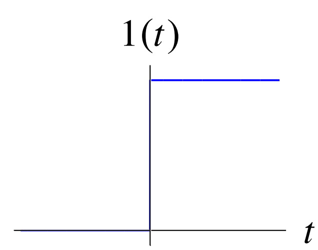
DYNAMIC RESPONSE

3.1: System response in the time domain

- We can now model dynamic systems with differential equations. What do these equations mean?
- We'll proceed by looking at a system's response to certain inputs in the time domain.
- Then, we'll see how the Laplace transform can make our lives a lot easier by simplifying the math.
- This will give insights into how we might specify the way the system *should* respond.
- Finally, we'll preview how adding dynamics (*e.g.*, a controller) can change how the system responds.

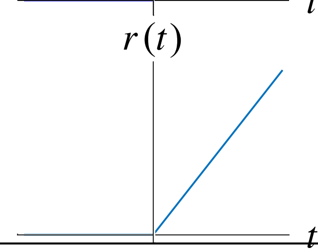
Some important input signals

- Several signals recur throughout this course.
- The unit step function:



$$1(t) = \begin{cases} 1, & t \ge 0; \\ 0, & \text{otherwise.} \end{cases}$$

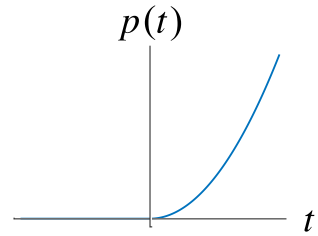
- The unit ramp function:



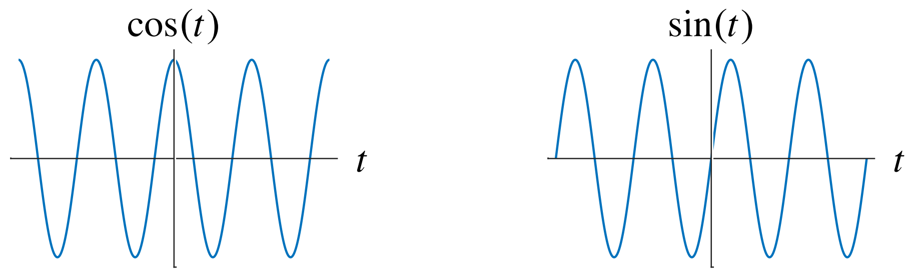
$$r(t) = \begin{cases} t, & t \ge 0; \\ 0, & \text{otherwise.} \end{cases}$$

- The unit parabola function:

$$p(t) = \begin{cases} \dfrac{t^2}{2}, & t \geq 0; \\ 0, & \text{otherwise.} \end{cases}$$



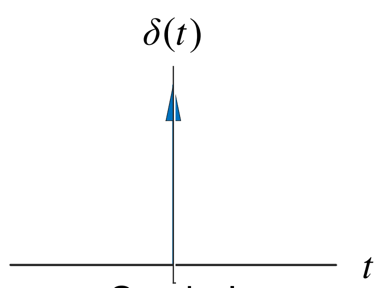
- The cosine/sine functions:



- The (ideal) impulse function, $\delta(t)$:
  - Very strange "generalized" function, defined only under an integral.

$$\delta(t) = 0, \quad t \neq 0 \qquad \text{zero duration}$$

$$\int_{-\infty}^{\infty} \delta(t)\,\mathrm{d}t = 1. \qquad \text{unit area.}$$



Symbol

  - Sifting property:[1]

$$\int_{-\infty}^{\infty} x(\tau)\delta(t-\tau)\,\mathrm{d}\tau = x(t).$$

[1] Assumes that $x(t)$ is continuous at $t = \tau$. Interpretation: no value of $x(t)$ matters except that over the short range where $\delta(t)$ occurs.

Time response of a linear time invariant system

- Let $y(t)$ be the output of an LTI system with input $x(t)$.

$$\begin{aligned}
y(t) &= \mathbb{T}\big[x(t)\big] \\
&= \mathbb{T}\left[\int_{-\infty}^{\infty} x(\tau)\delta(t-\tau)\,\mathrm{d}\tau\right] && \text{(sifting)} \\
&= \int_{-\infty}^{\infty} x(\tau)\mathbb{T}\big[\delta(t-\tau)\big]\,\mathrm{d}\tau. && \text{(linear)} \\
\text{Let } h(t,\tau) &= \mathbb{T}\big[\delta(t-\tau)\big] \\
&= \int_{-\infty}^{\infty} x(\tau)h(t,\tau)\,\mathrm{d}\tau
\end{aligned}$$

If the system is time invariant, $h(t,\tau) = h(t-\tau)$

$$\begin{aligned}
&= \int_{-\infty}^{\infty} x(\tau)h(t-\tau)\,\mathrm{d}\tau && \text{(time invariant)} \\
&\stackrel{\triangle}{=} x(t) * h(t).
\end{aligned}$$

- The output of an LTI system is equal to the convolution of its impulse response with the input.
- This makes life EASY (TRUST me!)

EXAMPLE: Finding an impulse response:

- Consider a first-order system, $\dot{y}(t) + ky(t) = u(t)$.
- Let $y(0^-) = 0$, $u(t) = \delta(t)$.
- For positive time we have $\dot{y}(t) + ky(t) = 0$. Recall from your differential-equation math course: $y(t) = Ae^{st}$, solve for A, s.

$$\begin{aligned}\dot{y}(t) &= Ase^{st}\\ Ase^{st} + kAe^{st} &= 0\\ s + k &= 0\\ s &= -k.\end{aligned}$$

- We have solved for s; now, solve for A.

$$\underbrace{\int_{0^-}^{0^+} \dot{y}(t)\,\mathrm{d}t}_{y(t)|_{0^-}^{0^+}} + \underbrace{k\int_{0^-}^{0^+} y(t)\,\mathrm{d}t}_{0} = \underbrace{\int_{0^-}^{0^+} \delta(t)\,\mathrm{d}t}_{1}$$

$$\begin{aligned}y(0^+) - y(0^-) &= 1\\ Ae^{-k0^+} - 0 &= 1\\ A &= 1.\end{aligned}$$

- Response to impulse: $h(t) = e^{-kt}$, $t > 0$.
- $h(t) = e^{-kt}1(t)$.
- Response of this system to general input:

$$\begin{aligned}y(t) &= \int_{-\infty}^{\infty} h(\tau)u(t-\tau)\,\mathrm{d}\tau\\ &= \int_{-\infty}^{\infty} e^{-k\tau}1(\tau)u(t-\tau)\,\mathrm{d}\tau\\ &= \int_{0}^{\infty} e^{-k\tau}u(t-\tau)\,\mathrm{d}\tau.\end{aligned}$$

3.2: Transfer functions

- Response to impulse = "impulse response": $h(t)$.
- Response to general input = messy convolution: $h(t) * u(t)$.
- To choose a simpler example, what is the response to a cosine?

$$A\cos(\omega t) = \frac{A}{2}\left(e^{j\omega t} + e^{-j\omega t}\right)$$

Break it down: What is the response to an exponential?

- Let $u(t) = e^{st}$, where s is complex.

$$\begin{aligned} y(t) = \int_{-\infty}^{\infty} h(\tau)u(t-\tau)\,\mathrm{d}\tau &= \int_{-\infty}^{\infty} h(\tau)e^{s(t-\tau)}\,\mathrm{d}\tau \\ &= \int_{-\infty}^{\infty} h(\tau)e^{st}e^{-s\tau}\,\mathrm{d}\tau \\ &= e^{st}\underbrace{\int_{-\infty}^{\infty} h(\tau)e^{-s\tau}\,\mathrm{d}\tau}_{\text{Transfer function, } H(s)} \\ &= e^{st}H(s). \end{aligned}$$

- An e^{st} input decouples the convolution into two *independent* parts: a part depending on e^{st} and a part depending on $h(t)$.

EXAMPLE: $\dot{y}(t) + ky(t) = u(t) = e^{st}$:

but, $y(t) = H(s)e^{st}, \quad \dot{y}(t) = sH(s)e^{st},$

$$sH(s)e^{st} + kH(s)e^{st} = e^{st}$$

$$H(s) = \frac{1}{s+k} \qquad \text{(I never integrated!)}$$

$$y(t) = \frac{e^{st}}{s+k}.$$

Response to a cosinusoid (revisited)

Let $s=j\omega$ $\quad u(t)=e^{j\omega t}$ $\quad y(t)=H(j\omega)e^{j\omega t}$

$s=-j\omega$ $\quad u(t)=e^{-j\omega t}$ $\quad y(t)=H(-j\omega)e^{-j\omega t}$

$u(t)=A\cos(\omega t)$ $\quad y(t)=\dfrac{A}{2}\left[H(j\omega)e^{j\omega t}+H(-j\omega)e^{-j\omega t}\right]$

Now,
$$H(j\omega) \triangleq Me^{j\phi}$$

$$H(-j\omega) = Me^{-j\phi} \quad \text{(can be shown for } h(t) \text{ real)}$$

$$y(t) = \frac{AM}{2}\left[e^{j(\omega t+\phi)} + e^{-j(\omega t+\phi)}\right]$$
$$= AM\cos(\omega t+\phi).$$

- The response of an LTI system to a sinusoid is a sinusoid! (of the *same* frequency).

EXAMPLE: Frequency response of our first order system:

$$H(s) = \frac{1}{s+k}$$

$$H(j\omega) = \frac{1}{j\omega+k}$$

$$M = |H(j\omega)| = \frac{1}{\sqrt{\omega^2+k^2}}$$

$$\phi = \angle H(j\omega) = -\tan^{-1}\left(\frac{\omega}{k}\right)$$

$$y(t) = \frac{A}{\sqrt{\omega^2+k^2}}\cos\left(\omega t - \tan^{-1}\left(\frac{\omega}{k}\right)\right).$$

- Can we use these results to simplify convolution and get an easier way to understand dynamic response?

Defining the Laplace $\mathcal{L}_-$ transform

- We have seen that if a system has an impulse response $h(t)$, we can compute a transfer function $H(s)$,
$$H(s) = \int_{-\infty}^{\infty} h(t)e^{-st}\,\mathrm{d}t.$$
- Since we deal with causal systems (possibly with an impulse at $t = 0$), we can integrate from 0^- instead of negative infinity.
$$H(s) = \int_{0^-}^{\infty} h(t)e^{-st}\,\mathrm{d}t.$$
- This is called the one-sided (uni-lateral) Laplace transform of $h(t)$.

Laplace Transforms of Common Signals

Name	Time function, $f(t)$	Laplace tx., $F(s)$
Unit impulse	$\delta(t)$	1
Unit step	$1(t)$	$\dfrac{1}{s}$
Unit ramp	$t \cdot 1(t)$	$\dfrac{1}{s^2}$
nth order ramp	$t^n \cdot 1(t)$	$\dfrac{n!}{s^{n+1}}$
Exponential	$\exp(-at)1(t)$	$\dfrac{1}{s+a}$
Ramped exponential	$t\exp(-at)1(t)$	$\dfrac{1}{(s+a)^2}$
Sine	$\sin(bt)1(t)$	$\dfrac{b}{s^2+b^2}$
Cosine	$\cos(bt)1(t)$	$\dfrac{s}{s^2+b^2}$
Damped sine	$e^{-at}\sin(bt)1(t)$	$\dfrac{b}{(s+a)^2+b^2}$
Damped cosine	$e^{-at}\cos(bt)1(t)$	$\dfrac{s+a}{(s+a)^2+b^2}$
Diverging sine	$t\sin(bt)1(t)$	$\dfrac{2bs}{(s^2+b^2)^2}$
Diverging cosine	$t\cos(bt)1(t)$	$\dfrac{s^2-b^2}{(s^2+b^2)^2}$

Properties of the Laplace transform

- Superposition: $\mathcal{L}\{af_1(t) + bf_2(t)\} = aF_1(s) + bF_2(s)$.
- Time delay: $\mathcal{L}\{f(t-\tau)\} = e^{-s\tau}F(s)$.
- Time scaling: $\mathcal{L}\{f(at)\} = \dfrac{1}{|a|}F\left(\dfrac{s}{a}\right)$.
  (useful if original equations are expressed poorly in time scale. *e.g.*, measuring disk-drive seek speed in hours).
- Differentiation:

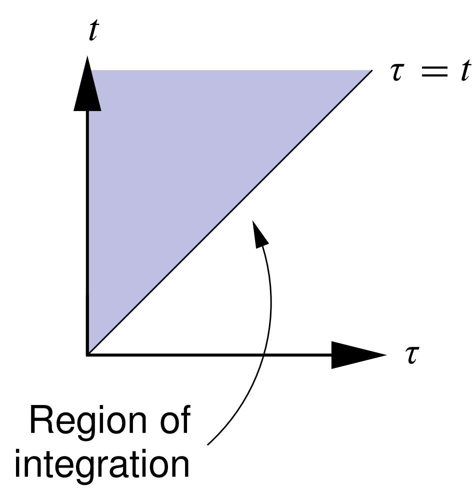
$$\mathcal{L}\left\{\dot{f}(t)\right\} = sF(s) - f(0^-)$$

$$\mathcal{L}\left\{\ddot{f}(t)\right\} = s^2F(s) - sf(0^-) - \dot{f}(0^-)$$

$$\mathcal{L}\left\{f^{(m)}(t)\right\} = s^mF(s) - s^{m-1}f(0^-) - \ldots - f^{(m-1)}(0^-).$$

- Integration: $\mathcal{L}\left\{\int_{0^-}^{t} f(\tau)\,\mathrm{d}\tau\right\} = \dfrac{1}{s}F(s)$.
- Convolution: Recall that $y(t) = h(t) * u(t)$

$$\begin{aligned}
Y(s) &= \mathcal{L}\{y(t)\} = \mathcal{L}\{h(t) * u(t)\} \\
&= \mathcal{L}\left\{\int_{\tau=0^-}^{t} h(\tau)u(t-\tau)\,\mathrm{d}\tau\right\} \\
&= \int_{t=0^-}^{\infty}\int_{\tau=0^-}^{t} h(\tau)u(t-\tau)\,\mathrm{d}\tau\; e^{-st}\,\mathrm{d}t \\
&= \int_{\tau=0^-}^{\infty}\int_{t=\tau^-}^{\infty} h(\tau)u(t-\tau)\; e^{-st}\,\mathrm{d}t\,\mathrm{d}\tau.
\end{aligned}$$



- Multiply by $e^{-s\tau}e^{s\tau}$

$$Y(s) = \int_{\tau=0^-}^{\infty} h(\tau)e^{-s\tau}\int_{t=\tau^-}^{\infty} u(t-\tau)e^{-s(t-\tau)}\,\mathrm{d}t\,\mathrm{d}\tau.$$

Let $t' = t - \tau$:

$$Y(s) = \int_{\tau=0^-}^{\infty} h(\tau)e^{-s\tau}\,\mathrm{d}\tau \int_{t'=0^-}^{\infty} u(t')e^{-st'}\,\mathrm{d}t'$$

$$Y(s) = H(s)U(s).$$

- The Laplace transform "unwraps" convolution for *general* input signals. Makes system easy to analyze.
- This is **the most** important property of the Laplace transform. This is why we use it. It converts differential equations into algebraic equations that we can solve quite readily.

3.3: The inverse Laplace transform

- The inverse Laplace transform converts $F(s) \to f(t)$.
- Once we get an intuitive feel for $F(s)$, we won't need to do this often.
- The main tool for ILT is partial-fraction-expansion.

Assume :
$$F(s) = \frac{b_0 s^m + b_1 s^{m-1} + \cdots + b_m}{s^n + a_1 s^{n-1} + \cdots + a_n}$$
$$= k\frac{\prod_{i=1}^{m}(s - z_i)}{\prod_{i=1}^{n}(s - p_i)} \quad \Leftarrow \quad \frac{\text{(zeros)}}{\text{(poles)}}$$
$$= \frac{c_1}{s - p_1} + \frac{c_2}{s - p_2} + \cdots + \frac{c_n}{s - p_n} \qquad \text{if } \{p_i\} \text{ distinct.}$$

so, $$(s - p_1)F(s) = c_1 + \frac{c_2(s - p_1)}{s - p_2} + \cdots + \frac{c_n(s - p_1)}{s - p_n}$$

let $s = p_1$:
$$c_1 = (s - p_1)F(s)|_{s=p_1}$$
$$c_i = (s - p_i)F(s)|_{s=p_i}$$
$$f(t) = \sum_{i=1}^{n} c_i e^{p_i t} 1(t) \qquad \text{since } \mathcal{L}\left[e^{kt}1(t)\right] = \frac{1}{s - k}.$$

EXAMPLE: $F(s) = \dfrac{5}{s^2 + 3s + 2} = \dfrac{5}{(s + 1)(s + 2)}$.

$$c_1 = (s + 1)F(s)\Big|_{s=-1} = \frac{5}{s + 2}\Big|_{s=-1} = 5$$
$$c_2 = (s + 2)F(s)\Big|_{s=-2} = \frac{5}{s + 1}\Big|_{s=-2} = -5$$
$$f(t) = (5e^{-t} - 5e^{-2t})1(t).$$

- If $F(s)$ has repeated roots, we must modify the procedure. *e.g.*, repeated three times:

$$\begin{aligned}
F(s) &= \frac{k}{(s-p_1)^3(s-p_2)\cdots} \\
&= \frac{c_{1,1}}{s-p_1} + \frac{c_{1,2}}{(s-p_1)^2} + \frac{c_{1,3}}{(s-p_1)^3} + \frac{c_2}{s-p_2} + \cdots \\
c_{1,3} &= (s-p_1)^3 F(s)\Big|_{s=p_1} \\
c_{1,2} &= \left[\frac{\mathrm{d}}{\mathrm{d}s}\left((s-p_1)^3 F(s)\right)\right|_{s=p_1} \\
c_{1,1} &= \frac{1}{2}\left[\frac{\mathrm{d}^2}{\mathrm{d}s^2}\left((s-p_1)^3 F(s)\right)\right|_{s=p_1} \\
c_{x,k-i} &= \frac{1}{i!}\left[\frac{\mathrm{d}^i}{\mathrm{d}s^i}\left((s-p_i)^k F(s)\right)\right|_{s=p_i}.
\end{aligned}$$

EXAMPLE: Find the ILT of

$$H(s) = \frac{s+2}{(s+1)^2(s+3)} = \frac{A}{s+1} + \frac{B}{(s+1)^2} + \frac{C}{s+3}.$$

- We start with B,

$$B = \left.\frac{s+2}{s+3}\right|_{s=-1} = \frac{1}{2}.$$

- Next, we find A,

$$\begin{aligned}
A &= \left[\frac{\mathrm{d}}{\mathrm{d}s}\left(\frac{s+2}{s+3}\right)\right|_{s=-1} \\
&= \left[\frac{\mathrm{d}}{\mathrm{d}s}(s+2)(s+3)^{-1}\right|_{s=-1} \\
&= \left[(s+2)(-1)(s+3)^{-2} + (s+3)^{-1}\right|_{s=-1} \\
&= \left[-\frac{s+2}{(s+3)^2} + \frac{1}{s+3}\right|_{s=-1} \\
&= -\frac{1}{4} + \frac{1}{2} = \frac{1}{4}.
\end{aligned}$$

- Lastly, we find C,

$$C = \left.\frac{s+2}{(s+1)^2}\right|_{s=-3} = -\frac{1}{4}.$$

- Therefore, the inverse Laplace transform we are looking for is

$$h(t) = \left[\frac{1}{2}te^{-t} + \frac{1}{4}e^{-t} - \frac{1}{4}e^{-3t}\right]1(t).$$

EXAMPLE: Find ILT of $\dfrac{s+3}{(s+1)(s+2)^2}$.

- ans: $f(t) = (2e^{-t} - 2e^{-2t} - \underbrace{te^{-2t}}_{\text{from repeated root}})1(t).$

- Note that this is quite tedious, but MATLAB can help.

- Try MATLAB with two examples; first, $F(s) = \dfrac{5}{s^2+3s+2}$.

Example 1.

```
>> Fnum = [0 0 5];
>> Fden = [1 3 2];
[r,p,k] = residue(Fnum,Fden);
  r = -5
       5
  p = -2
      -1
  k = []
```

Example 2.

```
>> Fnum = [0 0 1 3];
>> Fden = conv([1 1],conv([1 2],[1 2]));
[r,p,k] = residue(Fnum,Fden);
  r = -2
      -1
       2
  p = -2
      -2
      -1
  k = []
```

- When you use “residue” and get repeated roots, *BE SURE* to type “help residue” to correctly interpret the result.

Using the Laplace transform to solve problems

- We can use the Laplace transform to solve both homogeneous and forced differential equations.

EXAMPLE: $\ddot{y}(t) + y(t) = 0, \qquad y(0^-) = \alpha,\ \dot{y}(0^-) = \beta.$

- Take Laplace transforms, term by term:

$$s^2Y(s) - \alpha s - \beta + Y(s) = 0$$

$$\begin{aligned} Y(s)(s^2+1) &= \alpha s + \beta \\ Y(s) &= \frac{\alpha s + \beta}{s^2+1} \\ &= \frac{\alpha s}{s^2+1} + \frac{\beta}{s^2+1}. \end{aligned}$$

$$\frac{a}{s^2+a^2} \Longleftrightarrow \sin(at)1(t)$$
$$\frac{s}{s^2+a^2} \Longleftrightarrow \cos(at)1(t).$$

- From tables, $y(t) = [\alpha\cos(t) + \beta\sin(t)]1(t)$.
- If initial conditions are zero, things are very simple.

EXAMPLE:

$$\ddot{y}(t)+5\dot{y}(t)+4y(t) = u(t), \qquad y(0^-) = 0,\ \dot{y}(0^-) = 0, \quad u(t) = 2e^{-2t}1(t).$$

- Start with:

$$s^2Y(s) + 5sY(s) + 4Y(s) = \frac{2}{s+2}$$

- Rearrange:

$$\begin{aligned} Y(s) &= \frac{2}{(s+2)(s+1)(s+4)} \\ &= \frac{-1}{s+2} + \frac{2/3}{s+1} + \frac{1/3}{s+4}. \end{aligned}$$

$$H(s) \Longleftrightarrow h(t)$$
$$H(s+a) \Longleftrightarrow e^{-at}h(t).$$

- From tables, $y(t) = \left[-e^{-2t} + \frac{2}{3}e^{-t} + \frac{1}{3}e^{-4t}\right]1(t)$.

3.4: Time response versus pole locations

- If we wish to know how a system responds to some input (for example, an impulse response, or a step response), it seems like we need to do the following:
  1. Find the Laplace transform $U(s)$ of the input $u(t)$,
  2. Find the Laplace transform of the output $Y(s) = H(s)U(s)$,
  3. Find the time response by taking the inverse Laplace transform of $Y(s)$. That is, $y(t) = \mathcal{L}^{-1}(Y(s))$.
- This is true if we want a precise, *quantitative* answer.
- But, if we're interested only in a *qualitative* answer, we can learn a lot simply by looking at the pole locations of the transfer function.
- If we can represent $H(s) = \text{num}_H(s)/\text{den}_H(s)$ and $U(s) = \text{num}_U(s)/\text{den}_U(s)$, then we have

$$\begin{aligned} Y(s) &= \frac{\text{num}_H(s)\text{num}_U(s)}{\text{den}_H(s)\text{den}_U(s)} \\ &= \sum_k \frac{r_k}{s + p_k}, \end{aligned}$$

  where "pole" $s = -p_k$ is a root of either $\text{den}_H(s)$ or $\text{den}_U(s)$.
- So, some of the system's response is due to the poles of the input signal, and some is due to the poles of the plant.
- Here, we're interested in the contribution due to the poles of the plant.
  - Neglecting the residues r_k, which simply scale the output by some fixed amount, we're interested in "what does an output of the type $\dfrac{1}{s + p_k}$ look like?"

- That is, the poles *qualitatively* determine the behavior of the system; zeros (equivalently, residues) quantify this relationship.

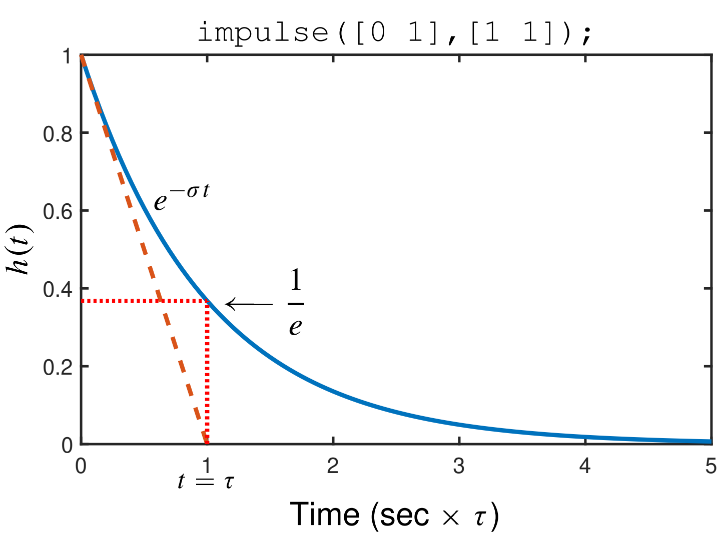
- Note that the poles p_k may be real, or they may occur in complex-conjugate pairs.
- So, in the next sections, we look at the time responses of real poles and of complex-conjugate poles.

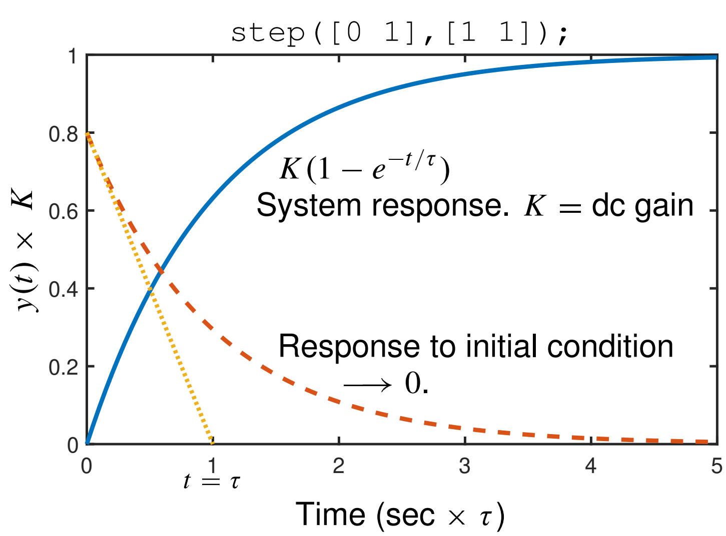
Time response due to a real pole

- Consider a transfer function having a single real pole:

$$H(s) = \frac{1}{s+\sigma} \quad \Rightarrow \quad h(t) = e^{-\sigma t}1(t).$$

- If $\sigma > 0$, pole is at $s < 0$, *STABLE i.e.*, impulse response decays, and any bounded input produces bounded output.
- If $\sigma < 0$, pole is at $s > 0$, *UNSTABLE*.
- σ is "time constant" factor: $\tau = 1/\sigma$.





Time response due to complex-conjugate poles

- Now, consider a second-order transfer function having complex-conjugate poles

$$H(s) = \frac{b_0}{s^2 + a_1 s + a_2} = K \underbrace{\frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}}_{\text{standard form}}.$$

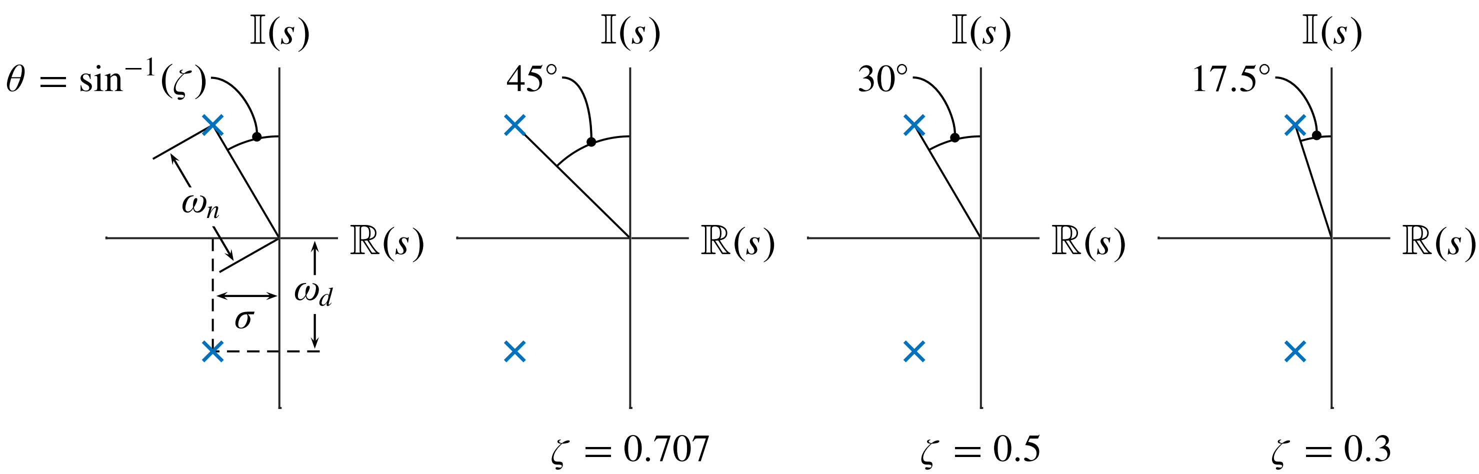
$$\zeta = \text{damping ratio.}$$

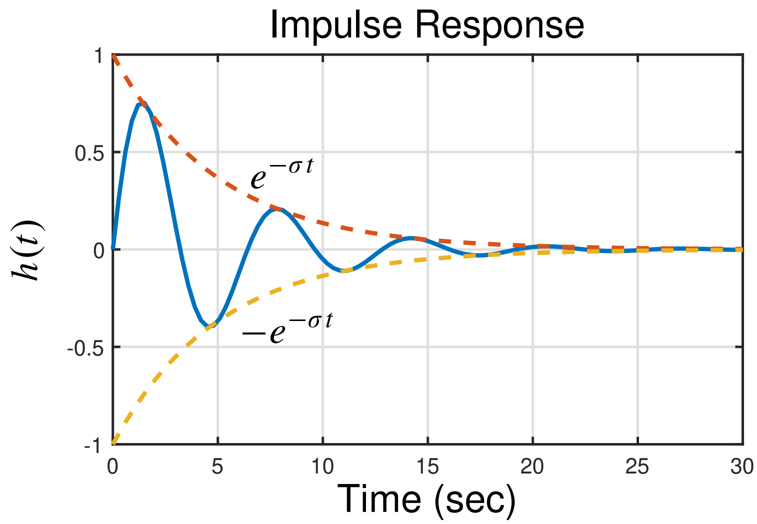
$$\omega_n = \text{natural frequency or undamped frequency.}$$

$$h(t) = \frac{\omega_n}{\sqrt{1-\zeta^2}} e^{-\sigma t} \left(\sin(\omega_d t)\right) 1(t),$$

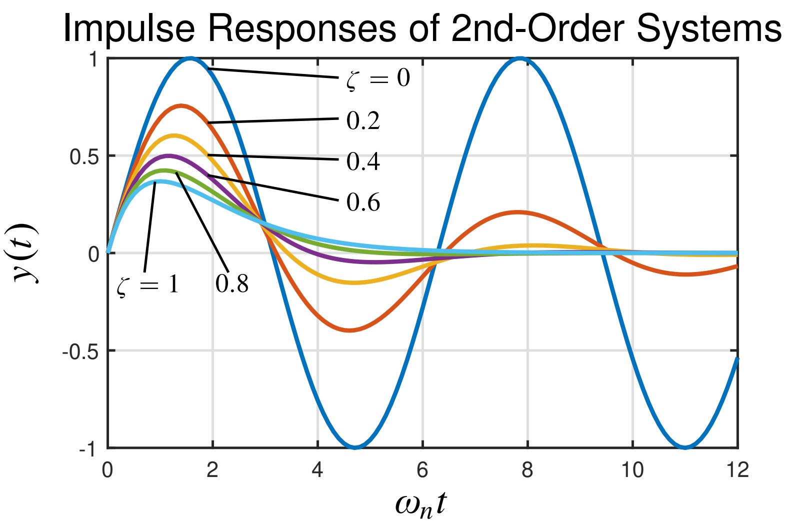
where, $\sigma = \zeta\omega_n,$

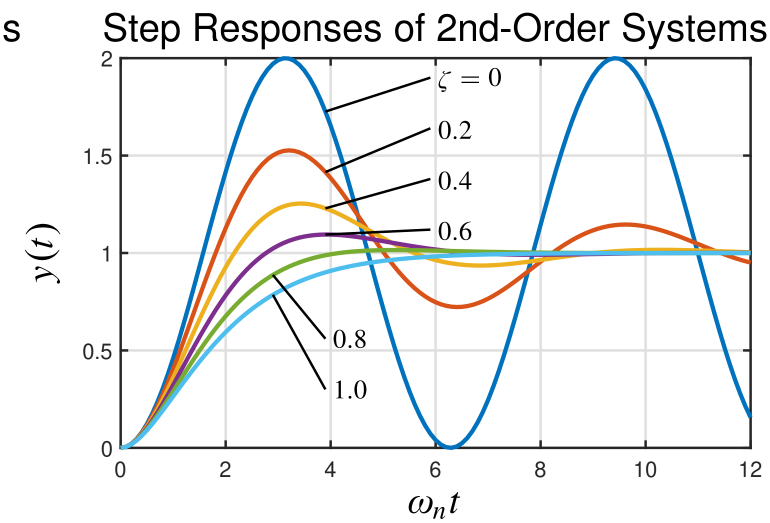
$$\omega_d = \omega_n\sqrt{1-\zeta^2} = \text{damped frequency.}$$



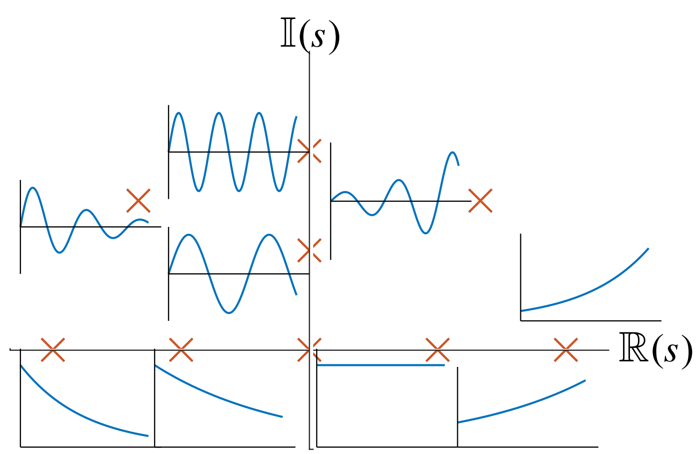


Envelope of sinusoid decays as $e^{-\sigma t}$

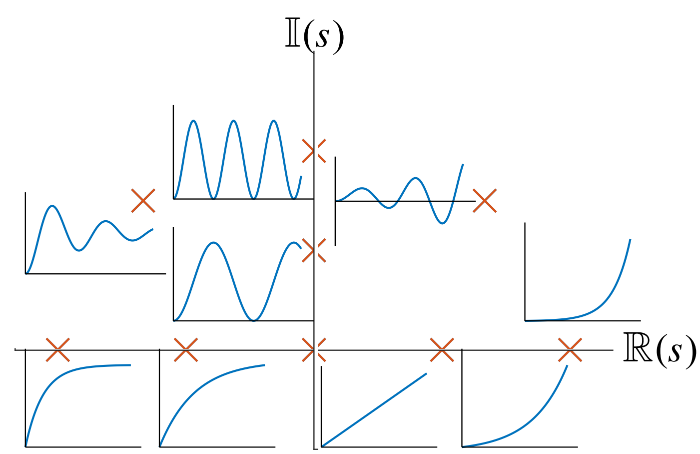




- Low damping, $\zeta \approx 0$, oscillatory; High damping, $\zeta \approx 1$, no oscillations.



Impulse responses vs. pole locations

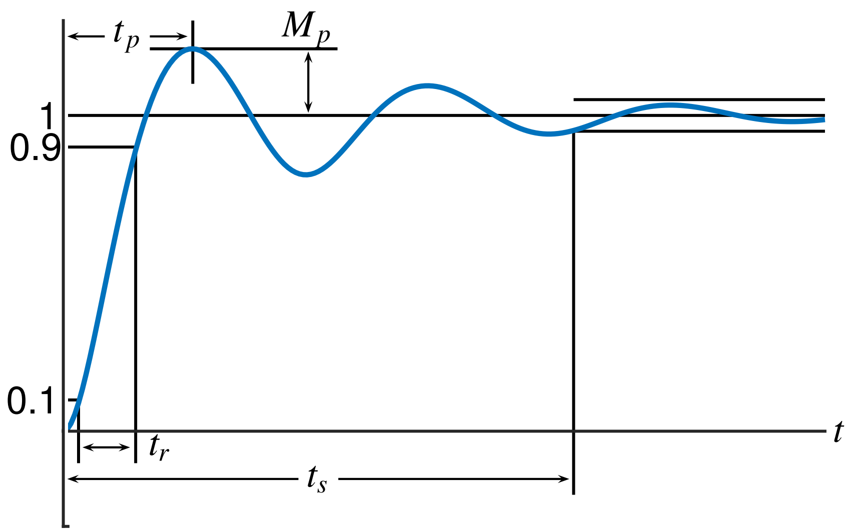


Step responses vs. pole locations

- $0 < \zeta < 1$ underdamped.
- $\zeta = 1$ critically damped, $\zeta > 1$ over-damped.

3.5: Time-domain specifications

- We have seen impulse and step responses for first- and second-order systems.
- Our control problem may be to specify exactly what the response *SHOULD* be.
- Usually expressed in terms of the step response.



- t_r = Rise time = time to reach vicinity of new set point.
- t_s = Settling time = time for transients to decay (to 5 %, 2 %, 1 %).
- M_p = Percent overshoot.
- t_p = Time to peak.

Rise Time

- All step responses rise in roughly the same amount of time (see pg. 3–17.) Take $\zeta = 0.5$ to be average.
  - ➠ time from 0.1 to 0.9 is approximately $\omega_n t_r = 1.8$:

$$t_r \approx \frac{1.8}{\omega_n}.$$

- We could make this more accurate, but note:
  - Only valid for 2nd-order systems with no zeros.
  - Use this as approximate design "rule of thumb" and iterate design until spec. is met.

Peak Time and Overshoot

- Step response can be found from ILT of $H(s)/s$.

$$y(t) = 1 - e^{-\sigma t}\left(\cos(\omega_d t) + \frac{\sigma}{\omega_d}\sin(\omega_d t)\right),$$

$$\omega_d = \omega_n\sqrt{1-\zeta^2}, \quad \sigma = \zeta\omega_n.$$

- Peak occurs when $\dot{y}(t) = 0$

$$\dot{y}(t) = \sigma e^{-\sigma t}\left(\cos(\omega_d t) + \frac{\sigma}{\omega_d}\sin(\omega_d t)\right) - e^{-\sigma t}\left(-\omega_d \sin(\omega_d t) + \sigma\cos(\omega_d t)\right)$$

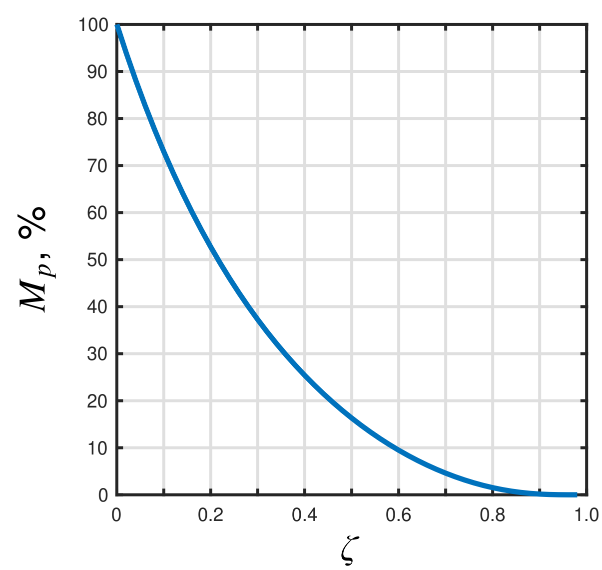
$$= e^{-\sigma t}\left(\frac{\sigma^2}{\omega_d}\sin(\omega_d t) + \omega_d \sin(\omega_d t)\right) = 0.$$

- So,

$$\omega_d t_p = \pi,$$

$$t_p = \frac{\pi}{\omega_d} = \frac{\pi}{\omega_n\sqrt{1-\zeta^2}}.$$

- $M_p = e^{-\zeta\pi/\sqrt{1-\zeta^2}} \times 100$.
- (common values: $M_p = 16\%$ for $\zeta = 0.5$; $M_p = 5\%$ for $\zeta = 0.7$).



Settling Time

- Determined mostly by decaying exponential

$$e^{-\omega_n \zeta t_s} = \epsilon \quad \ldots \qquad \epsilon = 0.01,\ 0.02,\ \text{or } 0.05$$

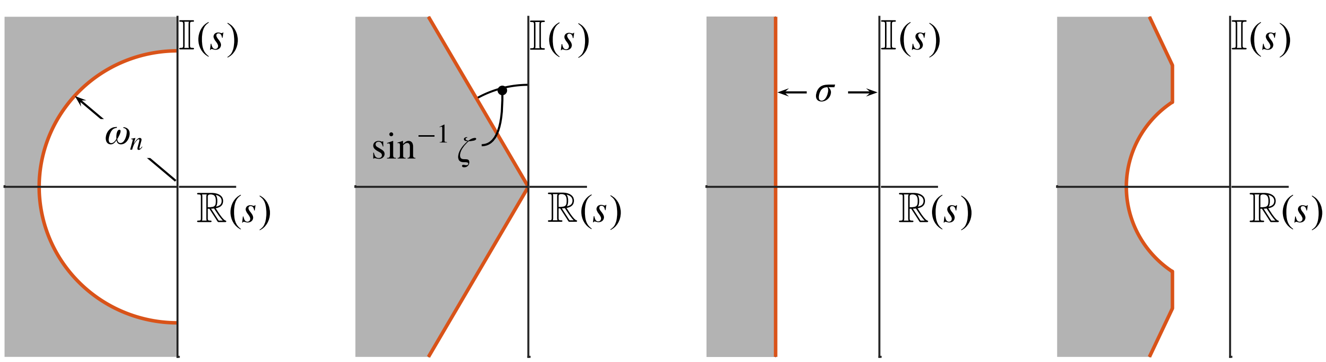
EXAMPLE:

$$\begin{aligned} \epsilon &= 0.01 \\ e^{-\omega_n \zeta t_s} &= 0.01 \\ \omega_n \zeta t_s &= 4.6 \\ t_s &= \frac{4.6}{\zeta \omega_n} = \frac{4.6}{\sigma} \end{aligned}$$

ϵ	t_s
0.01	$t_s = 4.6/\sigma$
0.02	$t_s = 3.9/\sigma$
0.05	$t_s = 3.0/\sigma$

Design synthesis

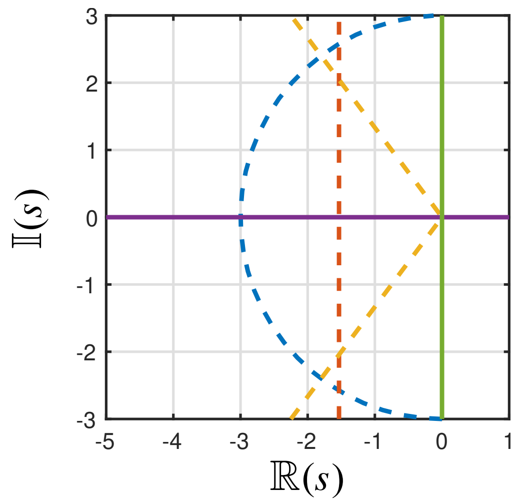
- Specifications on t_r, t_s, M_p determine pole locations.
- $\omega_n \geq 1.8/t_r$.
- $\zeta \geq \text{fn}(M_p)$. (read off of ζ versus M_p graph on page 3–19)
- $\sigma \geq 4.6/t_s$. (for example—settling to 1%)



EXAMPLE: Converting specs. to s-plane

- Specs: $t_r \leq 0.6$, $M_p \leq 10\%$, $t_s \leq 3$ sec. at 1%

- $\omega_n \geq 1.8/t_r = 3.0$ rad/sec.
- From graph of M_p versus ζ, $\zeta \geq 0.6$.
- $\sigma \geq 4.6/3 = 1.5$ sec.



EXAMPLE: Designing motor compensator

- Suppose a servo-motor system for a pen-plotter has transfer function

$$\frac{0.5K_a}{s^2 + 2s + 0.5K_a} = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}.$$

- Only one adjustable parameter K_a, so can choose only one spec: t_r, t_s or M_p ➠ Allow *NO* overshoot.
- $M_p = 0, \;\; \zeta = 1.$
- From transfer function: $2 = 2\zeta\omega_n$ ➠ $\omega_n = 1.$
- $\omega_n^2 = 1^2 = 0.5K_a, \quad K_a = 2.0$
- Note: $t_s = 4.6$ seconds. We will need a better controller than this for a pen plotter!

3.6: Time response vs. pole locations: Higher order systems

- We have looked at first-order and second-order systems without zeros, and with unity gain.

Non-unity gain

- If we multiply by K, the dc gain is K. t_r, t_s, M_p, t_p are not affected.

Add a zero to a second-order system

$$\begin{aligned} H_1(s) &= \frac{2}{(s+1)(s+2)} \\ &= \frac{2}{s+1} - \frac{2}{s+2} \end{aligned} \qquad \begin{aligned} H_2(s) &= \frac{2(s+1.1)}{1.1(s+1)(s+2)} \\ &= \frac{2}{1.1}\left(\frac{0.1}{s+1} + \frac{0.9}{s+2}\right) \\ &= \frac{0.18}{s+1} + \frac{1.64}{s+2} \end{aligned}$$

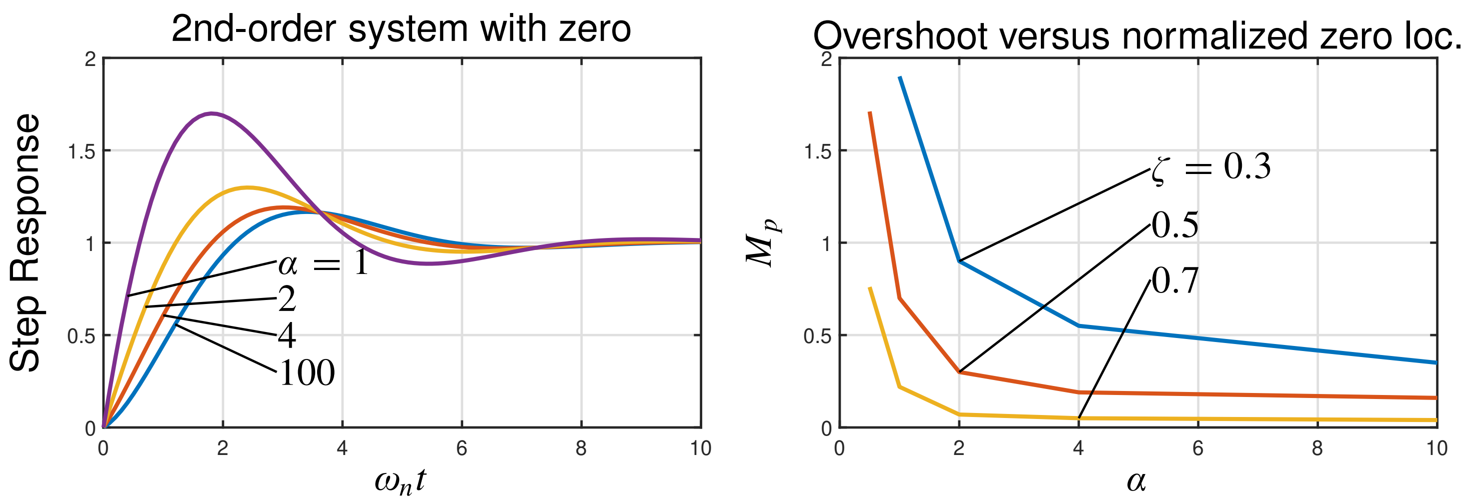
- Same dc gain (at $s = 0$).
- Coefficient of $(s+1)$ pole *GREATLY* reduced.
- General conclusion: a zero "near" a pole tends to cancel the effect of that pole.
- How about transient response?

$$H(s) = \frac{\frac{s}{\alpha\zeta\omega_n} + 1}{(s/\omega_n)^2 + 2\zeta s/\omega_n + 1}.$$

  - Zero at $s = -\alpha\sigma$ (since $\sigma = \zeta\omega_n$).
  - Poles at $\mathbb{R}(s) = -\sigma$.

- Large α, zero far from poles ➠ no effect.
- $\alpha \approx 1$, large effect.

- Notice that the overshoot goes up as $\alpha \to 0$.



- A little more analysis; set $\omega_n = 1$

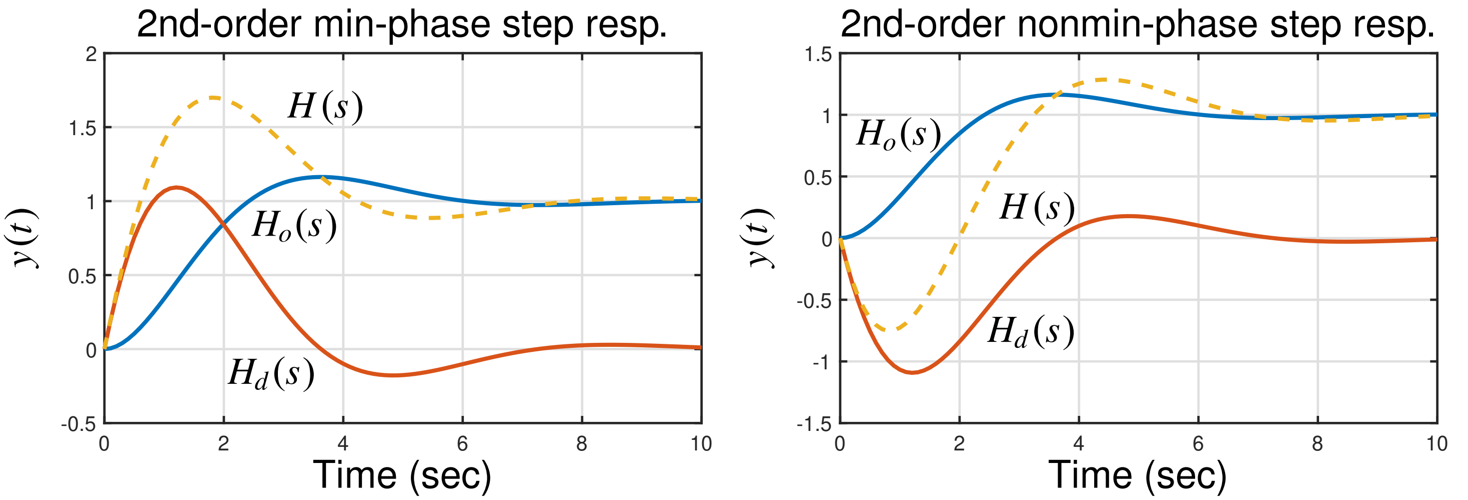
$$\begin{aligned} H(s) &= \frac{\frac{s}{\alpha\zeta} + 1}{s^2 + 2\zeta s + 1} \\ &= \frac{1}{s^2 + 2\zeta s + 1} + \left(\frac{1}{\alpha\zeta}\right)\frac{s}{s^2 + 2\zeta s + 1} \\ &= H_o(s) + H_d(s). \end{aligned}$$

- $H_o(s)$ is the original response, without the zero.
- $H_d(s)$ is the added term due to the zero. Notice that

$$H_d(s) = \frac{1}{\alpha\zeta} s H_o(s).$$

  The time response is a scaled version of the *derivative* of the time response of $H_o(s)$.

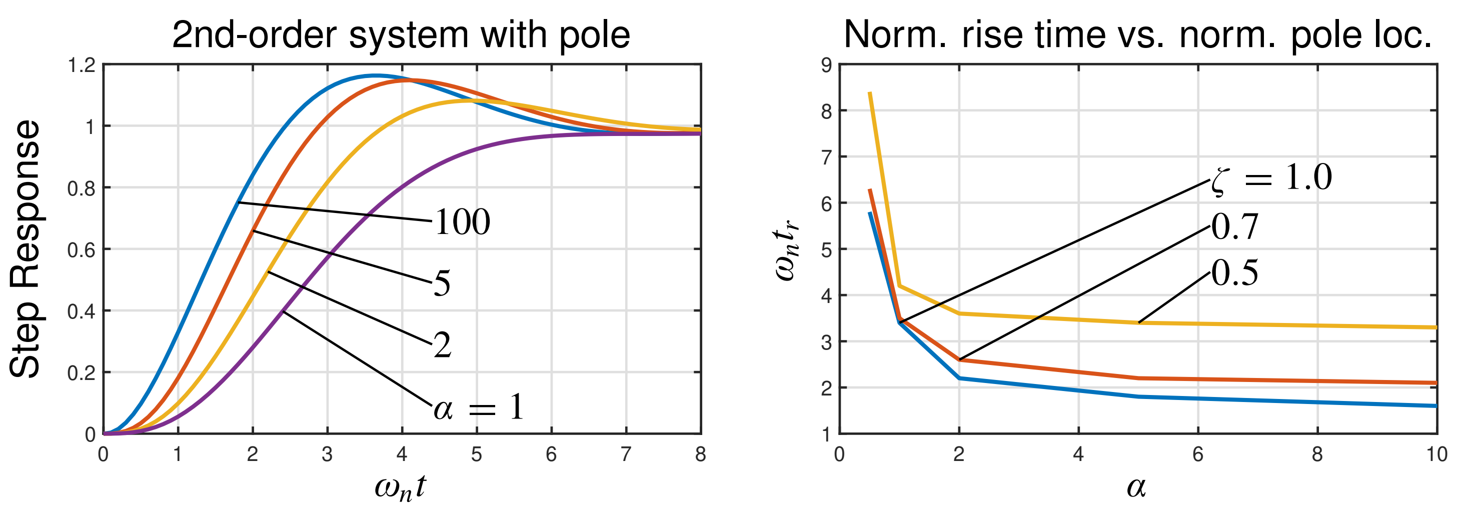
- If any of the zeros in RHP ($\alpha < 0$), system is nonminimum phase.



Add a pole to a second order system

$$H(s) = \frac{1}{\left(\frac{s}{\alpha\zeta\omega_n} + 1\right)[(s/\omega_n)^2 + 2\zeta s/\omega_n + 1]}.$$

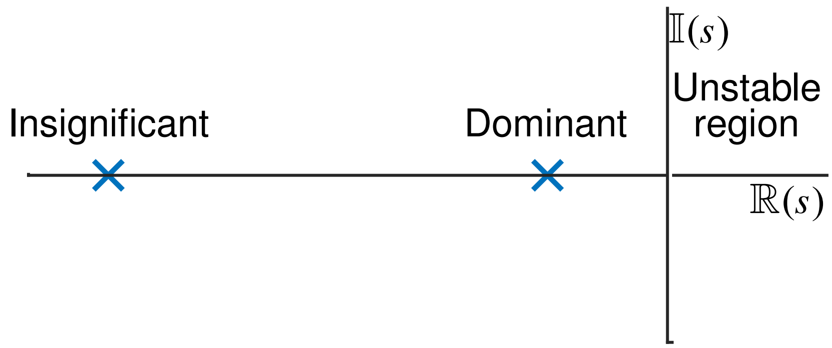
- Original poles at $\mathbb{R}(s) = -\sigma = -\zeta\omega_n$.
- New pole at $s = -\alpha\zeta\omega_n$.
- Major effect is an increase in rise time.



Summary of higher-order approximations

- Extra zero in LHP will increase overshoot if the zero is within a factor of $\approx$ 4 from the real part of complex poles.

- Extra zero in RHP depresses overshoot, and may cause step response to start in wrong direction. ***DELAY***.
- Extra pole in LHP increases rise-time if extra pole is within a factor of ≈ 4 from the real part of complex poles.



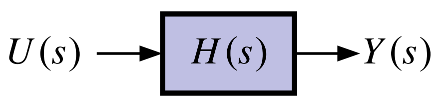
- MATLAB 'step' and 'impulse' commands can plot higher order system responses.
- Since a model is an approximation of a true system, it may be all right to reduce the order of the system to a first or second order system. If higher order poles and zeros are a factor of 5 or 10 time farther from the imaginary axis.
  - Analysis and design much easier.
  - Numerical accuracy of simulations better for low-order models.
  - 1st- and 2nd-order models provide us with great intuition into how the system works.
  - May be just as accurate as high-order model, since high-order model itself may be inaccurate.

3.7: Changing dynamic response

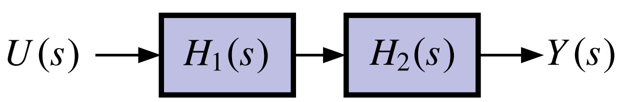
- Topic of the rest of the course.
- Important tool: block diagram manipulation.

Block-diagram manipulation

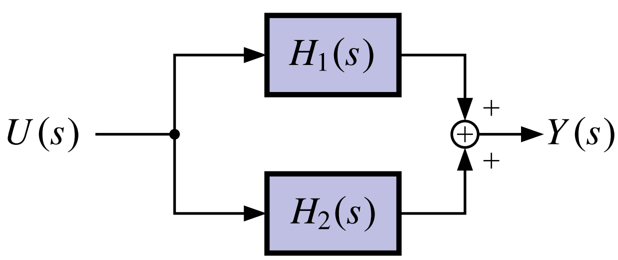
- We have already seen block diagrams (see pg. 1–4).
- Shows information/energy flow in a system, and when used with Laplace transforms, can simplify complex system dynamics.
- Four *BASIC* configurations:



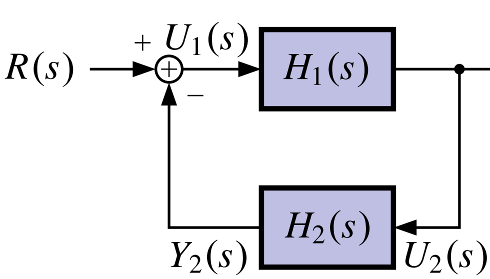
$$Y(s) = H(s)U(s)$$



$$Y(s) = [H_2(s)H_1(s)]\,U(s)$$



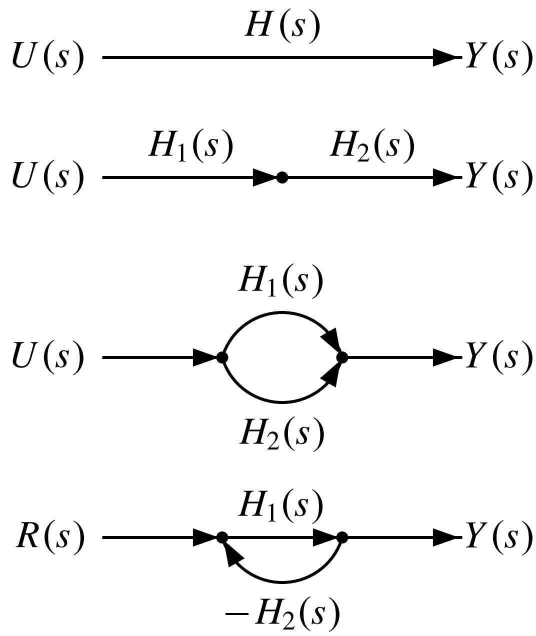
$$Y(s) = [H_1(s) + H_2(s)]\,U(s)$$



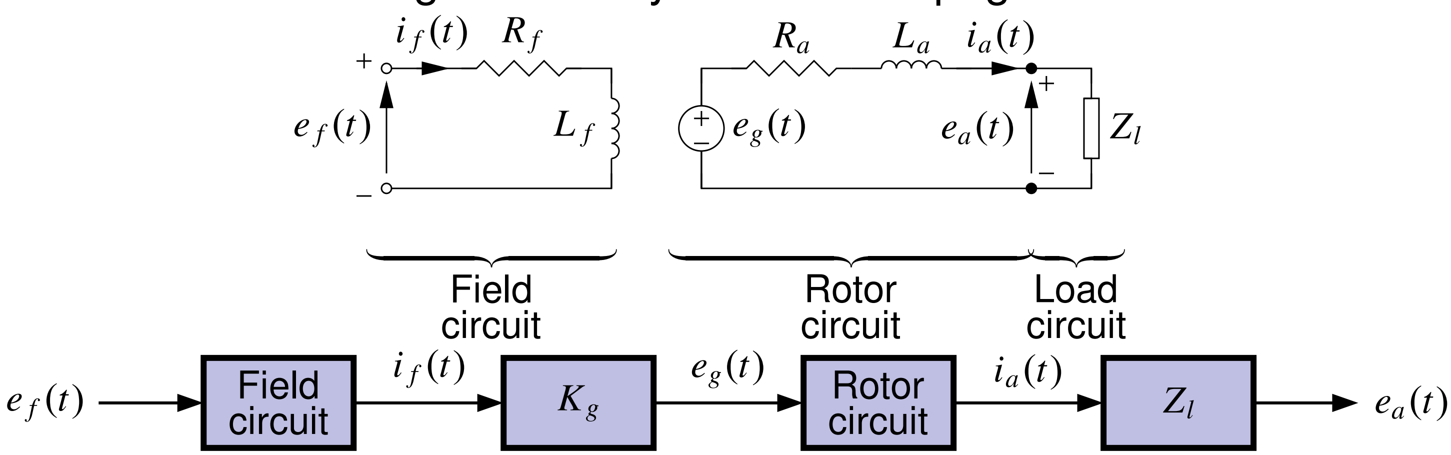
$$\begin{aligned} U_1(s) &= R(s) - Y_2(s) \\ Y_2(s) &= H_2(s)H_1(s)U_1(s) \end{aligned}$$

so, $$\begin{aligned} U_1(s) &= R(s) - H_2(s)H_1(s)U_1(s) \\ &= \frac{R(s)}{1 + H_2(s)H_1(s)} \\ Y(s) &= H_1(s)U_1(s) \\ &= \frac{H_1(s)}{1 + H_2(s)H_1(s)}R(s) \end{aligned}$$

- Alternate representation



EXAMPLE: Recall dc generator dynamics from page 2–19



- Compute the transfer functions of the four blocks.

$$e_f(t) = R_f i_f(t) + L_f \frac{\mathrm{d}}{\mathrm{d}t} i_f(t)$$

$$E_f(s) = R_f I_f(s) + L_f s I_f(s)$$

$$\frac{I_f(s)}{E_f(s)} = \frac{1}{R_f + L_f s}.$$

$$e_g(t) = K_g i_f(t)$$

$$E_g(s) = K_g I_f(s)$$

$$\frac{E_g(s)}{I_f(s)} = K_g.$$

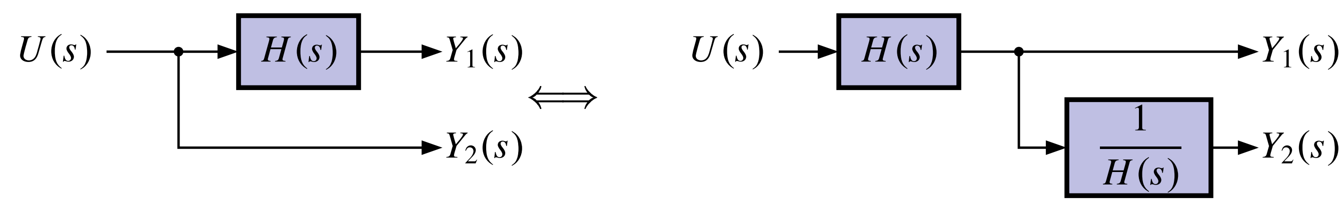
$$\begin{aligned} e_a(t) &= i_a(t) Z_l \\ E_a(s) &= Z_l I_a(s) \\ \frac{E_a(s)}{I_a(s)} &= Z_l. \end{aligned}$$

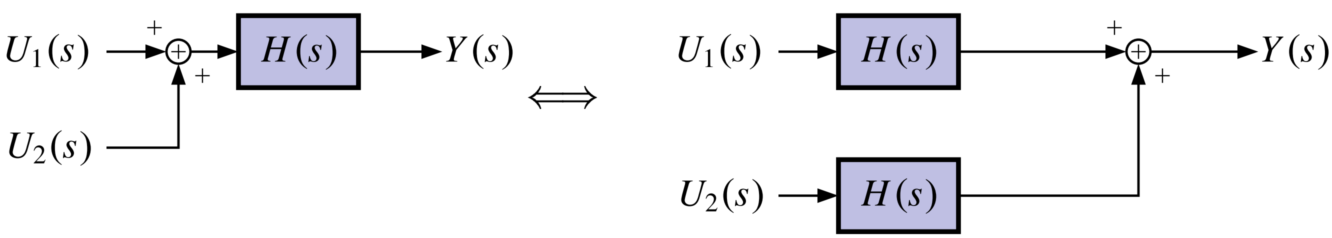
$$\begin{aligned} e_g(t) &= R_a i_a(t) + L_a \frac{\mathrm{d}}{\mathrm{d}t} i_a(t) + e_a(t) \\ E_g(s) &= R_a I_a(s) + L_a s I_a(s) + E_a(s) \\ &= (R_a + L_a s + Z_l)\, I_a(s) \\ \frac{I_a(s)}{E_g(s)} &= \frac{1}{L_a s + R_a + Z_l}. \end{aligned}$$

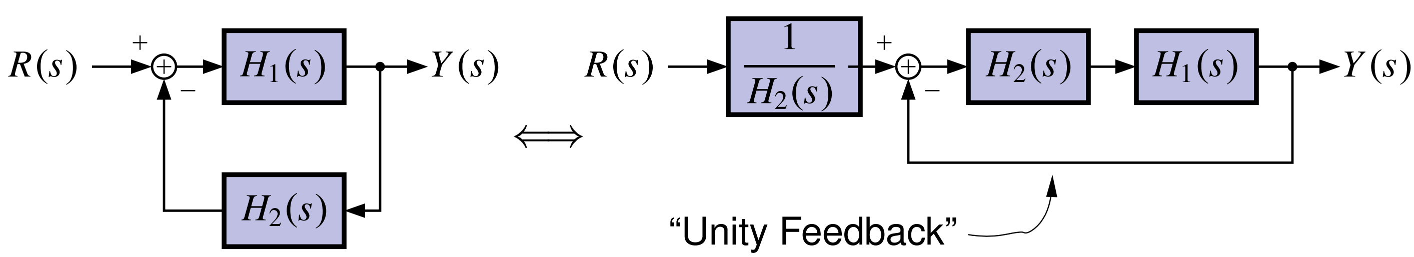
- Put everything together.

$$\begin{aligned} \frac{E_a(s)}{E_f(s)} &= \frac{E_a(s)}{I_a(s)} \frac{I_a(s)}{E_g(s)} \frac{E_g(s)}{I_f(s)} \frac{I_f(s)}{E_f(s)} \\ &= \frac{K_g Z_l}{\left(L_f s + R_f\right)\left(L_a s + R_a + Z_l\right)}. \end{aligned}$$

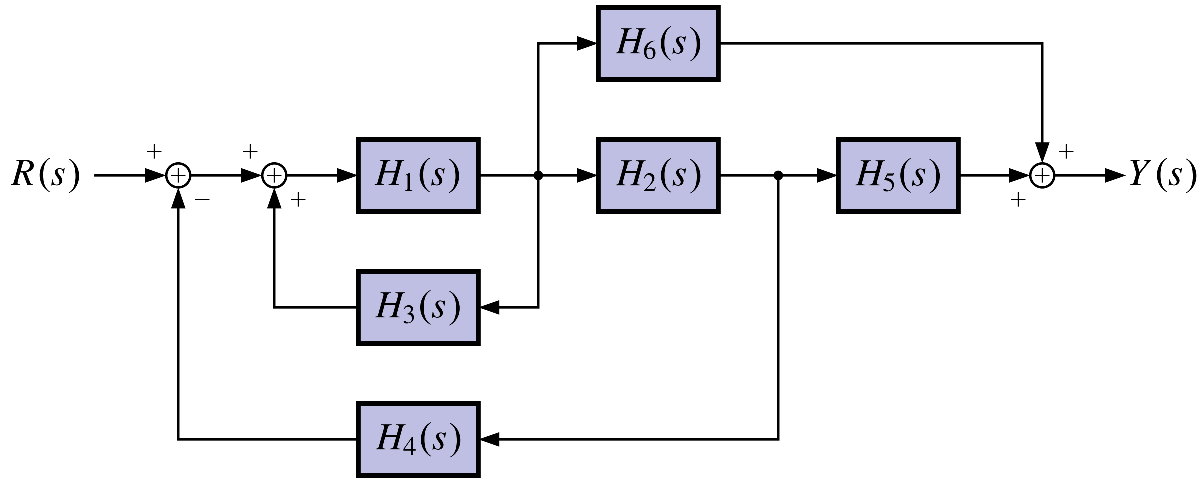
Block-diagram algebra

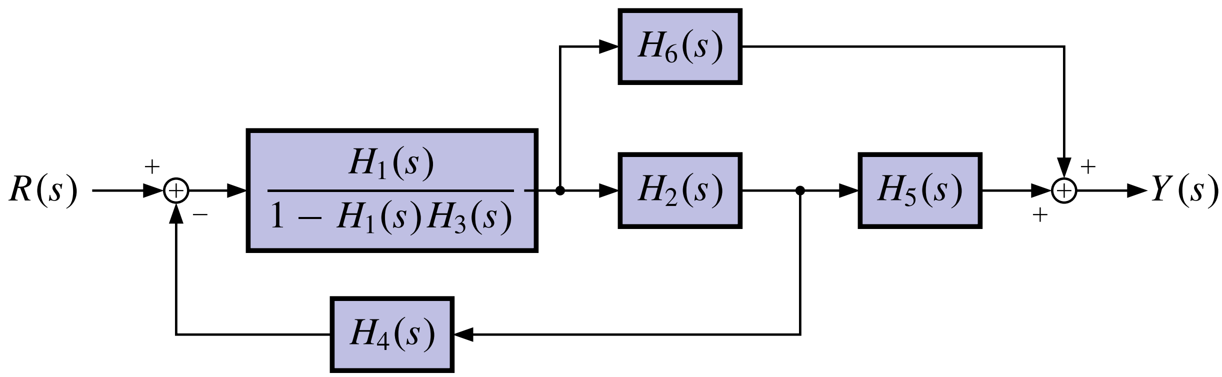


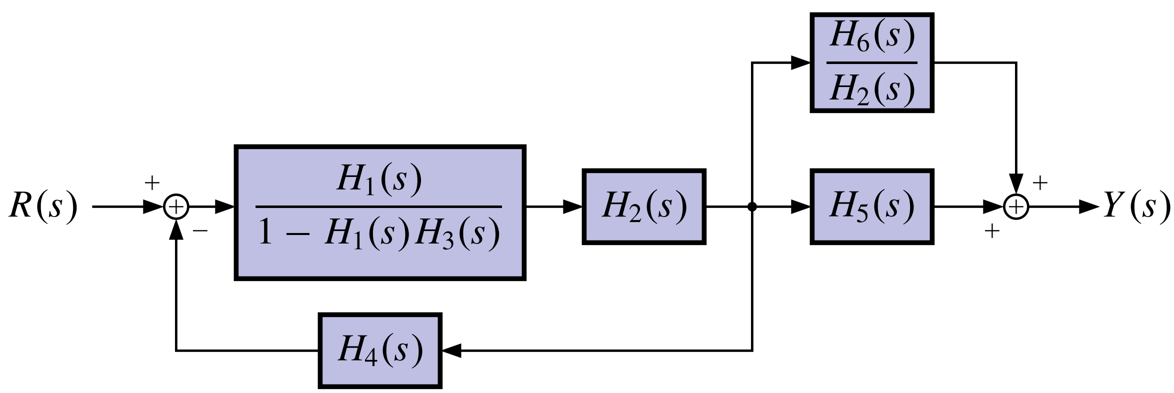




EXAMPLE: Simplify:







$$\frac{\left(\frac{H_1(s)H_2(s)}{1-H_1(s)H_3(s)}\right)}{\left(1+\frac{H_1(s)H_2(s)H_4(s)}{1-H_1(s)H_3(s)}\right)} \qquad H_5(s)+\frac{H_6(s)}{H_2(s)}$$