp input.}$$

- $K_a = \lim_{s\to 0} s^2G(s)$, and is nonzero for a type 2 system.
  - *At low frequency*, a type 2 system will have $G(s) \approx \dfrac{K_a}{s^2}$.
  - *At low frequency*, $|G(j\omega)| \approx \dfrac{K_a}{\omega^2}$. Slope of -40 dB/decade.
  - Again, use approximation to extend low-frequency asymptote to $\omega = 1$. The asymptote evaluated at $\omega = 1$ is K_a.

$$\Rightarrow \quad e_{ss} = \frac{1}{K_a} \quad \text{for parabolic input.}$$

- Similar for higher-order systems.



EXAMPLE 1:

- Horizontal as $\omega \to 0$, so we know this is type 0.
- Intercept = 6 dB... K_p = 6 dB = 2 [linear units].

EXAMPLE 2:

- Slope = −20 dB/decade as $\omega \to 0$, so we know this is type 1.
- Extend slope at low frequency to $\omega = 1$.
- Intercept = 20 dB... K_v = 20 dB = 10 [linear units].

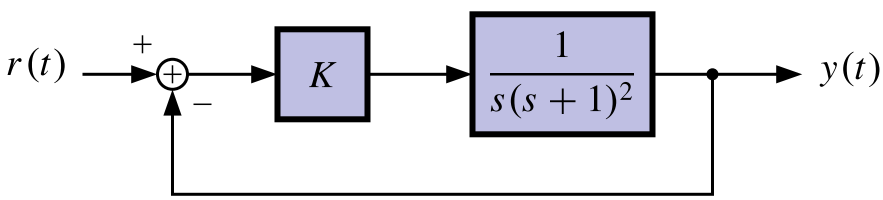
8.8: Stability revisited

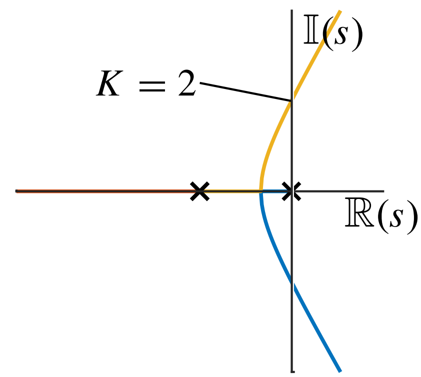
- If we know the closed-loop *transfer function* of a system in rational-polynomial form, we can use Routh to find stable ranges for K.
- **Motivation**: What if we only have open-loop *frequency* response?

A simple example

- Consider, for now, that we know the transfer-function of the system, and can plot the root-locus.

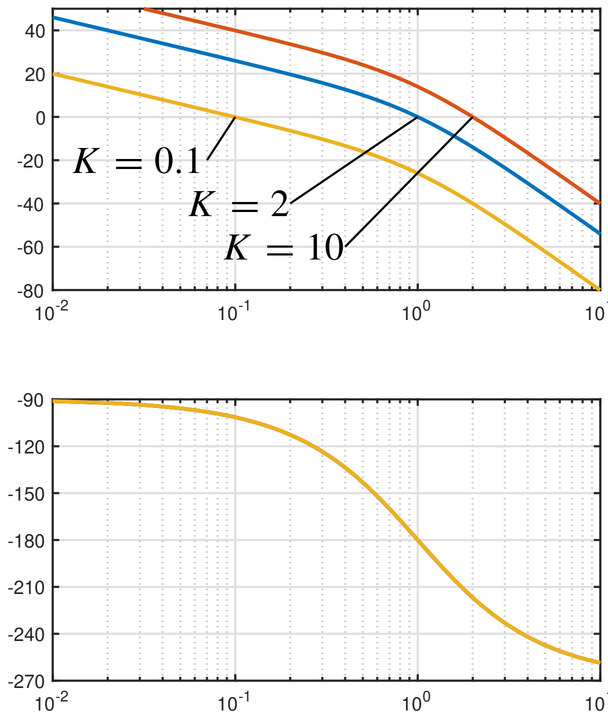
EXAMPLE:





- We see neutral stability at $K = 2$. The system is stable for $K < 2$ and unstable for $K > 2$.
- Recall that a point is on the root locus if $|KG(s)| = 1$ and $\angle G(s) = -180°$.
- If system is neutrally stable, $j\omega$-axis will have a point (points) where $|KG(j\omega)| = 1$ and $\angle G(j\omega) = -180°$.

- Consider the Bode plot of $KG(s)\ldots$
- A neutral-stability condition from Bode plot is: $|KG(j\omega_o)| = 1$ *AND* $\angle KG(j\omega_o) = -180^\circ$ at the same frequency ω_o.
- In this case, increasing $K \rightarrow$instability ➠ $|KG(j\omega)| < 1$ at $\angle KG(j\omega) = -180^\circ$ =stability.
- *In some cases*, decreasing $K \rightarrow$instability ➠ $|KG(j\omega)| > 1$ at $\angle KG(j\omega) = -180^\circ$ =stability.



KEY POINT: We can find neutral stability point on Bode plot, but don't (yet) have a way of determining if the system is stable or not. Nyquist found a frequency-domain method to do so.

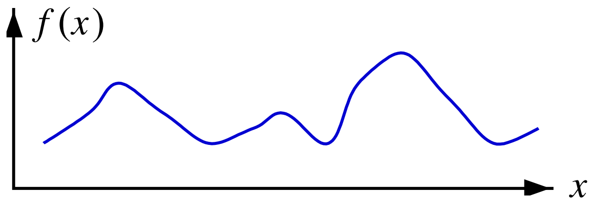
Nyquist stability

- Poles of closed-loop transfer function in RHP—the system is unstable.
- Nyquist found way to count closed-loop poles in RHP.
- If count is greater than zero, system is unstable.
- Idea:
  - First, find a way to count closed-loop poles inside a contour.
  - Second, make the contour equal to the RHP.
- Counting is related to complex functional mapping.

8.9: Interlude: Complex functional mapping

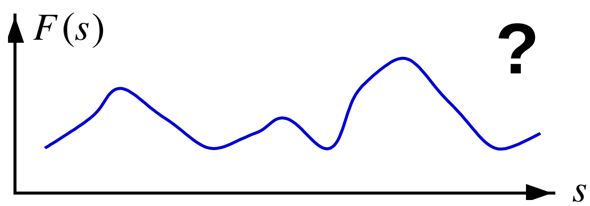
- Nyquist technique is a graphical method to determine system stability, regions of stability and *MARGINS* of stability.
- Involves graphing complex functions of s as a polar plot.

EXAMPLE: Plotting $f(x)$, a real function of a real variable x.



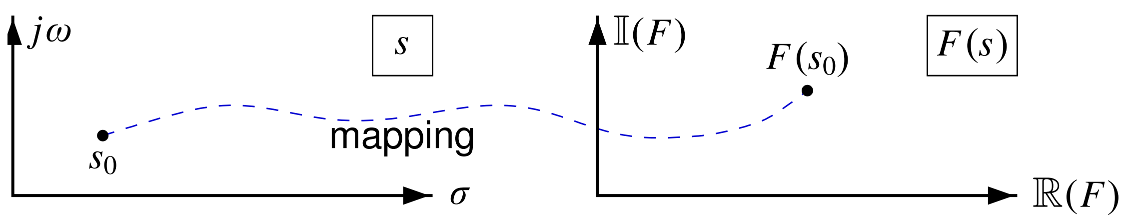
- This can be done.

EXAMPLE: Plotting $F(s)$, a complex function of a complex variable s.

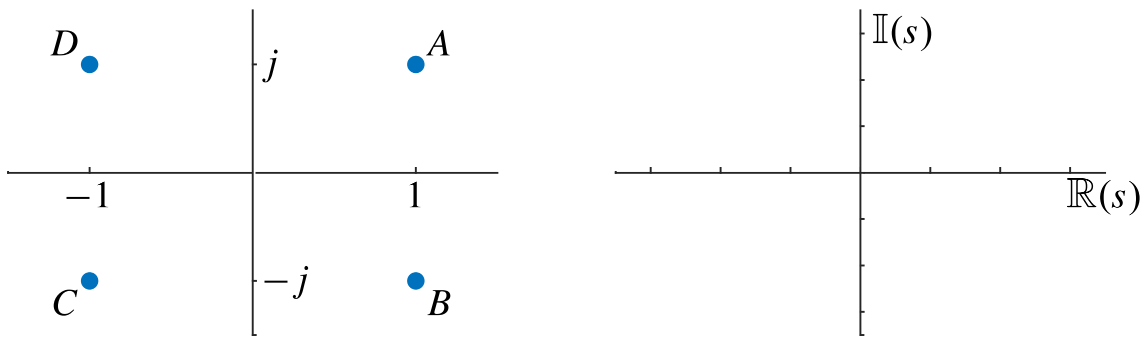


NO! This is wrong!

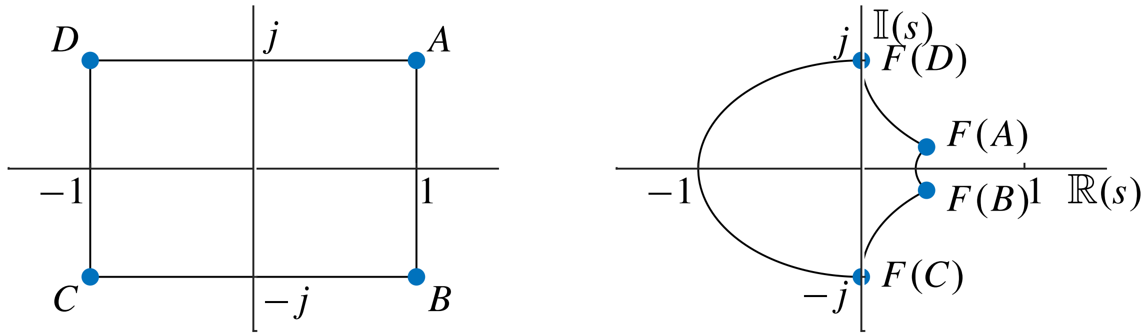
- Must draw mapping of points or lines from s-plane to $F(s)$-plane.



EXAMPLE: $F(s) = 2s + 1 \ldots$ "map the four points: A, B, C, D"



EXAMPLE: Map a square contour (closed path) by $F(s) = \dfrac{s}{s+2}$.



FORESHADOWING: By drawing maps of a specific contour, using a mapping function related to the plant open-loop frequency-response, we will be able to determine closed-loop stability of systems.

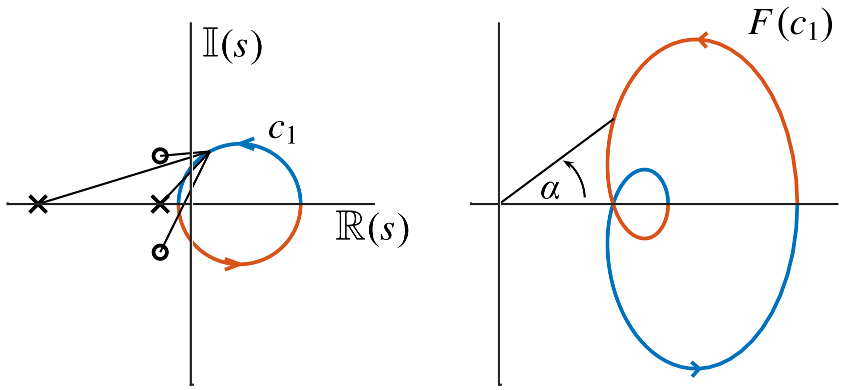
Mapping function: Poles of the function

- When we map a contour containing (encircling) poles and zeros of the mapping function, this *map* will give us information about how many poles and zeros are encircled by the contour.
- Practice drawing maps when we know poles and zeros. Evaluate

$$G(s)|_{s=s_o} = G(s_o) = |\vec{v}|e^{j\alpha}$$

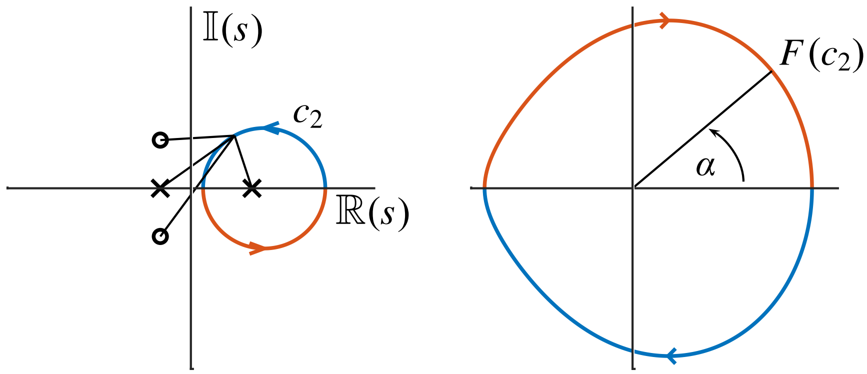
$$\alpha = \sum \angle(\text{zeros}) - \sum \angle(\text{poles}).$$

EXAMPLE:



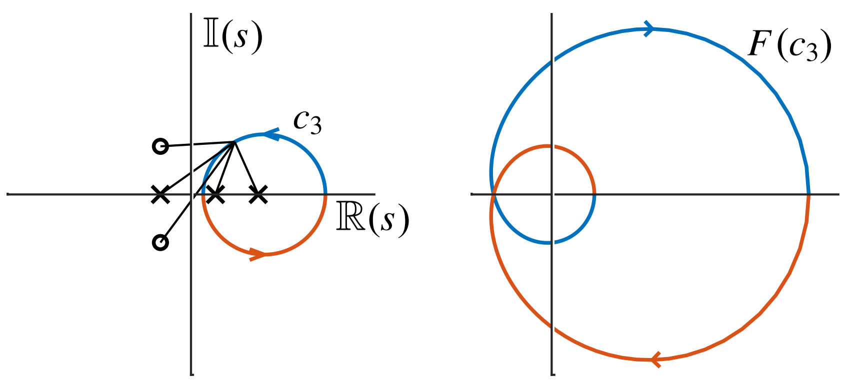
- In this example, there are no zeros or poles inside the contour. The phase α increases and decreases, but never undergoes a net change of 360° (does not encircle the origin).

EXAMPLE:



- One pole inside contour. Resulting map undergoes 360° net phase change. (Encircles the origin).

EXAMPLE:



- In this example, there are two poles inside the contour, and the map encircles the origin twice.

8.10: Cauchy's theorem and Nyquist's rule

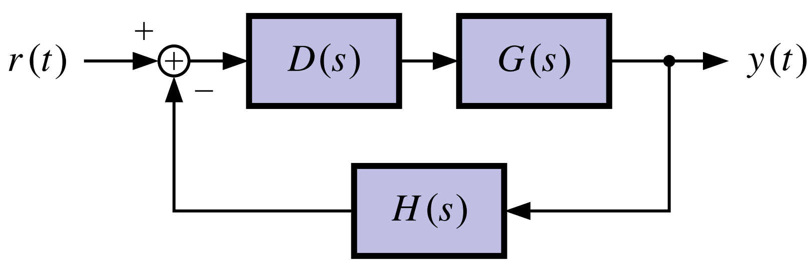
- These examples give heuristic evidence of the general rule: Cauchy's theorem

  "Let $F(s)$ be the ratio of two polynomials in s. Let the closed curve C in the s-plane be mapped into the complex plane through the mapping $F(s)$. If the curve C does not pass through any zeros or poles of $F(s)$ as it is traversed in the CW direction, the corresponding map in the $F(s)$-plane encircles the origin $N = Z - P$ times in the CW direction," where

$$Z = \#\text{ of zeros of } F(s) \text{ in } C,$$

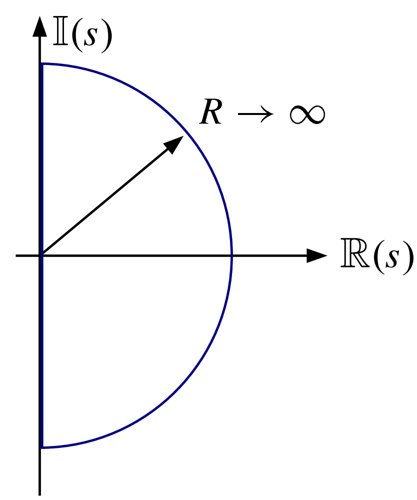
$$P = \#\text{ of poles of } F(s) \text{ in } C.$$

- Consider the following feedback system:



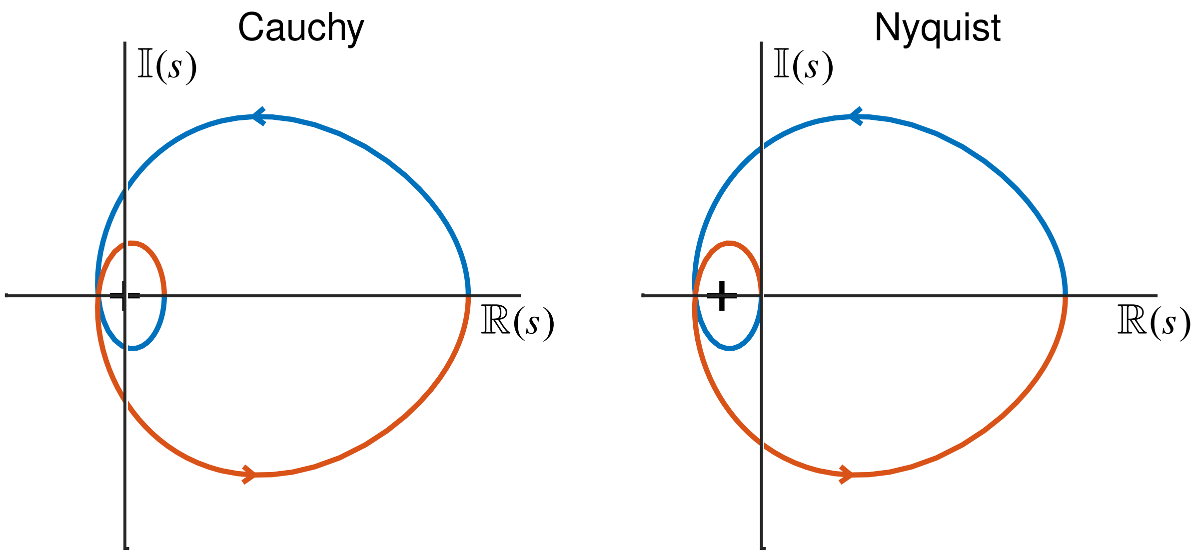
$$T(s) = \frac{D(s)G(s)}{1 + D(s)G(s)H(s)}.$$

- For closed-loop stability, no poles of $T(s)$ in RHP.
  - No zeros of $1 + D(s)G(s)H(s)$ in RHP.
  - Let $F(s) = 1 + D(s)G(s)H(s)$.
  - Count zeros in RHP using Cauchy theorem! (Contour=entire RHP).

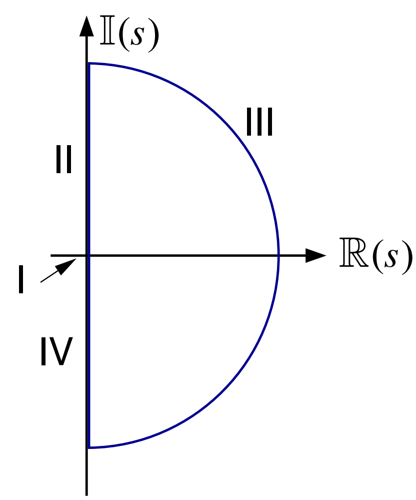


- The Nyquist criterion simplifies Cauchy's criterion for feedback systems of the above form.

- Cauchy: $F(s) = 1 + D(s)G(s)H(s)$. N = # of encirclements of origin.
- Nyquist: $F(s) = D(s)G(s)H(s)$. N = # of encirclements of -1.



- Simple? YES!!!
- Think of Nyquist path as four parts:



I. Origin. Sometimes a special case (later examples).

II. $+j\omega$-axis. FREQUENCY-response of O.L. system! Just plot it as a polar plot.

III. For physical systems=0.

IV. Complex conjugate of II.

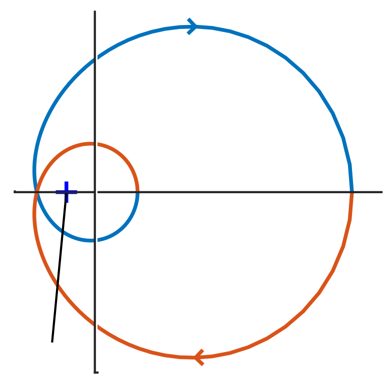
- So, for most physical systems, the Nyquist plot, used to determine *CLOSED-LOOP* stability, is merely a polar plot of *LOOP* frequency response $D(j\omega)G(j\omega)H(j\omega)$.
- We don't even need a mathematical model of the system. Measured data of $G(j\omega)$ combined with our known $D(j\omega)$ and $H(j\omega)$ are enough to determine closed-loop stability.

THE TEST:

- N = # encirclements of -1 point when $F(s) = D(s)G(s)H(s)$.
- P = # poles of $1 + F(s)$ in RHP= # of open-loop unstable poles. (assuming that $H(s)$ is stable—reasonable).
- Z = # of zeros of $1 + F(s)$ in RHP= # of closed-loop unstable poles.

$$Z = N + P$$

The system is stable iff $Z = 0$.



- Be careful counting encirclements!
- Draw line from -1 in any direction.
- Count # crossings of line and diagram.
- N = #CW crossings−#CCW crossings.

- Changing the gain K of $F(s)$ *MAGNIFIES* the entire plot.

ENHANCED TEST: Loop transfer function is $KD(s)G(s)H(s)$.

- N = # encirclements of $-1/K$ point when $F(s) = D(s)G(s)H(s)$.
- Rest of test is the same.
- Gives ranges of K for stability.

8.11: Nyquist test example

EXAMPLE: $D(s) = H(s) = 1$.

$$G(s) = \frac{5}{(s+1)^2}$$

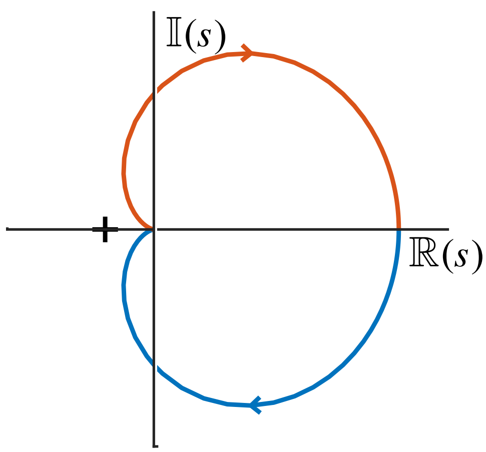
$$\text{or, } G(j\omega) = \frac{5}{(j\omega+1)^2}$$

I: At $s = 0$, $G(s) = 5$.

II: At $s = j\omega$, $G(j\omega) = \dfrac{5}{(1+j\omega)^2}$.

III: At $|s| = \infty$, $G(s) = 0$.

IV: At $s = -j\omega$, $G(s) = \dfrac{5}{(1-j\omega)^2}$.



ω	$\mathbb{R}(G(j\omega))$	$\mathbb{I}(G(j\omega))$
0.0000	5.0000	0.0000
0.0019	4.9999	−0.0186
0.0040	4.9998	−0.0404
0.0088	4.9988	−0.0879
0.0191	4.9945	−0.1908
0.0415	4.9742	−0.4135
0.0902	4.8797	−0.8872
0.1959	4.4590	−1.8172
0.4258	2.9333	−3.0513
0.9253	0.2086	−2.6856
2.0108	−0.5983	−0.7906
4.3697	−0.2241	−0.1082
9.4957	−0.0536	−0.0114
20.6351	−0.0117	−0.0011
44.8420	−0.0025	−0.0001
97.4460	−0.0005	−0.0000
500.0000	−0.0000	−0.0000

- No encirclements of -1, $N = 0$.
- No open-loop unstable poles $P = 0$.
- $Z = N + P = 0$. Closed-loop system is stable.
- No encirclements of $-1/K$ for any $K > 0$.
  - So, system is stable for any $K > 0$.

- Confirm by checking Routh array.
- Routh array: $a(s) = 1 + KG(s) = s^2 + 2s + 1 + 5K$.

$$\begin{array}{c|cc} s^2 & 1 & 1+5K \\ s^1 & 2 & \\ s^0 & 1+5K & \end{array}$$

- Stable for any $K > 0$.

EXAMPLE: $G(s) = \dfrac{50}{(s+1)^2(s+10)}$.

I: $G(0) = 50/10 = 5$.

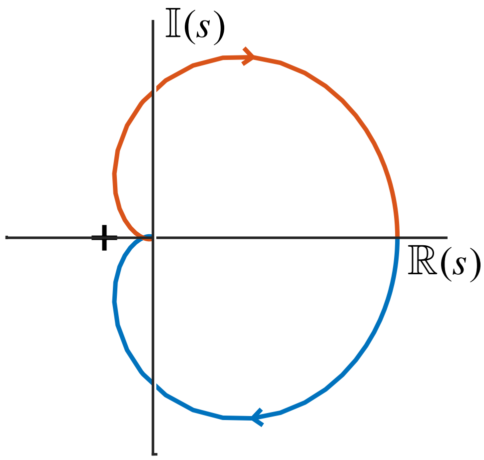
II: $G(j\omega) = \dfrac{50}{(j\omega+1)^2(j\omega+10)}$.

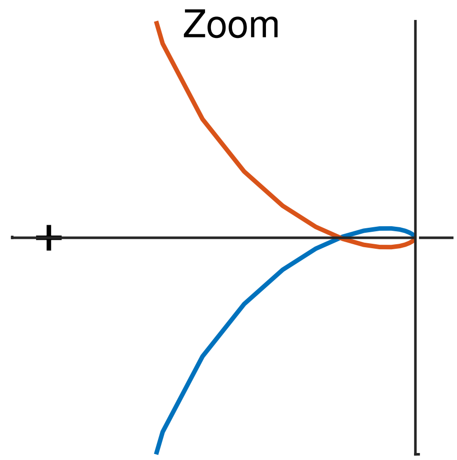
III: $G(\infty) = 0$.

IV: $G(-j\omega) = G(j\omega)^*$.

- Note loop to left of origin. System is *NOT* stable for all $K > 0$.

ω	$\mathbb{R}(G(j\omega))$	$\mathbb{I}(G(j\omega))$
0	5.0000	0
0.1	4.9053	−0.8008
0.2	4.4492	−1.8624
0.5	2.4428	−3.2725
1.2	−0.5621	−2.0241
2.9	−0.4764	−0.1933
7.1	−0.0737	0.0262
17.7	−0.0046	0.0064
43.7	−0.0002	0.0006
100.0	−0.0000	0.0000

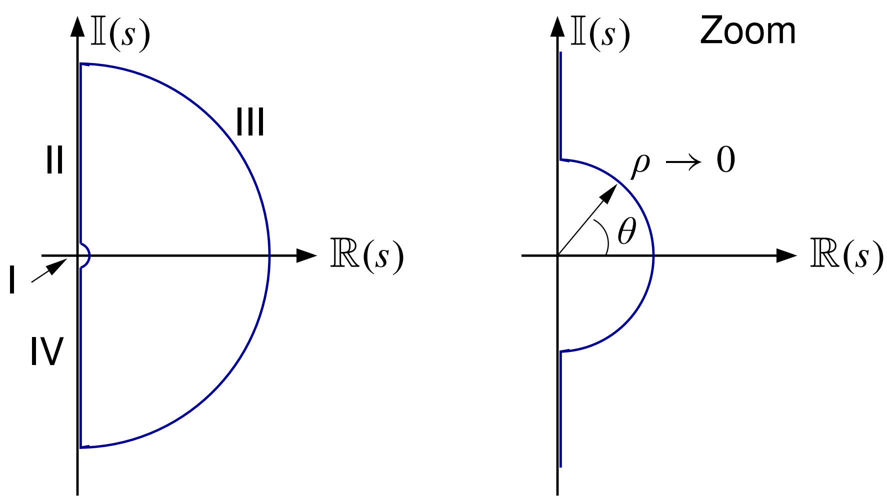




8.12: Nyquist test example with pole on $j\omega$-axis

EXAMPLE: Pole(s) at origin. $G(s) = \dfrac{1}{s(\tau s + 1)}$.

- *WARNING*! We cannot blindly follow procedure!
- Nyquist path goes through pole at zero! (Remember from Cauchy's theorem that the path cannot pass directly through a pole or zero.)
- Remember: We want to count closed-loop poles inside a "box" that encompasses the RHP.
- So, we use a slightly-modified Nyquist path.



- The bump at the origin makes a detour around the offending pole.
- Bump defined by curve: $s = \lim_{\rho \to 0} \rho e^{j\theta}, \qquad 0^\circ \le \theta \le 90^\circ$.
- From above,

$$G(s)|_{s=\rho e^{j\theta}} = \frac{1}{\rho e^{j\theta}(\tau \rho e^{j\theta} + 1)}, \qquad 0^\circ \le \theta \le 90^\circ$$

- Consider magnitude as $\rho \to 0$

$$\lim_{\rho \to 0} |G(s)|_{s=\rho e^{j\theta}} = \frac{1}{\rho|\tau \rho e^{j\theta} + 1|} \approx \frac{1}{\rho}.$$

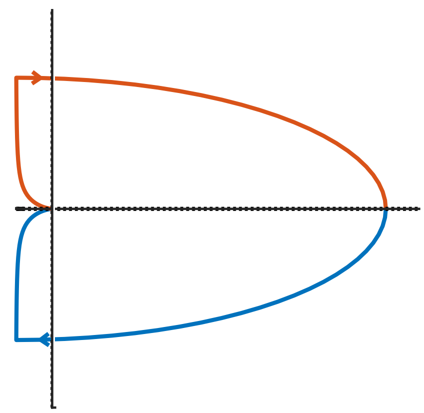
- Consider phase as $\rho \to 0$

$$\lim_{\rho\to 0} \angle\, G(s)|_{s=\rho e^{j\theta}} = -\theta - \angle(\tau\rho e^{j\theta}+1).$$

- So,

$$\lim_{\rho\to 0} G(\rho e^{j\theta}) = \lim_{\rho\to 0} \frac{1}{\rho}\angle -\theta^{+}$$

  - This is an arc of infinite radius, sweeping from 0° to $-90^{\circ+}$ (a little more than 90° because of contribution from $\dfrac{1}{(\tau s+1)}$ term).
  - *WE CANNOT DRAW THIS TO SCALE!*



- $Z = N + P$.
- N = # encirclements of -1. $N = 0$.
- P = # Loop transfer function poles inside *MODIFIED* contour. $P = 0$.
- $Z = 0$. Closed-loop system is stable.

EXAMPLE:

$$G(s) = \frac{1}{s^2(s+1)}$$

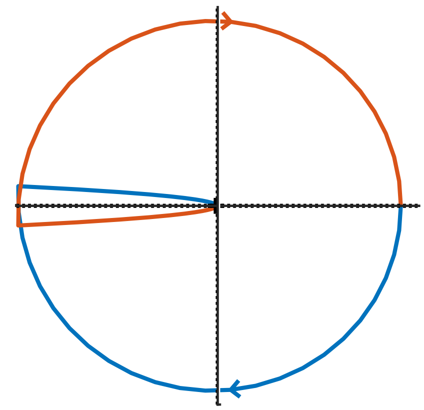
- Use modified Nyquist path again

I: Near origin

$$G(s)|_{s=\rho e^{j\theta}} = \frac{1}{\rho^2 e^{j2\theta}(1+\rho e^{j\theta})}.$$

- Magnitude: $\lim_{\rho\to 0} |G(\rho e^{j\theta})| = \dfrac{1}{\rho^2|1+\rho e^{j\theta}|} \approx \dfrac{1}{\rho^2}$.
- Phase: $\lim_{\rho\to 0} \angle G(\rho e^{j\theta}) = 0 - [2\theta + \angle(1+\rho e^{j\theta})] \approx -2\theta^{+}$. So,

$$\lim_{\rho\to 0} G(\rho e^{j\theta}) = \lim_{\rho\to 0} \frac{1}{\rho^2}\angle -2\theta^{+} \qquad 0^\circ \le \theta \le 90^\circ.$$



- Infinite arc from 0° to $-180^{\circ+}$ (a little more than -180° because of $\dfrac{1}{1+s}$ term.)
- $Z = N + P = 2 + 0 = 2$. Unstable for $K = 1$.
- In fact, unstable for any $K > 0$!
- Matlab for above

$$G(s) = \frac{1}{s^3 + s^2 + 0s + 0}$$

```
num=[0 0 0 1];
den=[1 1 0 0];
nyquist1(num,den);
axis([xmin xmax ymin ymax]);
```

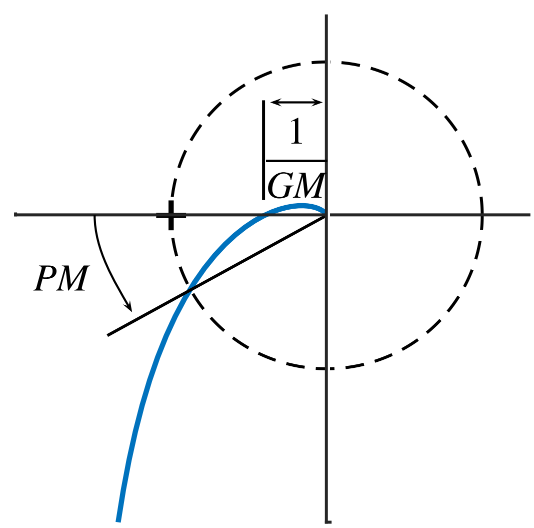
- “nyquist1.m” is available on course web site.
- It repairs the standard Matlab “nyquist.m” program, which doesn’t work when poles are on imaginary axis.
- “nyquist2.m” is also available. It draws contours around poles on the imaginary axis in the opposite way to “nyquist1.m”. Counting is different.

8.13: Stability (gain and phase) margins

- A large fraction of systems to be controlled are stable for small gain but become unstable if gain is increased beyond a certain point.
- The distance between the current (stable) system and an unstable system is called a "stability margin."
- Can have a gain margin and a phase margin.

GAIN MARGIN: Factor by which the gain is less than the neutral stability value.

- Gain margin measures "How much can we increase the gain of the loop transfer function $L(s) = D(s)G(s)H(s)$ and still have a stable system?"
- Many Nyquist plots are like this one. Increasing loop gain magnifies the plot.
- GM =1/(distance between origin and place where Nyquist map crosses real axis).
- If we increase gain, Nyquist map "stretches" and we may encircle -1.



- For a stable system, $GM > 1$ (linear units) or $GM >$ 0 dB.

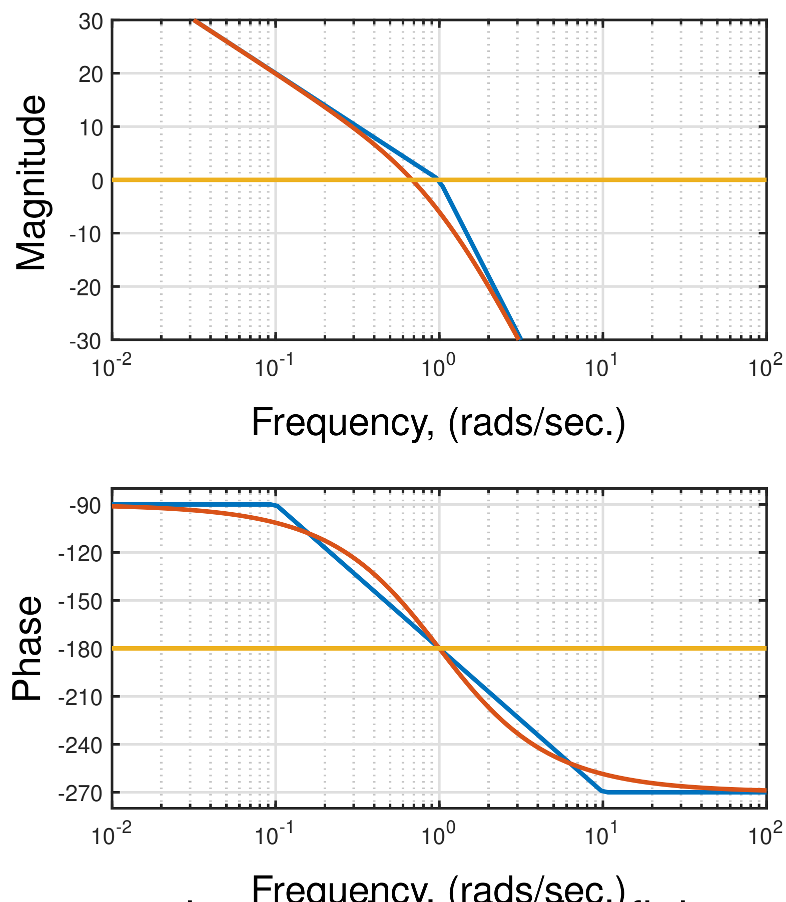
PHASE MARGIN: Phase factor by which phase is greater than neutral stability value.

- Phase margin measures "How much delay can we add to the loop transfer function and still have a stable system?"

- $PM =$ Angle to rotate Nyquist plot to achieve neutral stability = intersection of Nyquist with circle of radius 1.
- If we increase open-loop delay, Nyquist map "rotates" and we may encircle -1.
- For a stable system, $PM > 0°$.

IRONY: This is usually easiest to check on Bode plot, even though derived on Nyquist plot!

- Define gain crossover as frequency where Bode magnitude is 0 dB.
- Define phase crossover as frequency where Bode phase is $-180°$.



- $GM = 1/$(Bode gain at phase-crossover frequency) if Bode gain is measured in linear units.
- $GM = (-$ Bode gain at phase-crossover frequency) [dB] if Bode gain measured in dB.
- $PM =$ Bode phase at gain-crossover $-(-180°)$.
- We can also determine stability as K changes. Instead of defining gain crossover where $|G(j\omega)| = 1$, use the frequency where $|KG(j\omega)| = 1$.

- You need to be careful using this test.
  - It works if you apply it blindly and the system is minimum-phase.
  - You need to think harder if the system is nonminimum-phase.
  - Nyquist is the safest bet.

PM and performance

- A bonus of computing PM from the open-loop frequency response graph is that it can help us predict closed-loop system performance.
- PM is related to damping. Consider open-loop 2*nd*-order system

$$G(s) = \frac{\omega_n^2}{s(s+2\zeta\omega_n)}$$

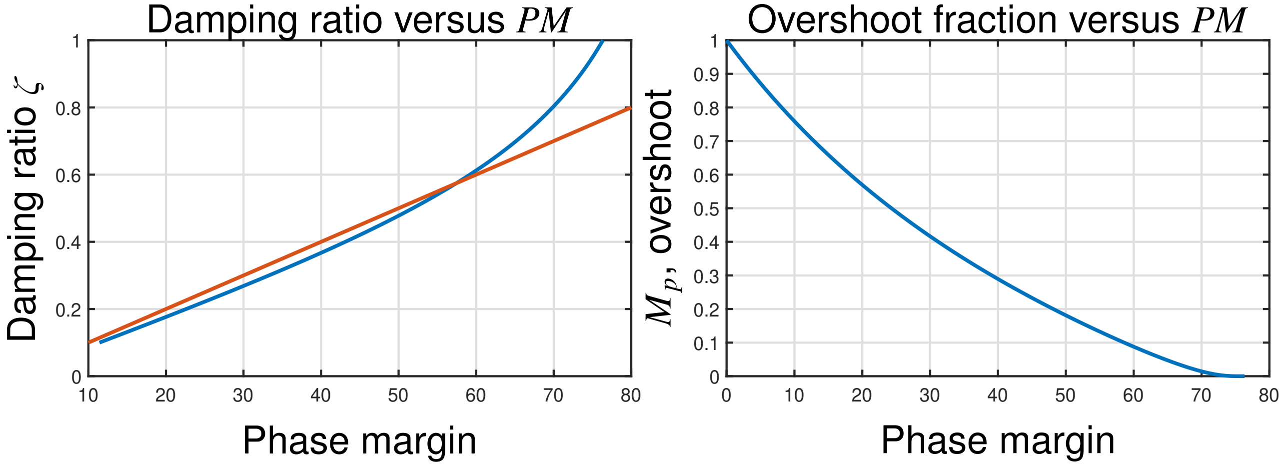
with unity feedback,

$$T(s) = \frac{\omega_n^2}{s^2+2\zeta\omega_n+\omega_n^2}.$$

- The relationship between PM and ζ is: (for this system)

$$PM = \tan^{-1}\left[\frac{2\zeta}{\sqrt{\sqrt{1+4\zeta^4}-2\zeta^2}}\right]$$

- For $PM \le 60^\circ$, $\zeta \approx \dfrac{PM}{100}$, so can also infer M_p from PM.



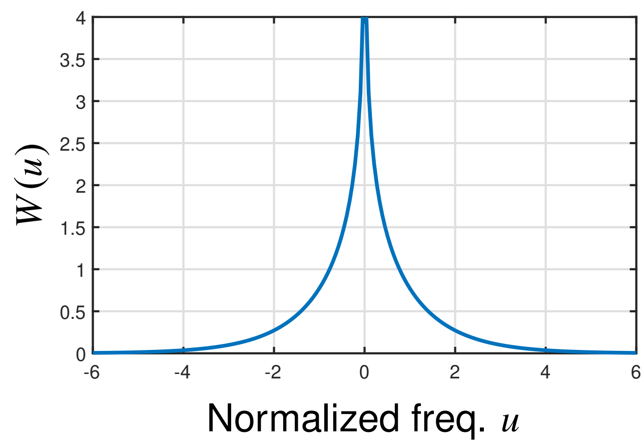
8.14: Preparing for control using frequency-response methods

Bode's gain-phase relationship

- "For any stable minimum-phase system (that is, one with no RHP zeros or poles), the phase of $G(j\omega)$ is uniquely related to the magnitude of $G(j\omega)$"
- Relationship: $\angle G(j\omega_o) = \dfrac{1}{\pi}\displaystyle\int_{-\infty}^{\infty}\left(\frac{\mathrm{d}M}{\mathrm{d}u}\right)W(u)\,\mathrm{d}u$ (in radians)

$$M = \ln|G(j\omega)|$$
$$u = \ln\left(\frac{\omega}{\omega_o}\right)$$
$$\frac{\mathrm{d}M}{\mathrm{d}u} \approx \text{slope } n \text{ of log-mag curve at } \omega = \omega_o$$
$$W(u) = \text{weighting function} = \ln(\coth|u|/2)$$

- $W(u) \approx \dfrac{\pi^2}{2}\delta(u)$. Using this relationship, $\angle G(j\omega) \approx n \times 90^\circ$ if slope of Bode magnitude-plot is constant in the decade-neighborhood of ω.



- So, if $\angle G(j\omega) \approx -90^\circ$ if $n = -1$ (or, in more familiar terms, if the slope of the Bode magnitude plot is -20 dB decade^{-1}).
- So, if $\angle G(j\omega) \approx -180^\circ$ if $n = -2$ (or, in more familiar terms, if the Bode-magnitude slope is -40 dB decade^{-1}).

KEY POINT: Want crossover $|G(j\omega)| = 1$ at a slope of about $n = -1$ (i.e., -20 dB decade^{-1}) for good PM. We'll soon see how to do this (design!).

Closed-loop frequency response

- Most of the notes in this section have used the open-loop frequency response to predict closed-loop behavior.
- How about closed-loop frequency response?

$$T(s) = \frac{KD(s)G(s)}{1 + KD(s)G(s)}.$$

- General approximations are simple to make. If,

$$|KD(j\omega)G(j\omega)| \gg 1 \qquad \text{for } \omega \ll \omega_c$$

$$\text{and } |KD(j\omega)G(j\omega)| \ll 1 \qquad \text{for } \omega \gg \omega_c$$

where ω_c is the cutoff frequency where open-loop magnitude response crosses magnitude=1.

$$|T(j\omega)| = \left|\frac{KD(j\omega)G(j\omega)}{1 + KD(j\omega)G(j\omega)}\right| \approx \begin{cases} 1, & \omega \ll \omega_c; \\ |KD(j\omega)G(j\omega)|, & \omega \gg \omega_c. \end{cases}$$

- Note: $\omega_c \le \omega_{\text{bw}} \le 2\omega_c$.