ROOT-LOCUS ANALYSIS

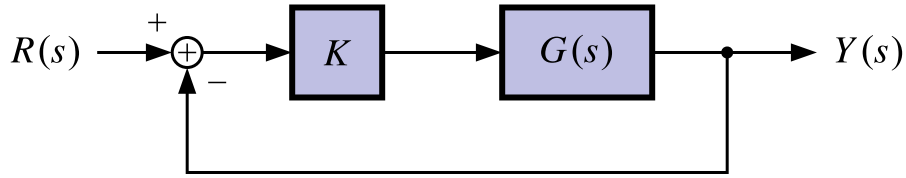
6.1: Manually plotting a root locus

- Recall step response: transfer-function pole locations determine performance characteristics such as rise and settling time, overshoot.
- We have also seen that feedback can change pole locations in the system transfer function and therefore performance is changed.
- Suppose that we have a single variable design parameter in our control system and we wish to select its value.
  - We could consider making a parametric plot of the system's pole locations as that parameter's value changes.
  - Then, select the value that gives "best" combination of pole locations (w.r.t. desired performance specs.).
- Poles are the *roots* of the denominator of the transfer function (*a.k.a.* the "characteristic polynomial."), and we are plotting all possible sets of locations (*i.e.*, a *locus*) of these poles versus a single parameter, the resulting plot is called an Evans root-locus plot.

VERY IMPORTANT NOTE: Root locus is a parametric plot (vs. K) of the roots of an equation

$$1 + K\frac{b(s)}{a(s)} = 0 \quad \text{or} \quad a(s) + Kb(s) = 0.$$

- For now, we specialize to a common control configuration: unity feedback, proportional gain (we'll generalize configuration later)



- Closed-loop transfer function

$$T(s) = \frac{KG(s)}{1 + KG(s)}.$$

- Poles = roots of $1 + KG(s) = 0$.

■ Assume plant transfer function $G(s)$ is rational polynomial:

$$G(s) = \frac{b(s)}{a(s)},$$

such that

$$b(s) = (s - z_1)(s - z_2) \cdots (s - z_m) \qquad (b(s) \text{ is monic.})$$

$$a(s) = (s - p_1)(s - p_2) \cdots (s - p_n) \qquad n \geq m \qquad (a(s) \text{ is monic.})$$

[$a(s)$ may be assumed monic without loss of generality. If $b(s)$ is not monic, then its gain is just absorbed as part of K in $1 + KG(s) = 0$]

- z_i are zeros of $G(s)$, the *OPEN-LOOP* transfer function.
- p_i are poles of $G(s)$, the *OPEN-LOOP* transfer function.

■ *CLOSED-LOOP* poles are roots of equation

$$1 + KG(s) = 0$$

$$a(s) + Kb(s) = 0,$$

which clearly move as a function of K.

■ Zeros are unaffected by feedback.

EXAMPLE: $G(s) = \dfrac{1}{s(s+2)}$. Find the root locus.

- $a(s) = s(s+2)$; $b(s) = 1$.
- Locus of roots: (aside: stable for all $K > 0$.)

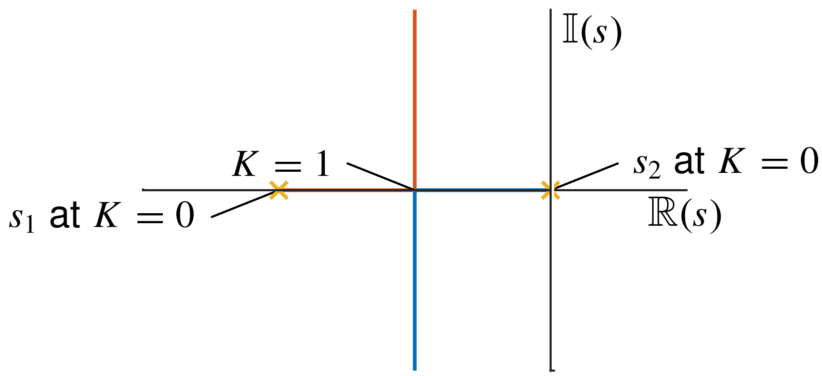
$$s(s+2) + K = 0$$

$$s^2 + 2s + K = 0.$$

- For this simple system we can easily solve for the roots.

$$s_{1,2} = \frac{-2 \pm \sqrt{4 - 4K}}{2} = -1 \pm \sqrt{1 - K}.$$

- Roots are real and negative for $0 < K < 1$.
- Roots are complex conjugates for $K > 1$.



- Suppose we want damping ratio $\zeta = 0.707$. We can recall that

$$\left.\begin{array}{rcl} s^2 + 2\zeta\omega_n s + \omega_n^2 & = & 0 \\ & \Updownarrow & \\ s^2 + 2s + K & = & 0 \end{array}\right\} \quad K = 2$$

or we can locate the point on the root locus where $|\mathbb{R}\{\text{poles}\}| = |\mathbb{I}\{\text{poles}\}|$.

$$|-1| = |\sqrt{1 - K}|$$

$$2 = K.$$

(or, $K = 0$, not an appropriate solution).

6.2: Root-locus plotting rule #1

- Factoring a quadratic is okay; factoring a cubic or quartic is painful; factoring a higher-order polynomial is not possible in closed form, in general.
- So, we seek methods to plot a root locus that do not require actually solving for the root locations for every value of K.
- Assuming still that $a(s)$ and $b(s)$ are monic (which may mean that a gain factor is absorbed into K), we first consider plotting the locus of roots for $0 \le K \le \infty$.
  - This is a standard "$180°$" root locus.
  - If $-\infty \le K \le 0$, then a similar procedure will yield a "$0°$" root locus.
- A point s_1 is on the root locus if $1 + KG(s_1) = 0$.
- Equivalently, $K = \dfrac{-1}{G(s)}$. Because $G(s)$ is complex, this is really two equations!

$$|K| = \left|\frac{1}{G(s)}\right| \tag{6.1}$$

$$\angle G(s) = \angle\left(\frac{-1}{K}\right). \tag{6.2}$$

- Since K is real and positive, $\angle K = 0$.
  Therefore, $\angle G(s) = 180° \pm l360°, \qquad l = 0, 1, 2, \ldots$
- So once we know a point on the root locus, we can use the magnitude equation Eq. (6.1) to find the gain K that produced it.
- We will use the angle equation Eq. (6.2) to plot the locus. *i.e.*, the locus of the roots = all points on s-plane where $\angle G(s) = 180° \pm l360°$.

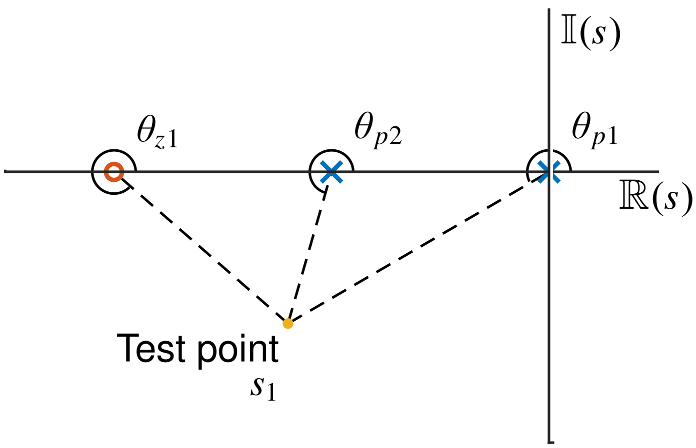
NOTES:

1. We'll learn techniques so we don't need to test each s-plane point!
2. The angle criteria explains the term "180° root locus" when $K \ge 0$.
3. It also explains the term "0° root locus" when $K \le 0$.

KEY TOOL: For any point on the s-plane

$$\angle G(s) = \sum \angle \text{ (due to zeros)} - \sum \angle \text{ (due to poles)}.$$

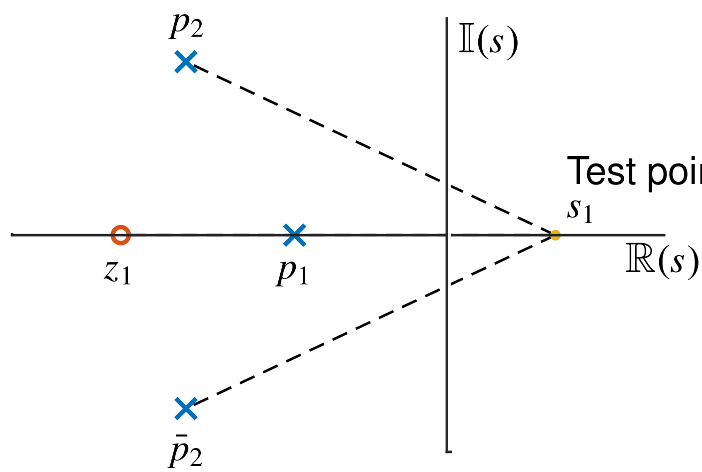
EXAMPLE: $G(s) = \dfrac{(s - z_1)}{(s - p_1)(s - p_2)}$.



$$\angle G(s_1) = \theta_{z_1} - \theta_{p_1} - \theta_{p_2}.$$

Locus on the real axis

- Consider a test point s_1 on the real axis.
- If the point is right of all poles and zeros of $G(s)$, then $\angle G(s) = 0$.
- *NOT ON THE LOCUS*.



$$\begin{aligned}\angle G(s) &= \angle z_1 - \angle p_1 - \angle p_2 - \angle \bar{p}_2 \\ &= 0 - 0 - \angle p_2 - \angle \bar{p}_2 \\ &= 0.\end{aligned}$$

OBSERVATION: *If* test point is on the real axis, complex-conjugate roots have equal and opposite angles that cancel and may be ignored.

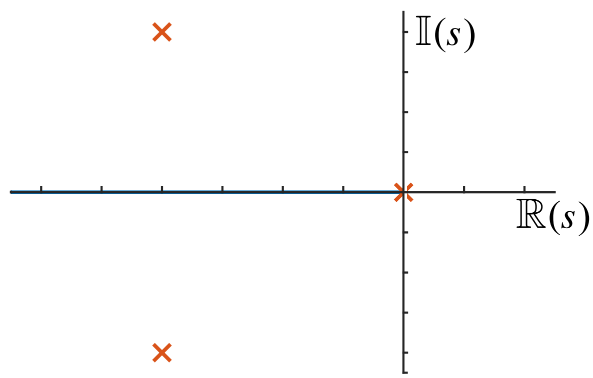
- If the test point is to the left of *ONE* pole or zero, the angle will be $-180°$ or $+180°$ $(= -180°)$ so that point *IS* on the locus.
- If the test point s_1 is to the left of
  - 1 pole and 1 zero: $\angle G(s_1) = 180° - (-180°) = 360° = 0°$.
  - 2 poles: $\angle G(s_1) = -180° - 180° = -360° = 0°$.
  - 2 zeros: $\angle G(s_1) = 180° + 180° = 360° = 0°$.

➠ *NOT ON THE LOCUS.*

> **GENERAL RULE #1**
> All points on the real axis to the *left* of an *odd* number of poles and zeros are part of the root locus.

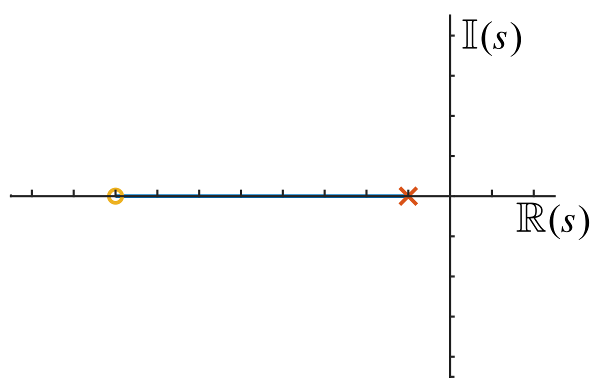
EXAMPLE:

$$G(s) = \frac{1}{s(s+4+4j)(s+4-4j)}.$$



EXAMPLE:

$$G(s) = \frac{s+8}{s+1}.$$



6.3: Root-locus plotting rule #2

Locus not *on the real axis*

OBSERVATION: Because our system models are rational-polynomial with real coefficients, poles must either be real or come as conjugate pairs.

- Thus, **the root locus is symmetric with respect to the real axis**.
- This helps, but is not sufficient for filling in what happens with complex-conjugate poles as K varies from 0 to ∞.
- As a first step in doing so, we consider what happens at limiting cases $K = 0$ and $K = \infty$. Note, we can write $1 + KG(s) = 0$ as

$$(s - p_1)(s - p_2) \cdots (s - p_n) + K(s - z_1)(s - z_2) \cdots (s - z_m) = 0.$$

- At $K = 0$ the closed-loop poles equal the open-loop poles.
- As K approaches ∞, we can rewrite the expression as

$$\frac{(s - p_1)(s - p_2) \cdots (s - p_n)}{K} + (s - z_1)(s - z_2) \cdots (s - z_m) = 0;$$

  the n closed-loop poles approach the zeros of the open-loop transfer function, *INCLUDING THE $n - m$ ZEROS AT $\mathbb{C}^{\infty}$*.
- The m closed-loop poles going to the m open-loop zeros are "easy."
- The $n - m$ remaining poles going to $\mathbb{C}^{\infty}$ are a little more tricky. (Plug $s = \infty$ into $G(s)$ and notice that it equals zero if $m < n$.)
- The idea of ∞ in the complex plane is a number with infinite magnitude and some angle.
- To find where they go as $K \to \infty$, consider that the m finite zeros have canceled m of the poles. Looking back at the remaining $n - m$

poles (standing at $\mathbb{C}^\infty$), we have approximately

$$1 + K\frac{1}{(s-\alpha)^{n-m}} = 0,$$

or $n-m$ poles clustered/centered at α.

- We need to determine α, the center of the locus, and the directions that the poles take.
- Assume $s_1 = Re^{j\phi}$ is on the locus, R large and fixed, ϕ variable. We use geometry to see what ϕ must be for s_1 to be on the locus.
- Since all of the open-loop poles are at approximately the same place α, the angle of $G(s_1)$ is 180° if the $n-m$ angles from α to s_1 sum to 180.

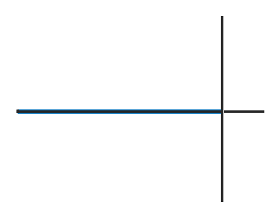
$$(n-m)\phi_l = 180^\circ + l360^\circ, \qquad l = 0, 1, \ldots, n-m-1$$

or

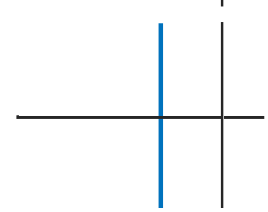
$$\phi_l = \frac{180^\circ + 360^\circ(l-1)}{n-m}, \qquad l = 1, 2, \ldots, n-m.$$

- So, if

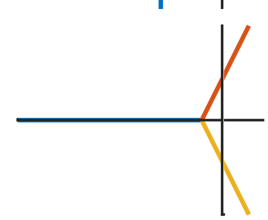
$n-m=1;$ $\phi = +180^\circ.$ There is one pole going to $\mathbb{C}^\infty$ along the negative real axis.



$n-m=2;$ $\phi = \pm 90^\circ.$ There are two poles going to $\mathbb{C}^\infty$ vertically.



$n-m=3;$ $\phi = \pm 60^\circ,\ 180^\circ.$ One goes left and the other two go at plus and minus 60°.



(etc)

- To find the center, α, note that roots of denominator of $G(s)$ satisfy:

$$s^n + a_1 s^{n-1} + a_2 s^{n-2} + \cdots + a_n = (s - p_1)(s - p_2)\cdots(s - p_n)$$

➠ $a_1 = -\sum p_i$.

- Note that the roots of the denominator of $T(s)$ are

$$s^n + a_1 s^{n-1} + a_2 s^{n-2} + \cdots + a_n + K(s^m + b_1 s^{m-1} + \cdots + b_m) = 0.$$

If $n - m > 1$ then the $(n-1)$st coefficient of the closed loop system is such that $a_1 = -\sum r_i$ where r_i are the closed-loop poles.

- We know that m poles go to the zeros of $G(s)$, and assume the other $n - m$ are clustered at $\dfrac{1}{(s-\alpha)^{n-m}}$. Therefore, the asymptotic sum of roots is $(n - m)\alpha + \sum z_i$.
- Putting this all together,

$$\sum r_i = (n - m)\alpha + \sum z_i = \sum p_i$$

or

$$\alpha = \frac{\sum p_i - \sum z_i}{(n - m)}.$$

GENERAL RULE #2

- All poles go from their open-loop locations at $K = 0$ to:
  - The zeros of $G(s)$, or
  - To $\mathbb{C}^\infty$.
- Those going to $\mathbb{C}^\infty$ go along asymptotes

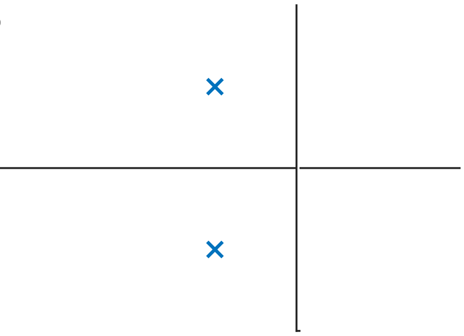
$$\phi_l = \frac{180^\circ + 360^\circ(l-1)}{n - m}$$

centered at

$$\alpha = \frac{\sum p_i - \sum z_i}{n - m}.$$

EXAMPLES:

①



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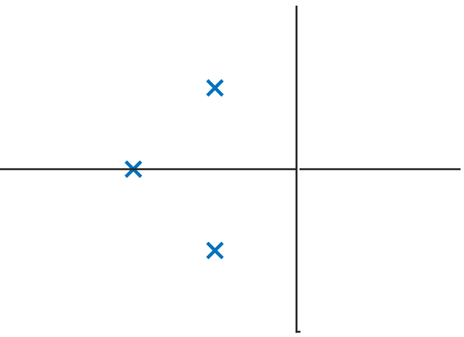

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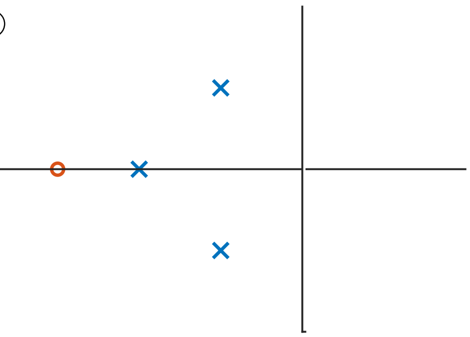

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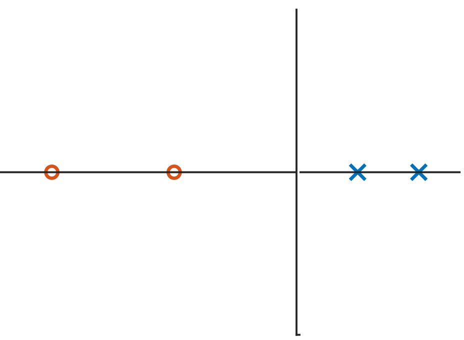
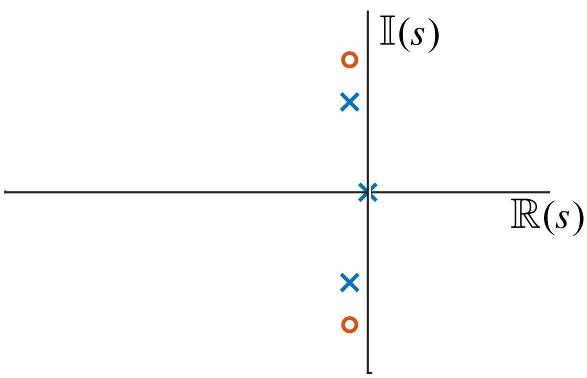


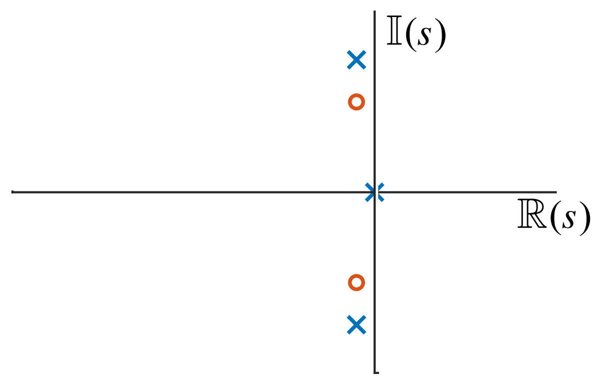
6.4: Additional techniques

- The two general rules given, plus some experience are enough to sketch root loci. Some additional rules help when there is ambiguity. (As in examples ②, ⑤)

Departure angles; arrival angles

- We know asymptotically where poles go, but need to know how they start, and how they end up there.
- Importance: One of the following systems is stable for all $K > 0$, the other is not. Which one?



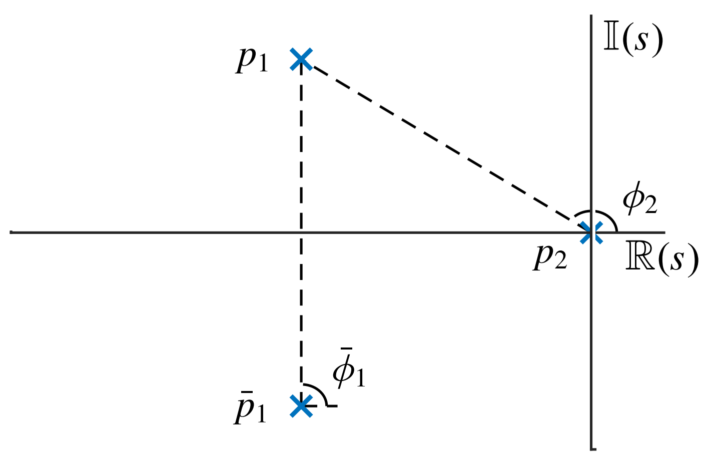


- We will soon be able to answer this. Consider an example:

EXAMPLE:

$$G(s) = \frac{1}{s(s+4+4j)(s+4-4j)}.$$

- Take a test point s_0 very close to p_1. Compute $\angle G(s_0)$.



- If on locus

$$-90^\circ - \phi_1 - 135^\circ = 180^\circ + 360^\circ l \qquad l = 0, 1, \ldots$$

where

$$\bar{\phi}_1 = \text{Angle from } \bar{p}_1 \text{ to } s_0 \approx 90^\circ.$$

$$\phi_1 = \text{Angle from } p_1 \text{ to } s_0.$$

$$\phi_2 = \text{Angle from } p_2 \text{ to } s_0 \approx 135^\circ.$$

$$\Rightarrow \phi_1 = -45^\circ.$$

- Can now draw "departure of poles" on locus.
- Single-pole departure rule:

$$\phi_{\text{dep}} = \sum \angle(\text{zeros}) - \sum \angle(\text{remaining poles}) - 180^\circ \pm 360^\circ l.$$

- Multiple-pole departure rule: (multiplicity $q \geq 1$)

$$q\phi_{\text{dep}} = \sum \angle(\text{zeros}) - \sum \angle(\text{remaining poles}) - 180^\circ \pm 360^\circ l.$$

- Multiple-pole arrival rule: (multiplicity $q \geq 1$)

$$q\,\psi_{\text{arr}} = \sum \angle(\text{poles}) - \sum \angle(\text{remaining zeros}) + 180^\circ \pm 360^\circ l.$$

- Note: The idea of adding $360^\circ l$ is to add enough angle to get the result within $\pm 180^\circ$. Also, if there is multiplicity, then l counts off the different angles.

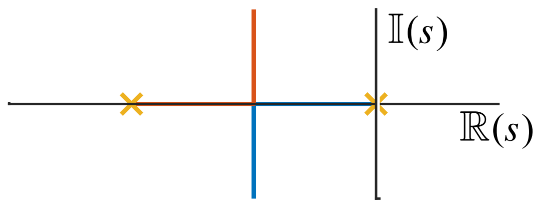
Imaginary axis crossings

- Routh stability test can be run to find value for $K = K'$ that causes marginal stability.

- Substitute K' and find roots of $a(s) + K'b(s) = 0$.
- Alternatively, substitute K', let $s = j\omega_0$, solve for $a(j\omega_0) + K'b(j\omega_0) = 0$. (Real and Imaginary parts)

Points with multiple roots

- Sometimes, branches of the locus intersect. (see ②, ⑤ on pg. 6–10).
- Computing points of intersection can clarify ambiguous loci.
- Consider two poles approaching each other on the real axis:



- As two poles approach each other gain is increasing.
- When they meet, they break away from the real axis and so K increases only for complex parts of the plane.
- Therefore gain K along a branch of the locus is maximum at breakaway point.
- Gain K is minimum along a branch of the locus for arrival points.
- Both are "saddle points" in the s-plane.
- So,

$$\frac{dK}{ds} = 0,$$

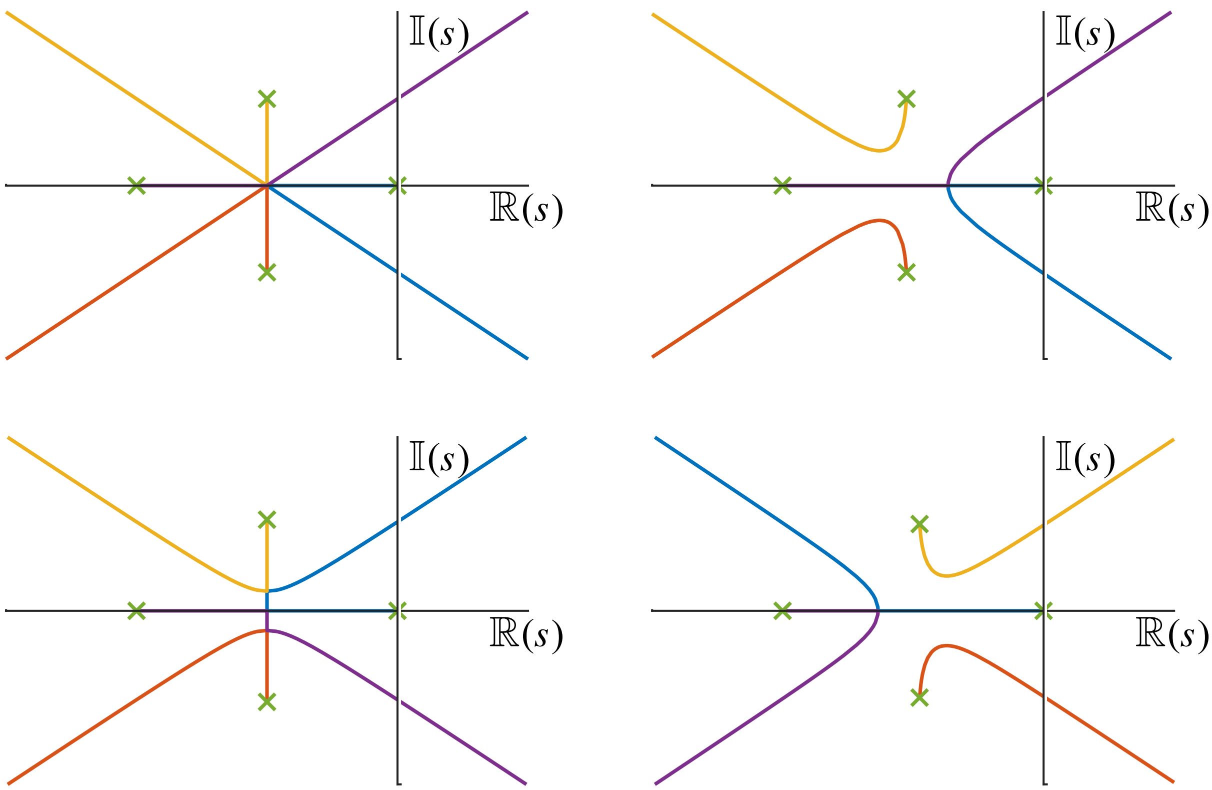
where $1 + KG(s) = 0$

$$K = \frac{-1}{G(s)}$$

$$\text{or,}\quad \frac{\mathrm{d}}{\mathrm{d}s}\left(\frac{-1}{G(s)}\right)_{s=s_0} = 0 \quad \Rightarrow \text{multiple root at } s_0.$$

[must verify that s_0 is on the root locus. May be an extraneous result.]

- Some similar loci for which knowing saddle points clarifies ambiguity:



Finding K for a specific locus point

- Recall the governing rules for a point to be on the root-locus (Eqs. (6.1) and (6.2))

$$|K| = \left|\frac{-1}{G(s)}\right| \qquad \text{and} \qquad \angle G(s) = \angle\left(\frac{-1}{K}\right).$$

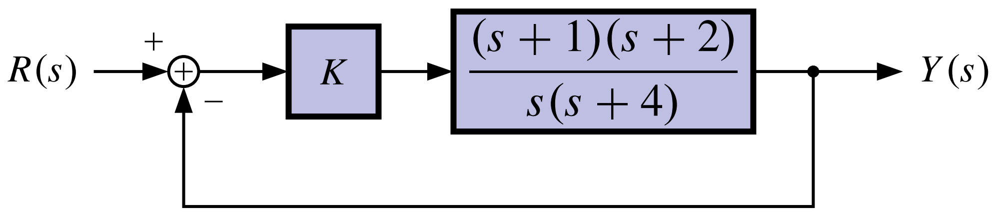
- The phase equation is used to plot the locus.
- The magnitude equation may be used to find the value of K to get a specific set of closed-loop poles.
- That is, $K = \left|\dfrac{-1}{G(s_0)}\right|$ is the gain to put a pole at s_0, if s_0 is on the locus.

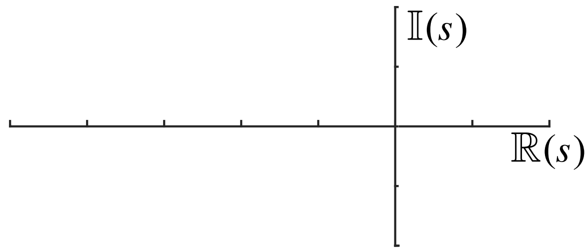
Summary: Root-locus drawing rules (180° locus)

- The steps in drawing a 180° root locus follow from the basic phase definition. This is the locus of
$$1+K\frac{b(s)}{a(s)}=0, \qquad K\geq 0 \qquad \left(\text{phase of } \frac{b(s)}{a(s)}=-180^\circ\right).$$
- They are
  - STEP 1: On the s-plane, mark poles (roots of $a(s)$) by an $\times$ and zeros (roots of $b(s)$) with an $\circ$. There will be a branch of the locus departing from every pole and a branch arriving at every zero.
  - STEP 2: Draw the locus on the real axis to the left of an odd number of real poles plus zeros.
  - STEP 3: Draw the asymptotes, centered at α and leaving at angles ϕ, where
$$\begin{aligned} n-m &= \text{number of asymptotes.}\\ n &= \text{order of } a(s)\\ m &= \text{order of } b(s)\\ \alpha &= \frac{\sum p_i - \sum z_i}{n-m} = \frac{-a_1+b_1}{n-m}\\ \phi_l &= \frac{180^\circ + (l-1)360^\circ}{n-m}, \quad l=1,2,\ldots n-m. \end{aligned}$$
For $n-m>0$, there will be a branch of the locus approaching each asymptote and departing to infinity.
  - STEP 4: Compare locus departure angles from the poles and arrival angles at the zeros where
$$\begin{aligned} q\phi_{\text{dep}} &= \sum \psi_i - \sum \phi_i - 180^\circ \pm l360^\circ\\ q\psi_{\text{arr}} &= \sum \phi_i - \sum \psi_i + 180^\circ \pm l360^\circ, \end{aligned}$$
where q is the order of the pole or zero and l takes on q integer values ($l=0,1,\ldots,q-1$) so that the angles are between ±180° . ψ_i is the angle of the line going from the ith zero to the pole or zero whose angle of departure or arrival is being computed. Similarly, ϕ_i is the angle of the line from the ith pole.
  - STEP 5: If further refinement is required at the stability boundary, assume $s_0=j\omega_0$ and compute the point(s) where the locus crosses the imaginary axis for positive K.
  - STEP 6: For the case of multiple roots, two loci come together at 180° and break away at $\pm 90^\circ$. Three loci segments approach each other at angles of 120° and depart at angles rotated by 60°.
  - STEP 7 Complete the locus, using the facts developed in the previous steps and making reference to the illustrative loci for guidance. The loci branches start at poles and end at zeros or infinity.
  - STEP 8 Select the desired point on the locus that meets the specifications (s_0), then use the magnitude condition to find that the value of K associated with that point is
$$K=\frac{1}{|b(s_0)/a(s_0)|}.$$

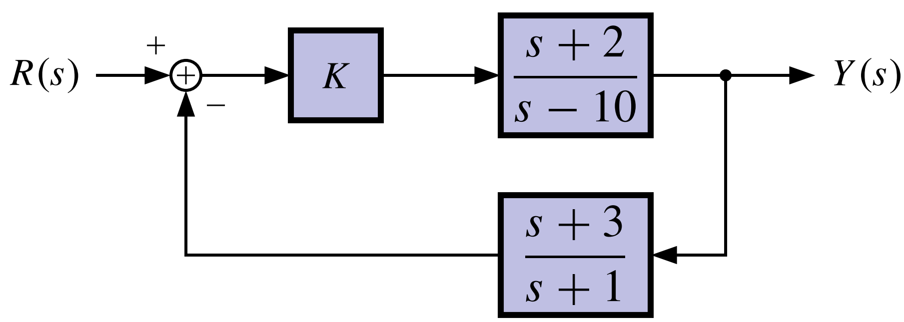
6.5: Some examples

EXAMPLE:





EXAMPLE: This example is *NOT* a unity-feedback case, so we need to be careful.



- Recall that the root-locus rules plot the locus of roots of the equation

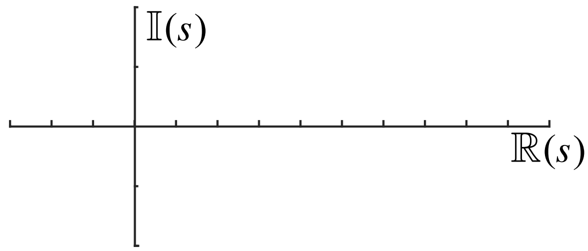
$$1 + K\frac{b(s)}{a(s)} = 0.$$

- Compute $T(s)$ to find the characteristic equation as a function of K.

$$T(s) = \frac{K\frac{s+2}{s-10}}{1 + K\frac{(s+2)(s+3)}{(s+1)(s-10)}} = \frac{K(s+2)(s+1)}{(s+1)(s-10) + K(s+2)(s+3)}.$$

- Poles of $T(s)$ at $(s+1)(s-10) + K(s+2)(s+3) = 0$ or

$$1 + K\frac{(s+2)(s+3)}{(s+1)(s-10)} = 0.$$



1) "Open loop" poles and zeros:

2) Real axis

3) Asymptotes

4) Departure angles

5) Stability boundary: Note, characteristic equation

$$
\begin{aligned}
&= (s+1)(s-10) + K(s+2)(s+3) \\
&= s^2 - 9s - 10 + Ks^2 + 5Ks + 6K \\
&= (K+1)s^2 + (5K-9)s + 6K - 10
\end{aligned}
$$

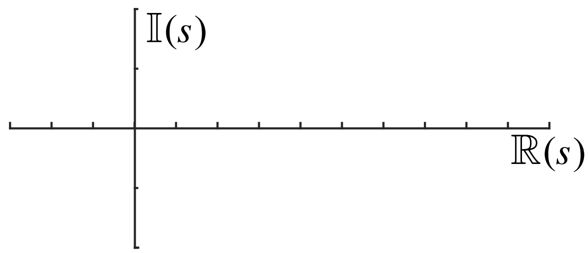
Routh Array

s^2	$K+1$	$6K-10$	$K > -1$	
s^1	$5K-9$		$K > \dfrac{9}{5} = 1.8$	$\leftarrow$ stability criterion
s^0	$6K-10$		$K > \dfrac{10}{6} = 1.66$	

- When $K = 9/5$, the s^1 row of the Routh array is zero—top row becomes a factor of the characteristic equation. Imag.-axis crossings where

$$
\left(\frac{9}{5}+1\right)s^2 + \left(6\frac{9}{5}-10\right) = 0
$$

$$14s^2 + 4 = 0 \qquad \ldots \qquad s = \pm j\sqrt{\frac{2}{7}}.$$



6) Breakaway points

$$K(s) = \frac{-(s+1)(s-10)}{(s+2)(s+3)} = \frac{-(s^2-9s-10)}{s^2+5s+6}$$

$$\frac{\mathrm{d}}{\mathrm{d}s}K(s) = \frac{-(s+1)(s-10)(2s+5)-(-2s+9)(s+2)(s+3)}{(\text{den})^2} = 0$$

$$= \left(-2s^3+18s^2+20s-5s^2+45s+50\right) -$$

$$\left(-2s^3-10s^2-12s+9s^2+45s+54\right) = 0$$

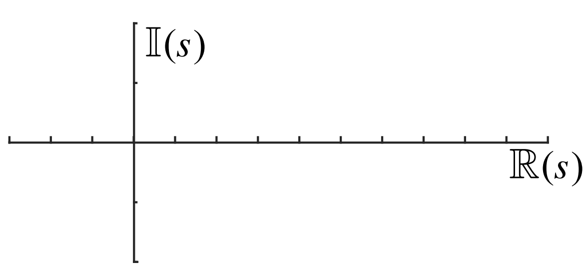
$$= (-2+2)s^3 + (18-5+10-9)s^2 +$$

$$(20+45+12-45)s + (50-54) = 0$$

$$= 14s^2 + 32s - 4 = 0$$

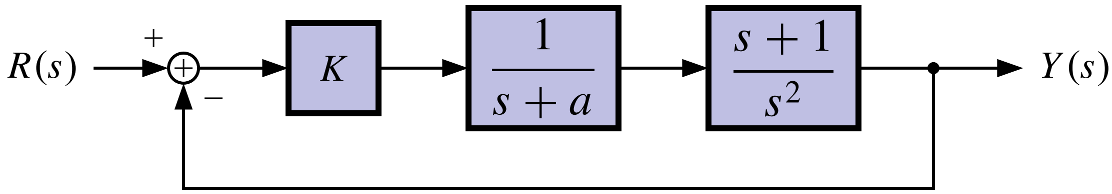
$$s = \frac{-32 \pm \sqrt{32^2 - 4(14)(-4)}}{28} = \frac{-32 \pm \sqrt{1248}}{28}$$

roots at $\{0.118,\ -2.40\}$. Now we can complete the locus:



6.6: A design example, and extensions

EXAMPLE:

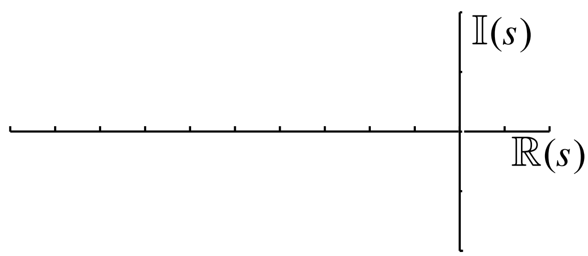


- Two design parameters: K, pole location $-a$.

$$G(s) = \frac{s+1}{s^2(s+a)}.$$

- Test $a = 2$, $a = 50$, $a = 9$.

1) Poles and zeros of $G(s)$.



2) Real axis.

3) Asymptotes:

$$\begin{aligned} n - m &= 2 \\ \alpha &= \frac{\sum p_i - \sum z_i}{2} = \frac{0 + 0 + (-a) - (-1)}{2} = \frac{1-a}{2} \\ \alpha &= \{-0.5,\ -24.5,\ -4\} \\ \phi_l &= \frac{180^\circ + 360^\circ(l-1)}{2} = \pm 90^\circ. \end{aligned}$$

4) Departure angles for two poles at $s = 0$.

$$2\phi_{\text{dep}} = \sum \angle(\text{zeros}) - \sum \angle(\text{remaining poles}) - 180^\circ - 360^\circ l$$

$$= 0 - 0 - 180^\circ - 360^\circ l$$

$$\phi_{\text{dep}} = -90^\circ - \frac{360^\circ}{2} l = \pm 90^\circ.$$

5) Will always be stable for $K > 0$.

6) Breakaway points:

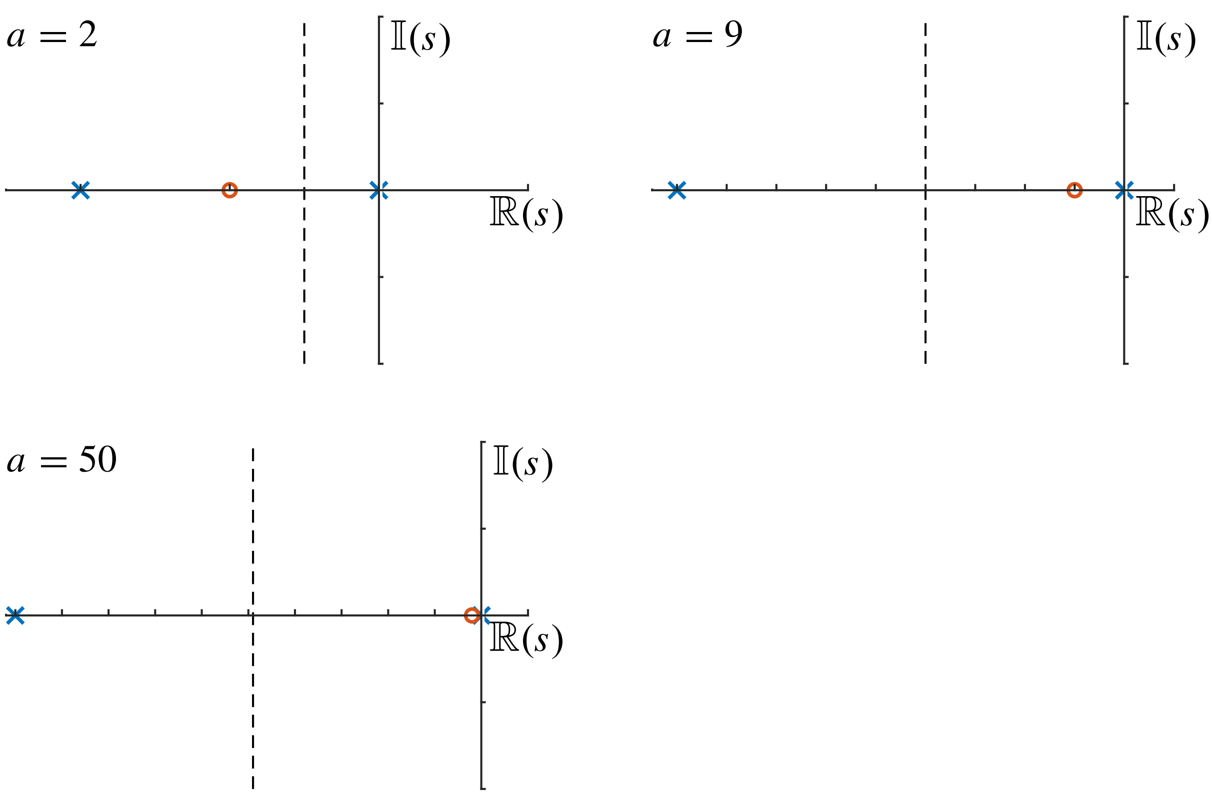
$$K(s) = \frac{-s^2(s+a)}{s+1}$$

$$\frac{\mathrm{d}}{\mathrm{d}s} K(s) = -\left(\frac{(s+1)(3s^2 + 2as) - s^2(s+a)(1)}{(s+1)^2} \right) = 0$$

$$= s\left(2s^2 + (a+3)s + 2a\right) = 0$$

$$\text{breakaway at } \left\{ 0, \frac{-(a+3) \pm \sqrt{(a+3)^2 - 4(2)(2a)}}{2(2)} \right\}$$

$$s_{a=\{2,50,9\}} = \{\underbrace{-1.25 \pm j\sqrt{7/16}}_{\text{not on locus}},\ (-2.04 \text{ and } -24.46),\ -3\} \text{ and } 0$$



Matlab and root loci

- If $G(s) = \dfrac{b(s)}{a(s)} = \dfrac{b_0 s^m + b_1 s^{m-1} + \cdots + b_m}{a_0 s^n + a_1 s^{n-1} + \cdots + a_n}$.

```
b = [b0 b1 b2 ... bm];
a = [a0 a1 a2 ... an];
rlocus(b,a);    % plots the root locus
```

- Also, `k=rlocfind(b,a);` returns the value of K for a specific point on the root locus (graphical, with a mouse).

Extensions to root-locus method—Time delay

- A system with a time delay has the form

$$G(s) = e^{-\tau_d s} G'(s)$$

where τ_d is the delay and $G'(s)$ is the non-delayed system.

- This is *not* in rational-polynomial form. We *cannot* use root locus techniques directly.

METHOD 1:

- Approximate $e^{-\tau_d s}$ by $\left[\dfrac{b_0 s + b_1}{a_0 s + 1}\right]$, a polynomial.
- Padé approximation.

$$\begin{aligned} e^{-\tau_d s} &\approx \frac{1 - (\tau_d s/2)}{1 + (\tau_d s/2)} && \text{First-order approximation} \\ &\approx \frac{1 - \tau_d s/2 + (\tau_d s)^2/12}{1 + \tau_d s/2 + (\tau_d s)^2/12} && \text{Second-order approximation.} \\ &\approx \frac{1}{1 + \tau_d s} && \text{Very crude.} \end{aligned}$$

- *Extremely* important for digital control!!!

METHOD 2:

- Directly plot locus using phase condition.
- *i.e.*, $\angle G(s) = (-\tau_d \omega) + \angle G'(s)$ if $s = \sigma + j\omega$.
- *i.e.,* look for places where $\angle G'(s) = 180° + \tau_d \omega + 360° l$.
- Fix ω, search horizontally for locus.

Summary: Root-locus drawing rules ($0°$ locus)

- We have assumed that $0 \le K < \infty$, and that $G(s)$ has monic $b(s)$ and $a(s)$. If $K < 0$ or $a(s)/b(s)$ has a negative sign preceding it, we must change our root-locus plotting rules.
- Recall that we plot

$$\begin{aligned} 1 + KG(s) &= 0 \\ G(s) &= \frac{-1}{K} \\ \angle G(s) &= \begin{cases} 180°, & \text{if } K \text{ positive;} \\ 0°, & \text{if } K \text{ negative.} \end{cases} \end{aligned}$$

- In the following summary of drawing a $0°$ root locus, the steps which have changed are highlighted.
- The steps in drawing a $0°$ root locus follow from the basic phase definition. This is the locus of

$$1 + K\frac{b(s)}{a(s)} = 0, \qquad K \le 0 \qquad \left(\text{phase of } \frac{b(s)}{a(s)} = 0°\right).$$

- They are
  - STEP 1: On the s-plane, mark poles (roots of $a(s)$) by an $\times$ and zeros (roots of $a(s)$) with an $\circ$. There will be a branch of the locus departing from every pole and a branch arriving at every zero.
  - ***STEP 2:*** Draw the locus on the real axis to the left of an ***even*** number of real poles plus zeros.

- ***STEP 3:*** Draw the asymptotes, centered at α and leaving at angles ϕ, where

$$
\begin{aligned}
n-m &= \text{number of asymptotes.}\\
n &= \text{order of } a(s)\\
m &= \text{order of } b(s)\\
\alpha &= \frac{\sum p_i - \sum z_i}{n-m} = \frac{-a_1+b_1}{n-m}\\
\Rightarrow\quad \phi_l &= \frac{(l-1)360^\circ}{n-m}, \quad l = 1, 2, \ldots n-m.
\end{aligned}
$$

  For $n-m>0$, there will be a branch of the locus approaching each asymptote and departing to infinity.

- ***STEP 4:*** Compare locus departure angles from the poles and arrival angles at the zeros where

$$
\begin{aligned}
\Rightarrow\quad q\phi_{\text{dep}} &= \sum \psi_i - \sum \phi_i \pm l360^\circ\\
\Rightarrow\quad q\psi_{\text{arr}} &= \sum \phi_i - \sum \psi_i \pm l360^\circ,
\end{aligned}
$$

  where q is the order of the pole or zero and l takes on q integer values so that the angles are between $\pm 180^\circ$. ψ_i is the angle of the line going from the ith zero to the pole or zero whose angle of departure or arrival is being computed. Similarly, ϕ_i is the angle of the line from the ith pole.

- STEP 5: If further refinement is required at the stability boundary, assume $s_0 = j\omega_0$ and compute the point(s) where the locus crosses the imaginary axis for positive K.
- STEP 6: For the case of multiple roots, two loci come together at 180° and break away at $\pm 90^\circ$. Three loci segments approach each other at angles of 120° and depart at angles rotated by 60°.
- STEP 7: Complete the locus, using the facts developed in the previous steps and making reference to the illustrative loci for guidance. The loci branches start at poles and end at zeros or infinity.
- STEP 8: Select the desired point on the locus that meets the specifications (s_0), then use the magnitude condition to find that the value of K associated with that point is

$$K = \frac{1}{|b(s_0)/a(s_0)|}.$$